MATEMATIKA PRE OPTOMETROV

cvičenie 3

príklad 1. Je dané $r_1 = 2+3i$, $r_2 = 4-3i$. Vypočítajte

a) $r_1 + r_2$ b) $r_1 - r_2$ c) $|r_1|$ d) $r_1 r_2$ e) $\frac{r_1}{r_2}$

a) $(2+3i) + (4-3i) = 6$

b) $(2+3i) - (4-3i) = -2+6i$

c) $|r_1| = |2+3i| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

d) $(2+3i)(4-3i) = \cancel{4}\ 8 - 6i + 12i - 9i^2 = 17 + 6i$

e) $\frac{2+3i}{4-3i} = \frac{2+3i}{4-3i} \cdot \frac{4+3i}{4+3i} = \frac{(2+3i)(4+3i)}{(4-3i)(4+3i)} =$

$= \frac{8+6i+12i+9i^2}{16+12i-12i-9i^2} = \frac{-1+18i}{25} = -\frac{1}{25} + \frac{18}{25}i$

príklad 2. Nájdite algebraický tvar nasledujúcich komplexných čísel (upravte ich):

a) $\frac{2+i}{3-i} + (-2+i)(4-i)$

b) $\frac{i+1}{i} : \frac{i-1}{2i+3}$

a) $\ldots = \frac{2+i}{3-i} \cdot \frac{3+i}{3+i} + (-8+2i+4i-i^2) =$

$= \frac{6+2i+3i+i^2}{3^2+1^2} + (-7+6i) =$

$= \frac{5+5i}{10}\ \cancel{+} -7+6i = -6{,}5 + 6{,}5i$

b) $\ldots = \frac{i+1}{i} \cdot \frac{2i+3}{i-1} = \frac{2i^2+3i+2i+3}{i^2-i} =$

$$= \frac{1+5i}{-1-i} = \frac{1+5i}{-1-i} \cdot \frac{-1+i}{-1+i} = \frac{-1+i-5i+5i^2}{(-1)^2+1^2} =$$

$$= \frac{-6-4i}{2} = -3-2i$$

príklad 3. Vypočítajte i^{27}.

$i^1 = i \quad i^2 = -1 \quad i^3 = i \cdot i^2 = -i \quad i^4 = i^2 \cdot i^2 = 1$

$i^5 = i \cdot i^4 = i \quad i^6 = i^2 i^4 = -1 \quad \ldots\ldots$

$$\Rightarrow \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i, \quad i^{4n+4} = 1 \qquad n \in \mathbb{N}_0$$

$$\Rightarrow \quad i^{27} = i^{4\cdot 6+3} = -i$$

príklad 4. Nájdite reálne čísla x, y, ktoré sú riešením rovnice $(5+2i)x + (3-5i)y = 124$.

$$(5+2i)x + (3-5i)y = 124$$
$$(5x+3y) + i(2x-5y) = 124 + 0 \cdot i$$

Dve komplexné čísla z_1 a z_2 sa rovnajú práve vtedy, keď $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$ a zároveň ~~Re~~ $\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$

$\Rightarrow$ I. $5x + 3y = 124$

II. $2x - 5y = 0$

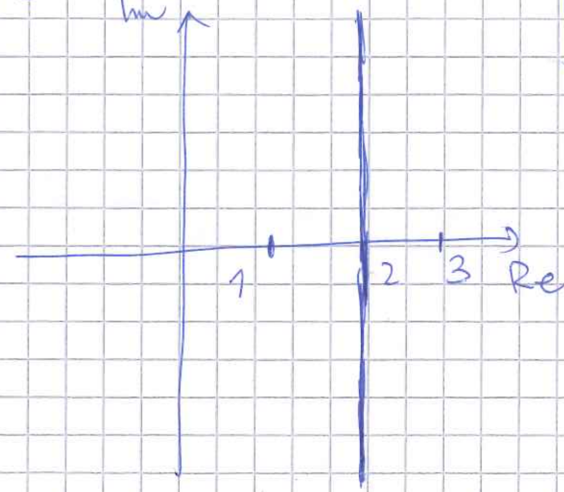
$2\mathrm{I} - 5\mathrm{II}.: \quad 31y = 248 \Rightarrow \boxed{y = \frac{248}{31} = 8}$

$\mathrm{II}.: \quad 2x = 5 \cdot 8 = 40 \Rightarrow \boxed{x = 20}$

príklad 5. V Gaussovej rovine znázornite obor pravdivosti nasledujúcich rovníc a nerovníc.

