Foundation course - PHYSICS

Lecture 6-2: **Momentum; rotational motion**

Dynamics of solid bodies

- Center of mass
- Linear momentum
- Impulse
- Rotational motion
- Torque
- Angular momentum

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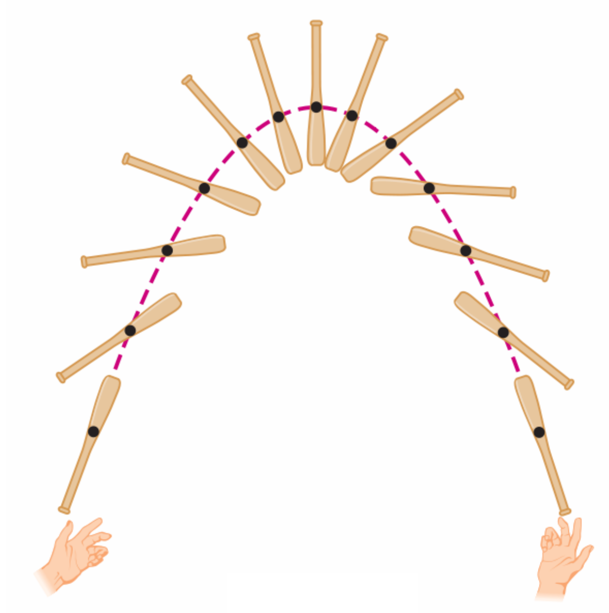
Center of mass



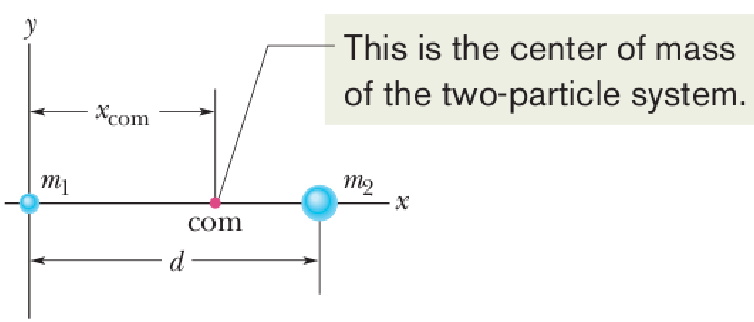
The center of mass of a system of particles is the point that moves as though:

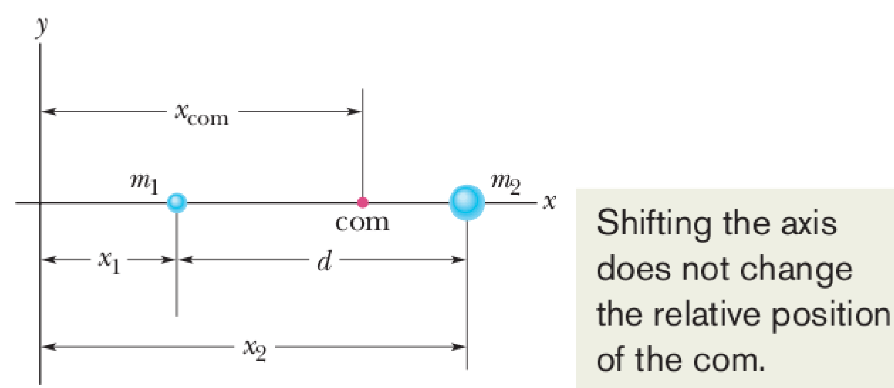
- all of the **system's mass** were concentrated there
- all **external forces** were applied there

System of two particles:

The center of mass position:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$





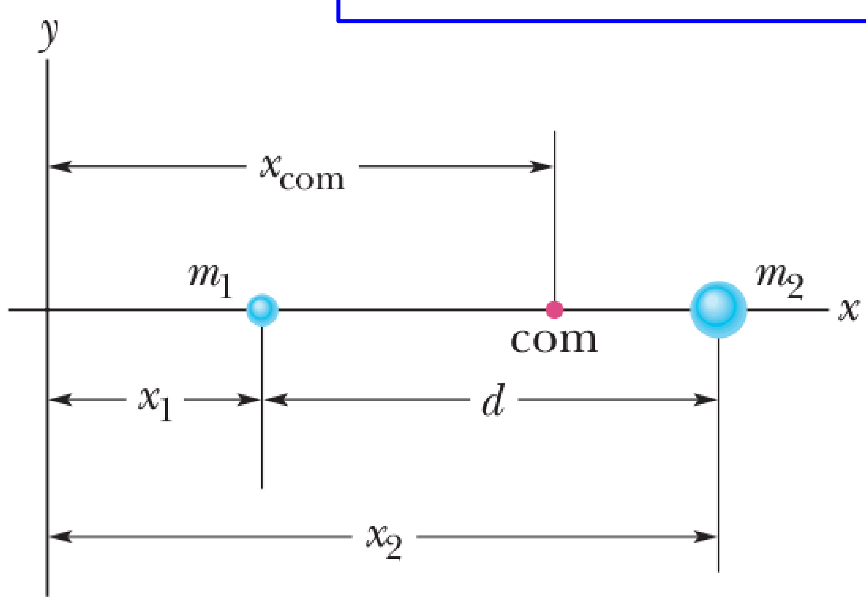
Center of mass

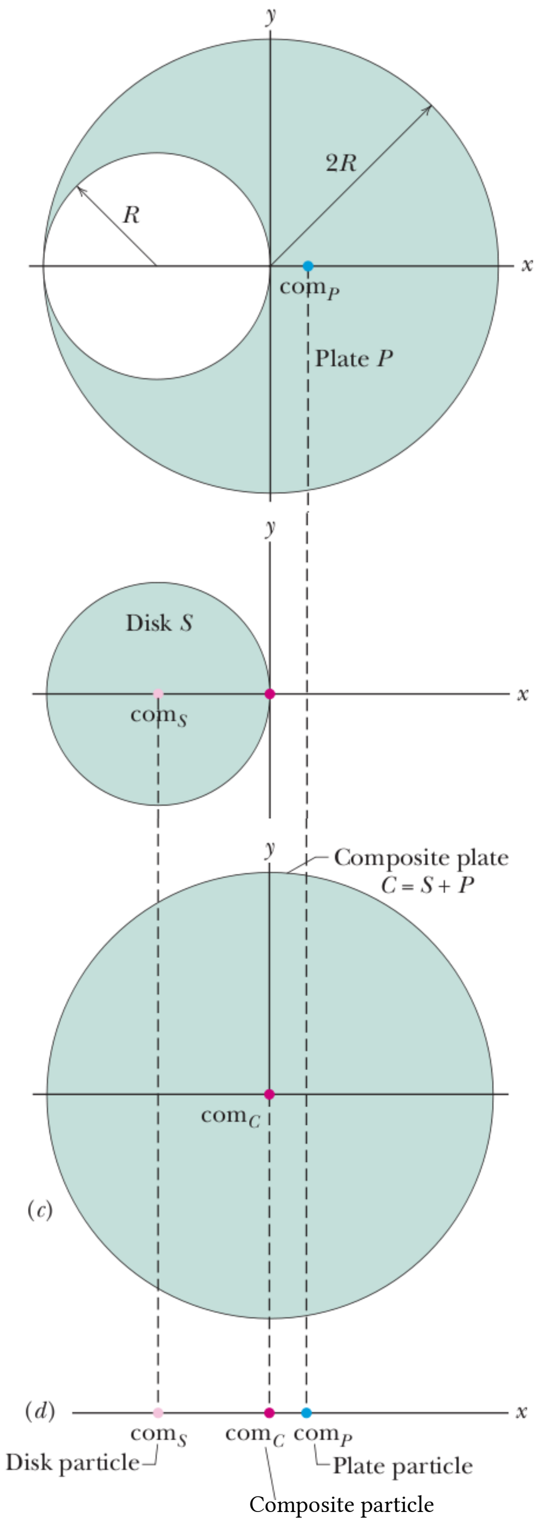
The center of mass (COM):

- all of the **system's mass** were concentrated there
- all **external forces** were applied there

The center of mass position:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$





Linear momentum

Linear momentum of a particle is a vector quantity that is defined as:

$$\boldsymbol{p} = m\,\boldsymbol{v}$$

Newton's second law expressed in terms of momentum:

$$\boldsymbol{F}_{net} = \frac{d\,\boldsymbol{p}}{d\,t}$$

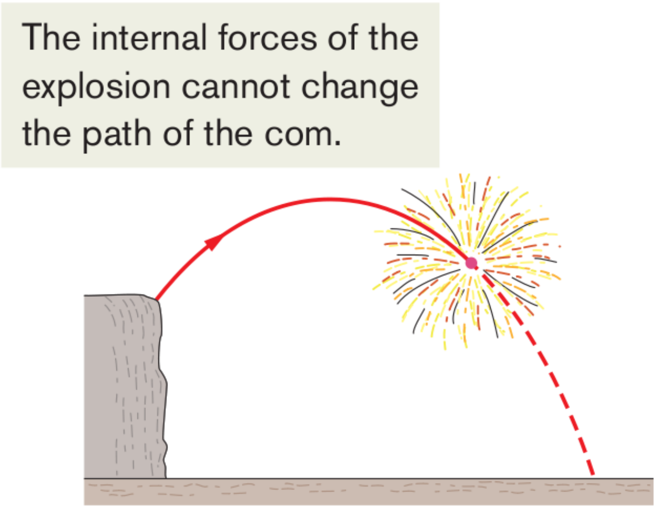
$$\boldsymbol{F}_{net} = m\frac{d\,\boldsymbol{v}}{d\,t} = m\,\boldsymbol{a}$$

System of two and more particles:

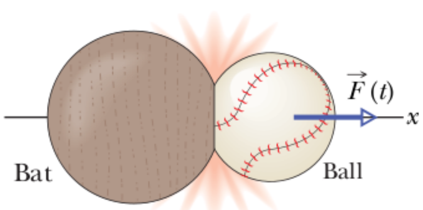
$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$$

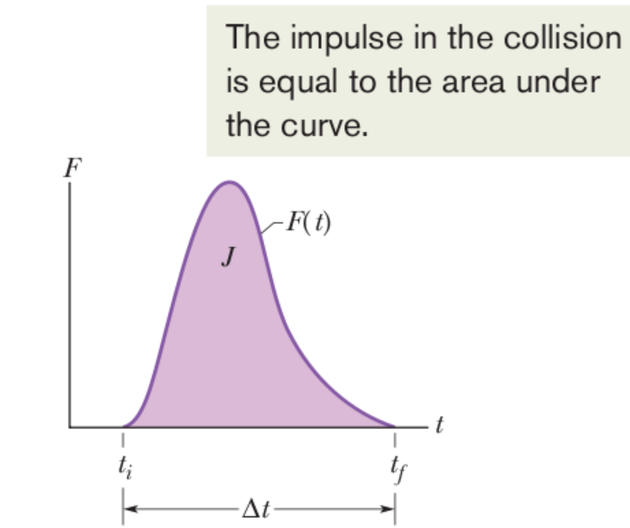
$$\boldsymbol{P} = M\,\boldsymbol{v}_{com} \qquad \boldsymbol{F}_{net} = M\,\boldsymbol{a}_{com}$$

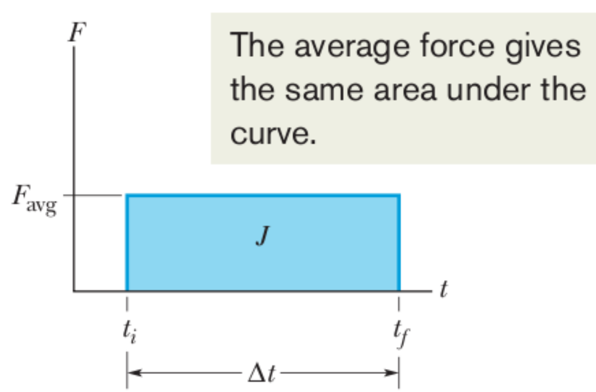
M is a total mass of the system



Collision and impulse







Collision:

The external force acting on a body is brief, has large magnitude, and suddenly changes the body's momentum

Impulse:

The impulse on an object is the product of the average force on an object and the time interval over which it acts.

Impulse-momentum theorem

Newton's second law of motion: $\boldsymbol{F} = m\boldsymbol{a} = m\left(\dfrac{\Delta \boldsymbol{v}}{\Delta t}\right)$

$$\boldsymbol{F}\Delta t = m\Delta \boldsymbol{v} = m\boldsymbol{v}_f - m\boldsymbol{v}_i = \boldsymbol{p}_f - \boldsymbol{p}_i = \Delta \boldsymbol{p}$$

Impulse-momentum theorem: $\boldsymbol{J} = \boldsymbol{F}_{avg}\Delta t = \Delta \boldsymbol{p}$

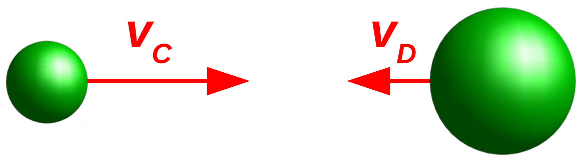
The impulse on an object is equal to the change in its momentum.

- a large impulse causes a large change in momentum
- the large impulse could result either from a large ***F*** acting over a short *Δt* or from a smaller ***F*** acting over a longer *Δt*

<u>Example:</u> Air bags in cars reduce injuries by making the force on an passenger less, by increasing the time interval of force acting and by spreading the force over a larger area of the person's body.