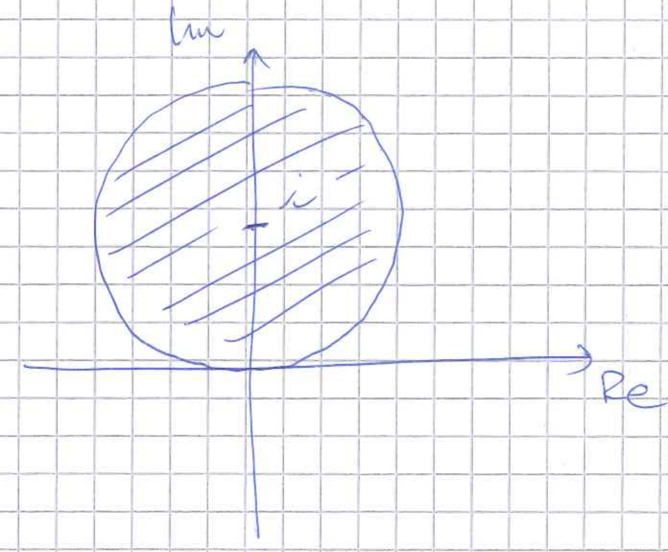
a) $|z-1| = |z-3|$

Vzdialenosti kompl. čísla z od 1 a od 3 sa rovnajú:

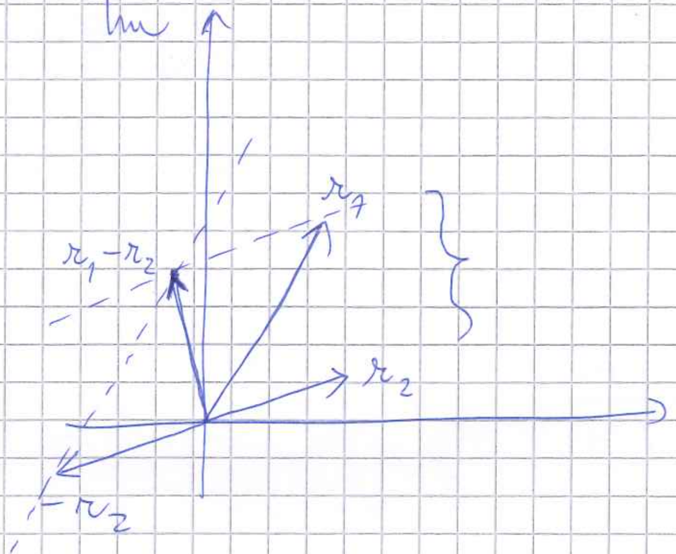


b) $|z-i| \leq 1$

Vzdialenosť kompl. čísla z od i je menšia alebo rovná 1.



Vysvetlenie:



Veľkosť (absolútna hodnota) komplexného čísla $z_1 - z_2$ je vzdialenosť komplexných čísel z_1 a z_2 v Gaussovej rovine.

príklad 6 Nájdite goniometrický tvar nasledujúcich komplexných čísel.

a) i b) $\frac{3}{2} + \frac{3}{2}\sqrt{3}\,i$

a) $|i| = |0+1i| = \sqrt{0^2+1^2} = 1$

$i = 1(0+1i) = 1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) =$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

b) $\left|\frac{3}{2} + \frac{3}{2}\sqrt{3}\,i\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9+27}{4}} =$

$= \sqrt{\frac{36}{4}} = \frac{6}{2} = 3$

$\frac{3}{2} + \frac{3\sqrt{3}}{2}i = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

príklad 7 Vypočítajte $(1+\sqrt{3}\,i)^5$.