Conservation of momentum

Collision of two balls (closed and isolated system):



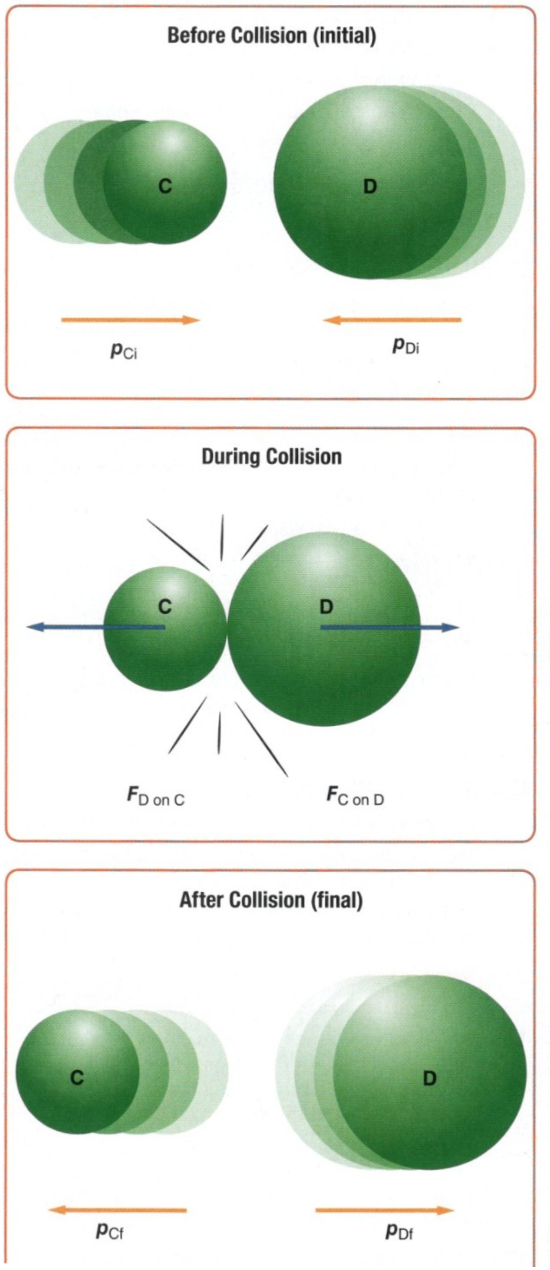
$$F_{DC} = -F_{CD} \qquad (F\Delta t)_{DC} = -(F\Delta t)_{CD}$$

$$p_{Cf} - p_{Ci} = -(p_{Df} - p_{Di}) \qquad p_{Cf} + p_{Df} = p_{Ci} + p_{Di}$$

A system with conserved mass = **closed system**
A system with the zero net external force (only internal forces are included) = **isolated system**

Law of conservation of momentum:
Momentum of any closed, isolated system does not change.

$$P = constant$$



Checkpoint question:

An initially stationary device lying on a frictionless floor explodes into two pieces, which then slide across the floor. One piece slides in the positive direction of an x axis. (a) What is the sum of the momenta of the two pieces after the explosion? (b) Can the second piece move at an angle to the x axis? (c) What is the direction of the momentum of the second piece?

Collisions: momentum and kinetic energy

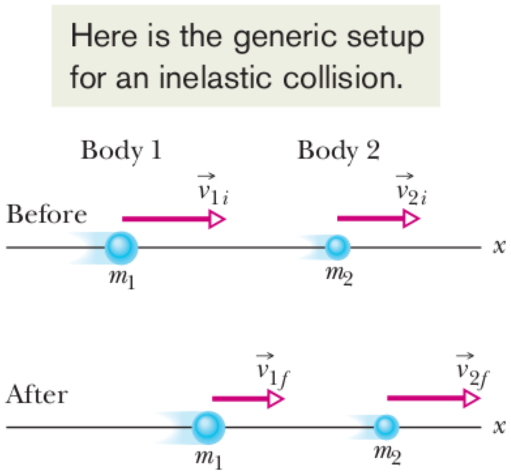
<u>Closed and isolated system</u>: **momentum** of the system is **constant**

Collision is **elastic**: kinetic energy of the system is **conserved**

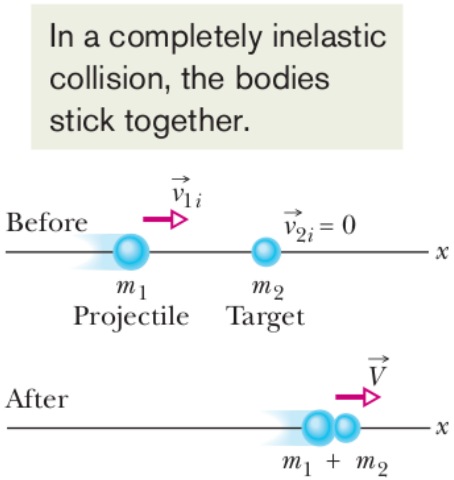
Collision is **inelastic**: kinetic energy of the system is **not** conserved

Inelastic collisions in one dimension

$$\boldsymbol{p}_{1i} + \boldsymbol{p}_{2i} = \boldsymbol{p}_{1f} + \boldsymbol{p}_{2f}$$



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

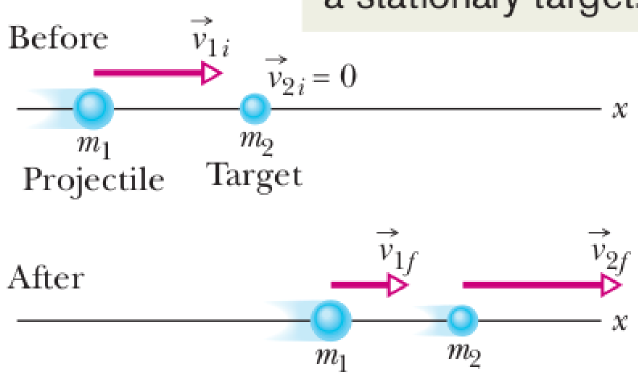


$$m_1 v_{1i} = (m_1 + m_2) V$$

Collisions: momentum and kinetic energy

Elastic collisions in one dimension

Here is the generic setup for an elastic collision with a stationary target.



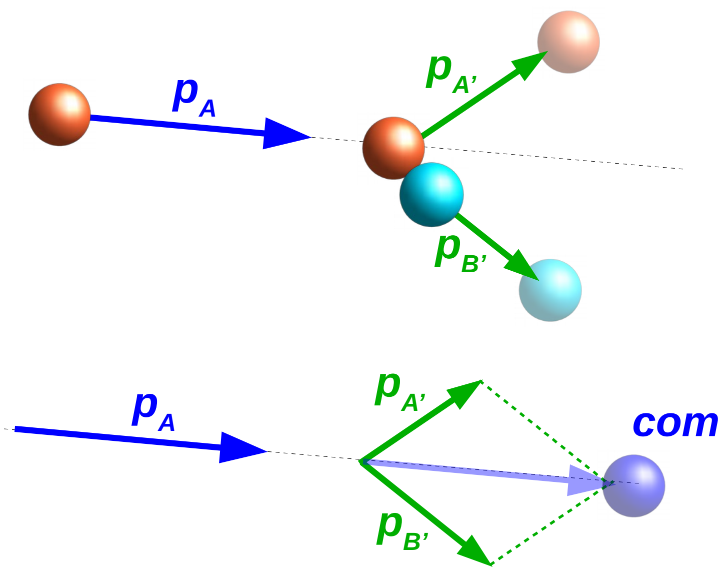
$$\boldsymbol{p}_{1i} + \boldsymbol{p}_{2i} = \boldsymbol{p}_{1f} + \boldsymbol{p}_{2f}$$

$$E_{k1i} + E_{k2i} = E_{k1f} + E_{k2f}$$

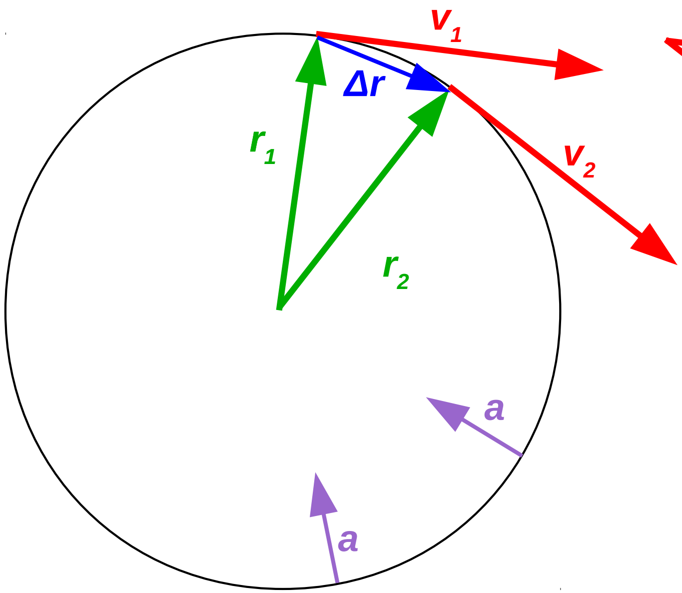
$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

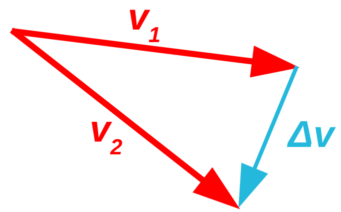
$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Collisions in two dimensions



Circular motion





Average velocity: $\bar{\boldsymbol{v}} = \dfrac{\Delta \boldsymbol{r}}{\Delta t}$

Average acceleration: $\bar{\boldsymbol{a}} = \dfrac{\Delta \boldsymbol{v}}{\Delta t}$

Centripetal acceleration: $a_c = \dfrac{v^2}{r}$

Period of revolution T:

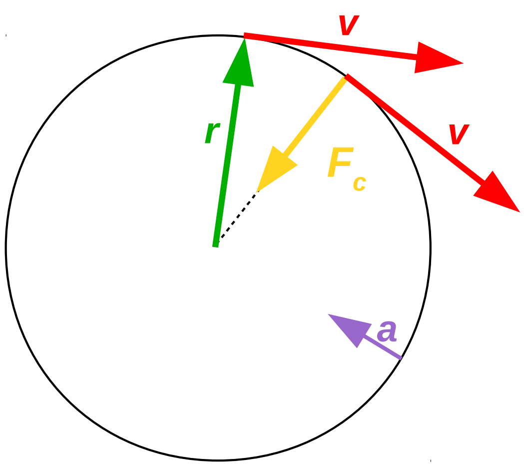
- time needed for the object to make one complete revolution
- during this time the object travels a distance equal to the circumference of the circle (*2πr*)

$$v = \frac{2\pi r}{T} \qquad a_c = \frac{(2\pi r/T)^2}{r} = \frac{4\pi^2 r}{T^2}$$

Centripetal force

Centripetal force

- because the acceleration of an object is always in the direction of the net force acting on it, the net force must be toward the center of the circle



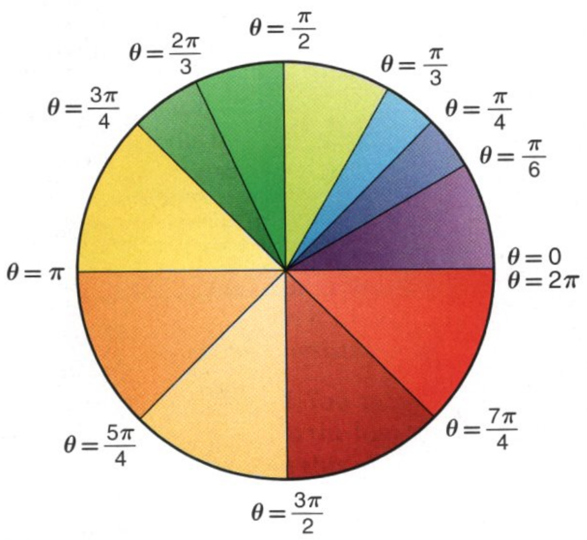
Newton's second law for circular motion

$$F_c = m a_c \;\rightarrow\; \boxed{F_c = \frac{m v^2}{R}}$$

Examples:

- Earth circling the Sun – F_c is Sun's gravitational force
- Hammer thrower swings the hammer – F_c is the tension in the chain attached to the ball

Angular displacement



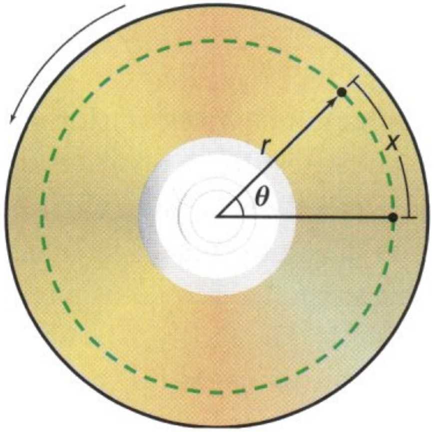
Fraction of one revolution can be measured:

- in degrees (one complete revolution is 360°)
- in radians (one complete revolution is 2π)

Radian is related to the ratio of the circumference of a circle to its radius.

Measuring distance:

- one complete revolution … $x = 2\pi r$
- generally for an angle θ … $x = \theta r$



Radians are dimensionless.
Clockwise rotation is negative, counterclockwise rotation is positive.

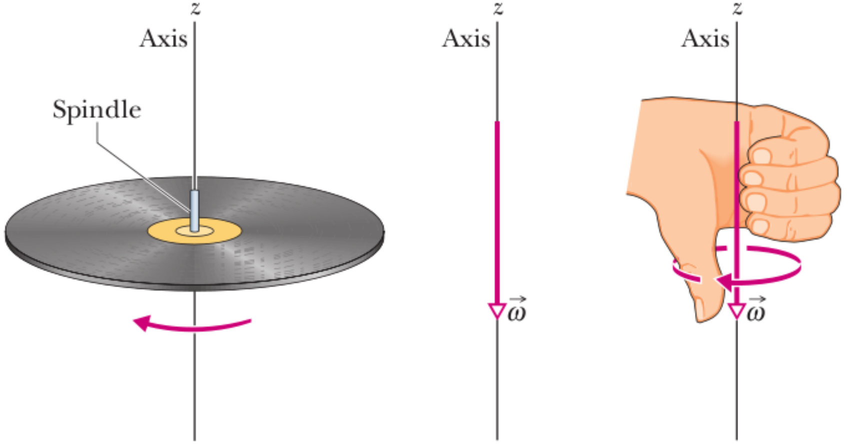
Angular displacement θ:

- the change in the angle if an object rotates

$$\theta = \frac{x}{r}$$

Angular velocity

- is defined as and angular displacement divided by the time taken to make the angular displacement



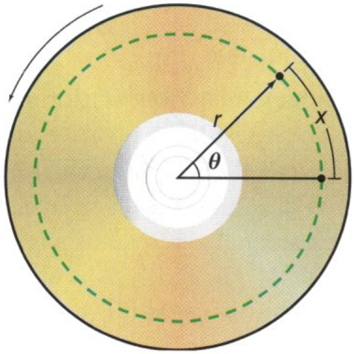
Average angular velocity ω:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

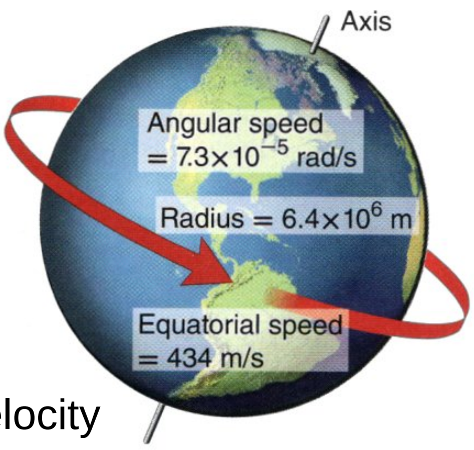
Instantaneous angular velocity equals the slope of a graph of angular position versus time.

Direction of angular velocity: the right-hand rule

Linear velocity of a point at a distance *r* from the axis of rotation:



$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{r\,\Delta\theta}{\Delta t} = r\,\omega$$



Earth's angular velocity

Angular acceleration

Angular acceleration is the change in angular velocity divided by the time required to make the change

Average angular acceleration *α*:

$$\alpha = \frac{\Delta \omega}{\Delta t}$$

Angular frequency:
number of complete revolutions made by an object in 1 s

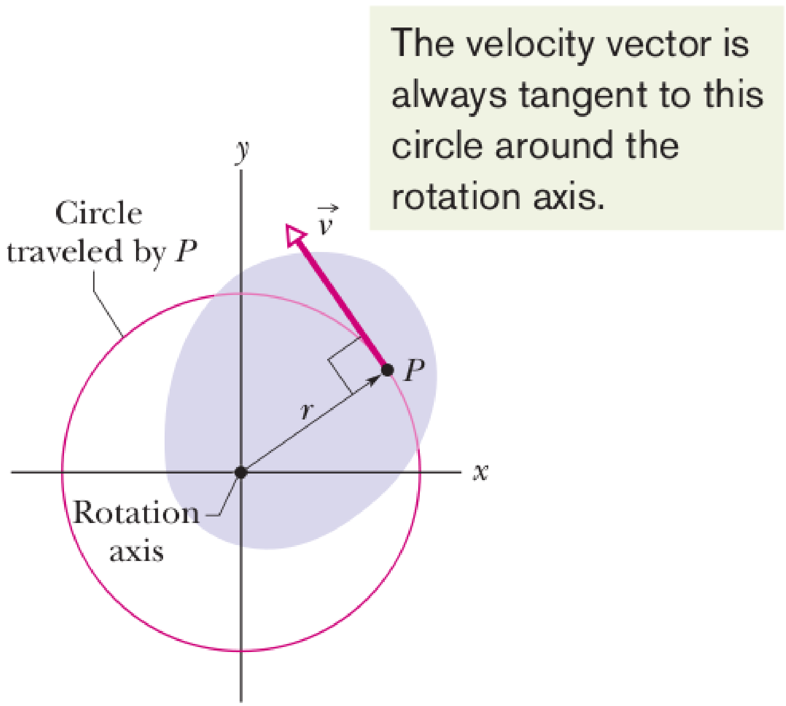
$$f \equiv \frac{\omega}{2\pi}$$

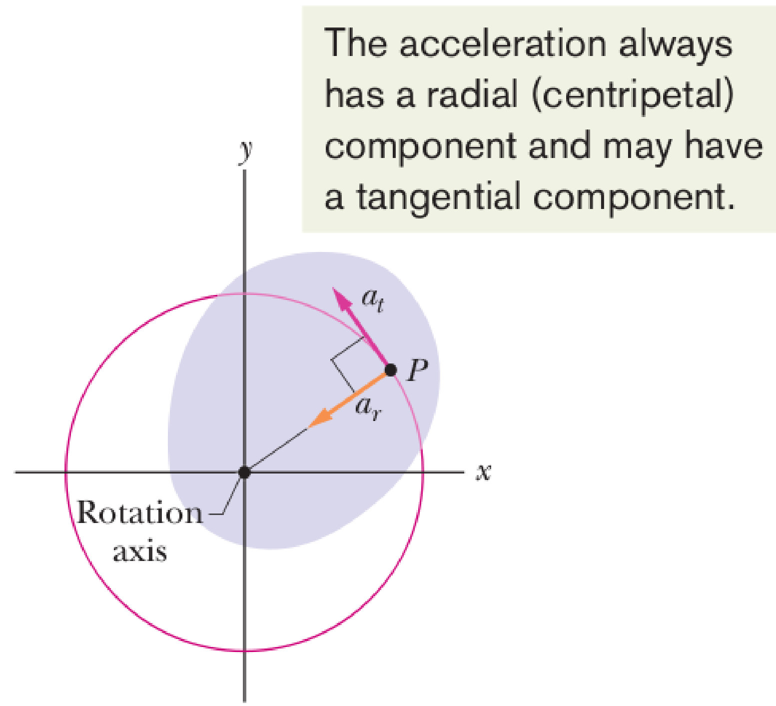
Linear and angular measurements

Quantity	*Linear*	*Angular*	*Relationship*
displacement	$\boldsymbol{x}$ (m)	$\boldsymbol{\theta}$ (rad)	$\boldsymbol{x} = \boldsymbol{r}\,\boldsymbol{\theta}$
velocity	$\boldsymbol{v}$ (m/s^1)	$\boldsymbol{\omega}$ (rad/s^1)	$\boldsymbol{v} = \boldsymbol{r}\,\boldsymbol{\omega}$
acceleration	$\boldsymbol{a}$ (m/s^2)	$\boldsymbol{\alpha}$ (rad/s^2)	$\boldsymbol{a} = \boldsymbol{r}\,\boldsymbol{\alpha}$

Acceleration

Linear speed: $v = \omega r$





Tangential acceleration a_t:

$$a_t = \alpha r$$

Radial acceleration a_r:

$$a_r = -\frac{v^2}{r} = -\omega^2 r$$

Kinetic energy of rotation

$$E_k = \frac{1}{2} m v^2$$ → This equation is valid only for a particle

Rigid body is a collection of particles with different speeds:

$$E_k = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots = \sum \frac{1}{2} m_i v_i^2$$

$$E_k = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \left(\sum m_i r_1^2\right) \omega^2$$

Moment of inertia *I*:

$$I = \sum m_i r_1^2$$

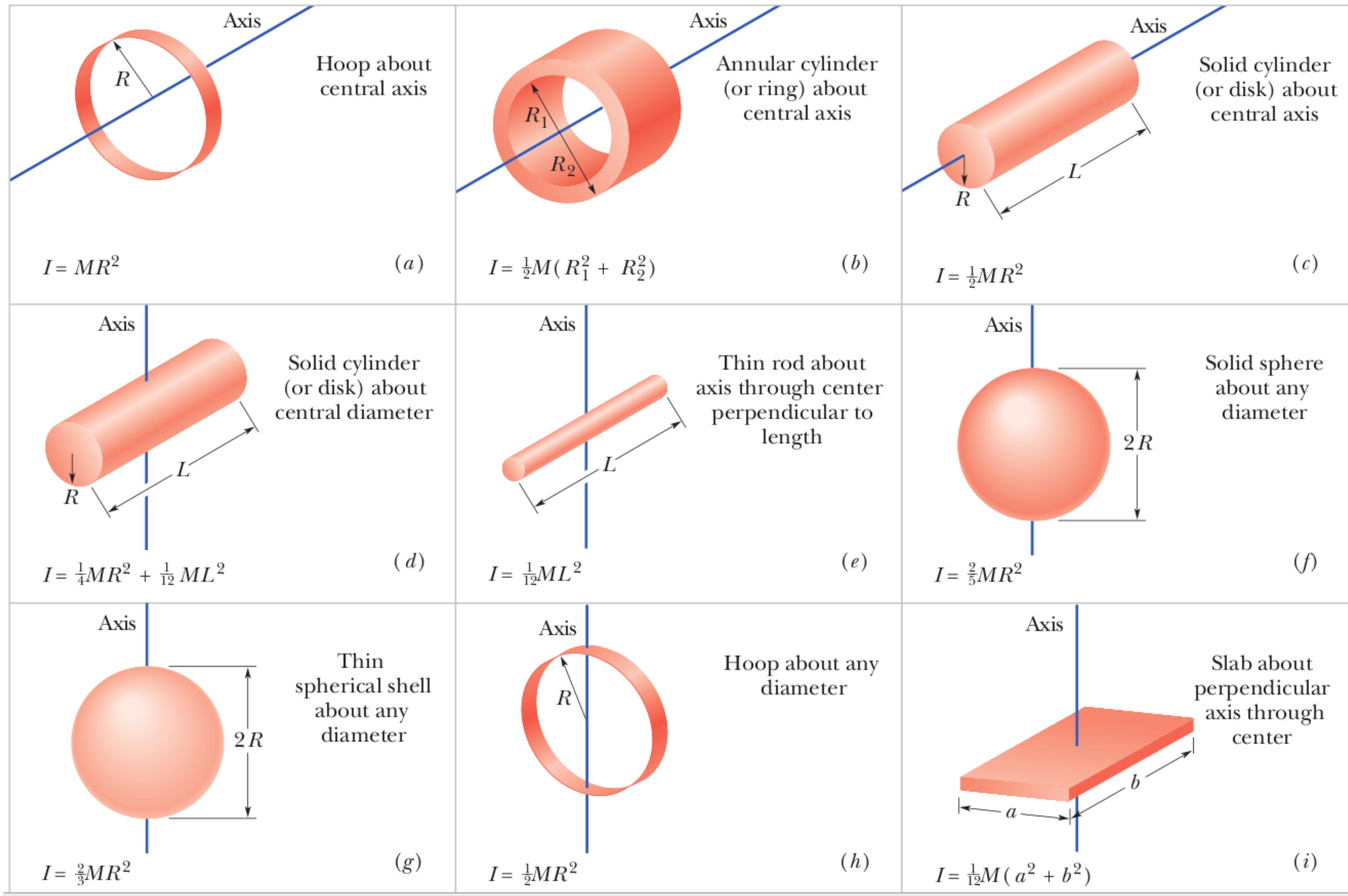
Rotational kinetic energy:

$$E_k = \frac{1}{2} I \omega^2$$

The **moment of inertia** characterizes the resistance to rotation

Moment of inertia of a point mass: $I = m r^2$

The moment of inertia

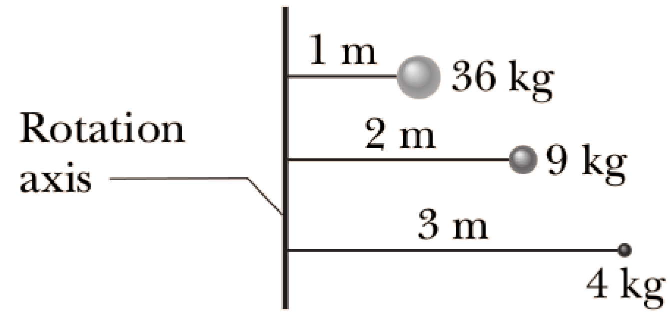


Parallel-axis theorem: $I = I_{com} + Mh^2$

h...distance between parallel axis and axis through the center of mass

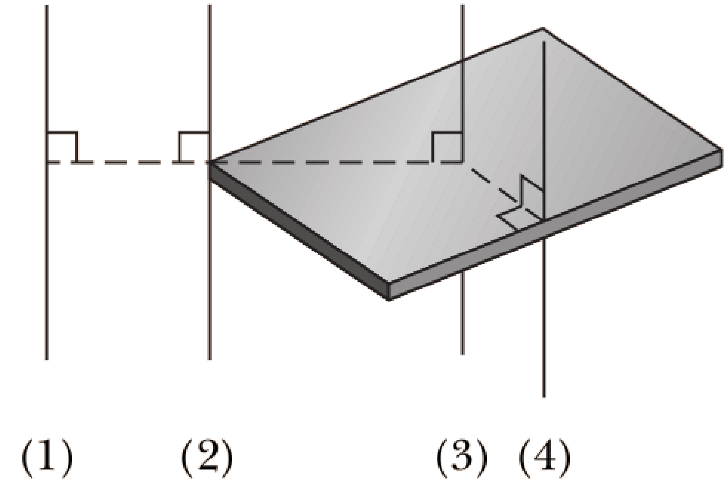
Checkpoint question:

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



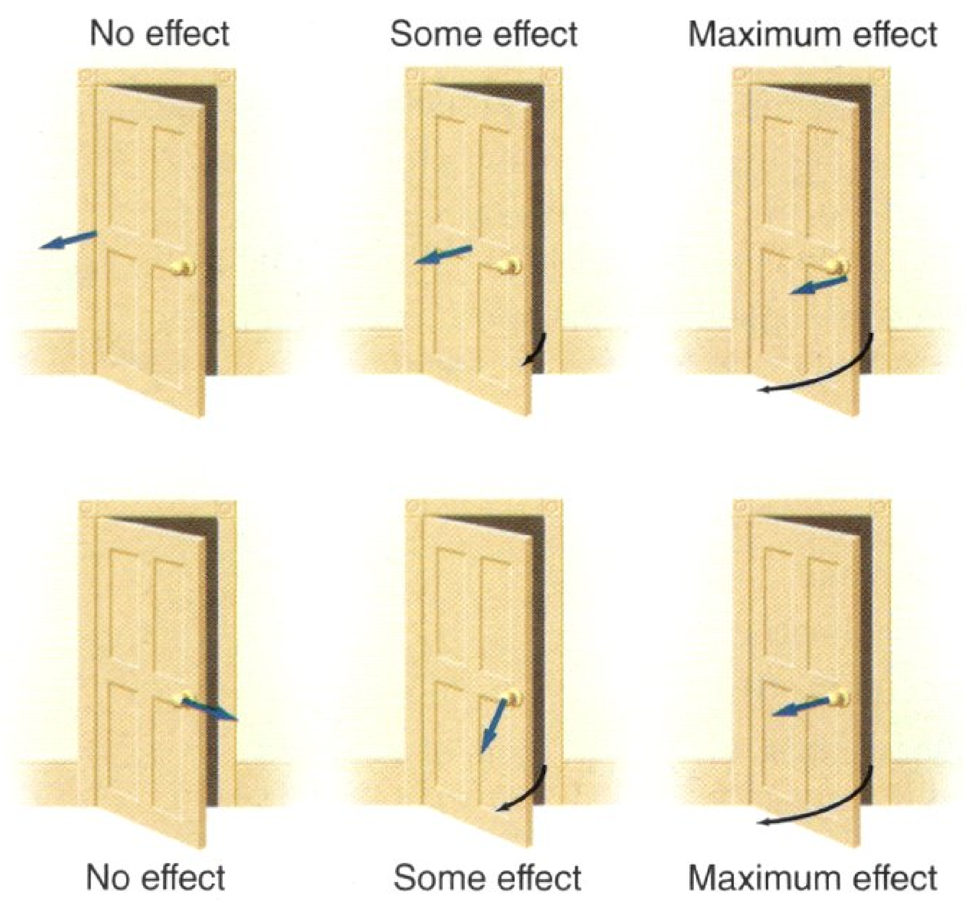
Checkpoint question:

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.



Torque

How to open a door most easily – how to get the most effect from the least force



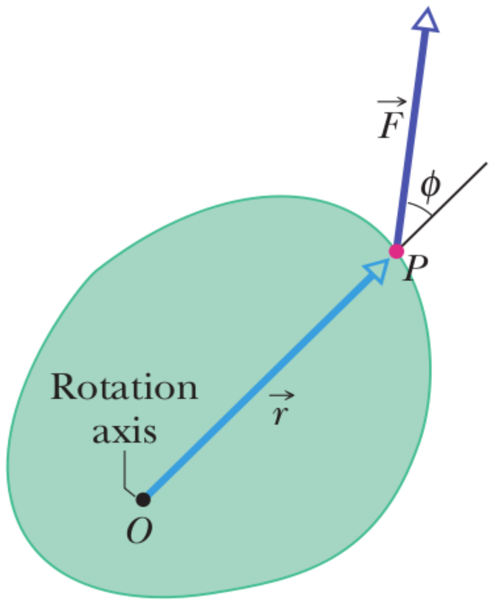
Application of the force **farthest** from the hinges is most effective.

Application of the force at an angle **perpendicular** to the door is most effective.

Torque

Torque *τ*:

- quantity characterizing the ability of the force ***F*** to rotate the body
- it depends on the magnitude of F_t and how far from *O* the force is applied



The torque due to this force causes rotation around this axis (which extends out toward you).

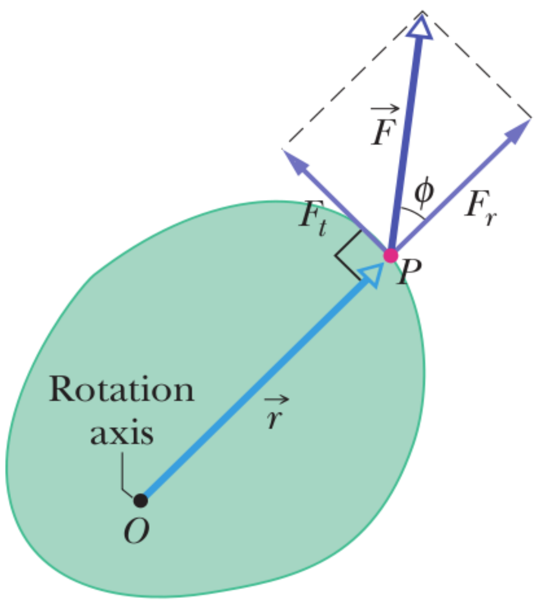
Torque is:

- is a vector
- is a vector product of ***r*** and ***F***
- measured in units N.m

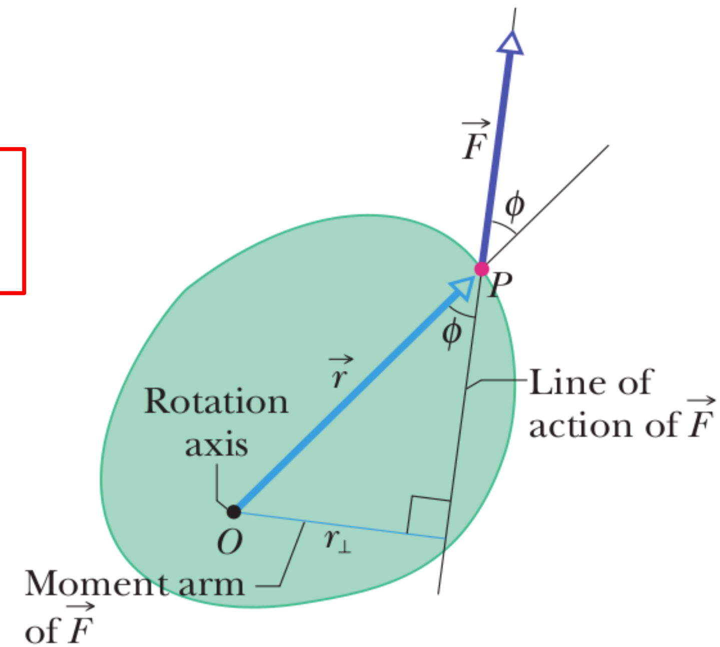
$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

$$\tau = \boldsymbol{r}\,\boldsymbol{F} \sin\phi$$

Torque



$$\tau = r F \sin\phi$$



But actually only the *tangential* component of the force causes the rotation.

You calculate the same torque by using this moment arm distance and the full force magnitude.

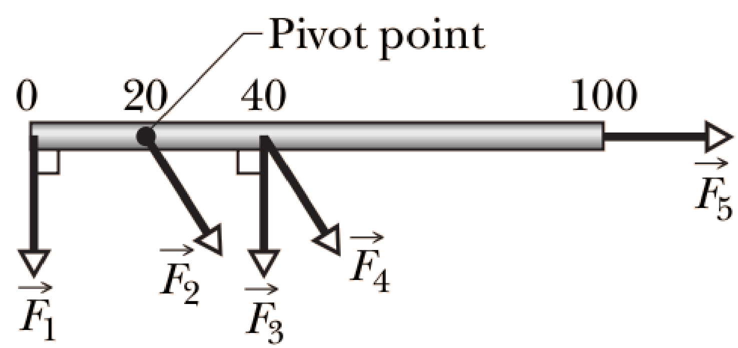
Two equivalent ways of computing the torque:

$$\tau = (r)(F\sin\phi) = r F_t$$

$$\tau = (r\sin\phi)(F) = r_\perp F$$

Checkpoint question:

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.



Torque

Direction of torque:

- clockwise rotation = torque is **negative**
- counterclockwise rotation = torque is **positive**

When several torques act on a body, the **net torque** is the sum of the individual torques.

Torque is a vector and a vector product of ***r*** and ***F***:

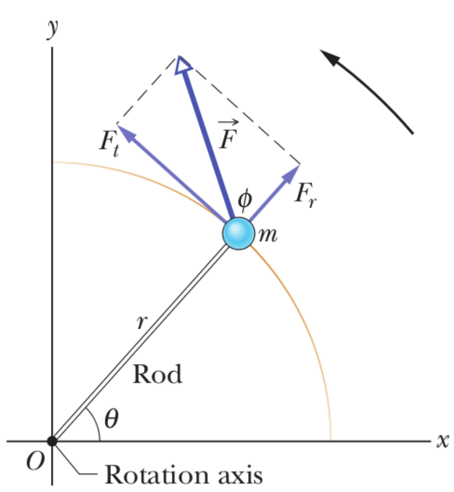
$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

Newton's second law for rotation:

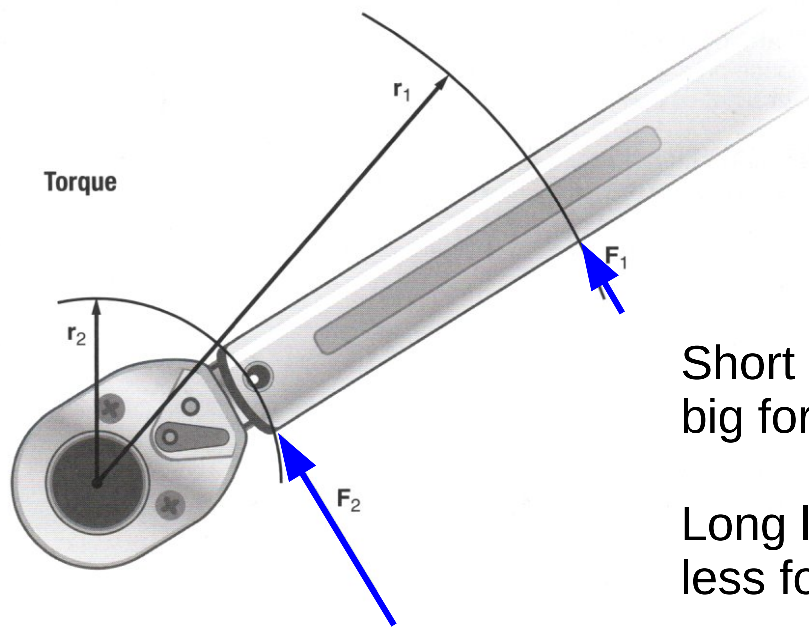
$$F_{net} = m\,a \;\Rightarrow\; \boxed{\tau_{net} = I\,\alpha}$$

$$\tau = F_t r = m a_t r = m(\alpha r) r = (m r^2)\alpha = I\alpha$$

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

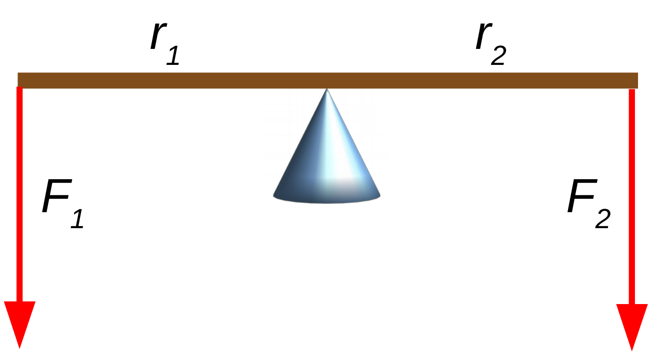


Finding net torque



Short lever arm requires to exert big force

Long lever arm requires to exert less force



$$\tau_N = \tau_1 + \tau_2$$
$$r_1 = r_2 \text{ and } F_1 = F_2$$
$$\tau_N = F_1 r_1 - F_2 r_2 = 0$$

System is in equilibrium and does not rotate

Angular momentum

$$\tau = I\alpha = I\frac{\Delta\omega}{\Delta t}$$

$$\tau\Delta t = I\Delta\omega = I\omega_f - I\omega_i$$

Angular momentum *L*:

$$L = I\omega$$

units: 1 kg.m^2.s^{-1}

The angular momentum is defined as a product of the object's moment of inertia and the object's angular velocity.

Corresponding variables:

Translational motion		**Rotational motion**	
Position	***x***	Angular position	***θ***
Velocity	***v***	Angular velocity	***ω***
Acceleration	***a***	Angular acceleration	***α***
Mass	***m***	Moment of inertia	***I***
Force	***F***	Torque	***τ***
Linear momentum	***p***	Angular momentum	***L***

Conservation of angular momentum

An isolated system's initial angular momentum is equal to its final angular momentum.

$$L_i = L_f$$

$$L = constant$$

angular velocity: $\omega_1 < \omega_2$

Spinning slowly

Spinning quickly

$L = I\,\omega$ … is constant

<u>Increased</u> **angular velocity** is accompanied by a <u>decreased</u> **moment of inertia.**

Moment of inertia can be decreased by decreasing the radius of rotation

$$I \sim r^2$$

The direction of rotation of a spinning object can be changed only by applying a torque.