Najrýchlejšie je počítať s goniometrickým tvarom:

$|1+\sqrt{3}\,i| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$

$1+\sqrt{3}\,i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$(1+\sqrt{3}\,i)^5 = 2^5\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right) =$

$= 32\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 16(1-\sqrt{3}\,i)$

príklad 8 Vypočítajte a výsledky zobrazte:

a) $\sqrt[5]{1}$ b) $\sqrt[3]{-2+2i}$

a) $1 = 1(1+0\cdot i) = 1\cdot(\cos 0 + i\sin 0)$

$\sqrt[5]{1} = \{r_1, r_2, r_3, r_4, r_5\}$

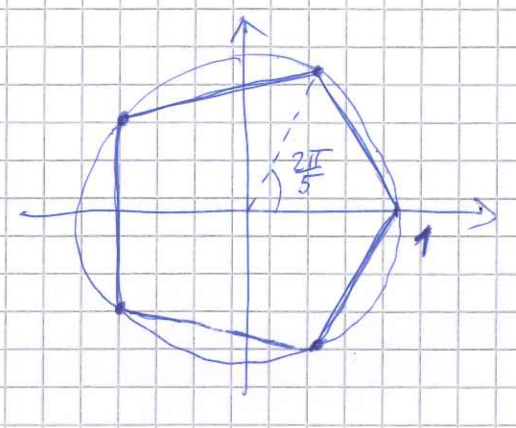
$r_1 = \sqrt[5]{1}\left(\cos\left(\frac{0}{5} + \frac{0\cdot 2\pi}{5}\right) + i\sin\left(\frac{0}{5} + \frac{0\cdot 2\pi}{5}\right)\right) = 1$

$r_2 = \sqrt[5]{1}\left(\cos\left(\frac{0}{5} + \frac{1\cdot 2\pi}{5}\right) + i\sin\left(\frac{0}{5} + \frac{1\cdot 2\pi}{5}\right)\right) = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$r_3 = \sqrt[5]{1}\left(\cos\left(\frac{0}{5} + \frac{2\cdot 2\pi}{5}\right) + i\sin\left(\frac{0}{5} + \frac{2\cdot 2\pi}{5}\right)\right) = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$r_4 = \sqrt[5]{1}\left(\cos\left(\frac{0}{5} + \frac{3\cdot 2\pi}{5}\right) + i\sin\left(\frac{0}{5} + \frac{3\cdot 2\pi}{5}\right)\right) = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$r_5 = \sqrt[5]{1}\left(\cos\left(\frac{0}{5} + \frac{4\cdot 2\pi}{5}\right) + i\sin\left(\frac{0}{5} + \frac{4\cdot 2\pi}{5}\right)\right) = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$



b) $|-2+2i| = \sqrt{(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$-2+2i = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$= 2\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

$\sqrt[3]{-2+2i} = \{r_1, r_2, r_3\}$

$$r_1 = \sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{4}+\frac{0\cdot 2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{0\cdot 2\pi}{3}\right)\right)$$

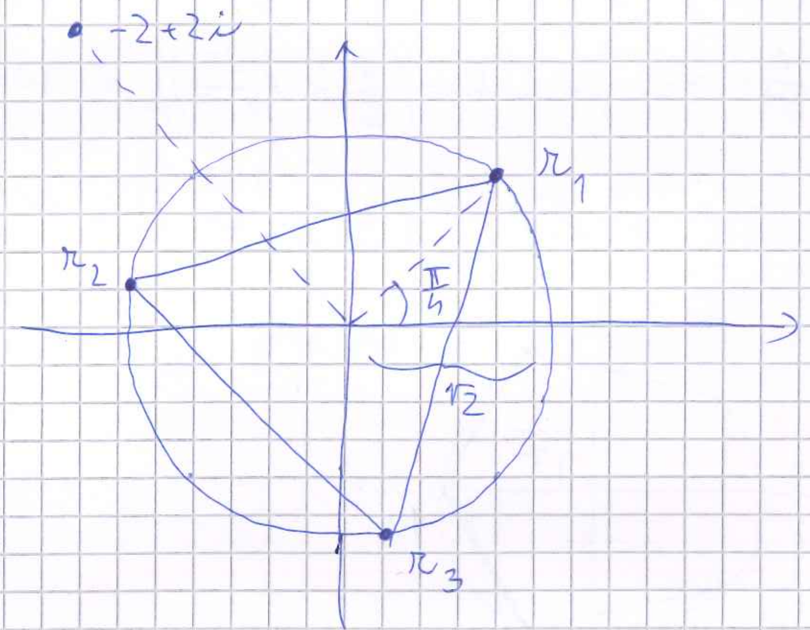
$$= \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$$

$$r_2 = \sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{4}+\frac{1\cdot 2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{1\cdot 2\pi}{3}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{11}{12}\pi + i\sin\frac{11}{12}\pi\right)$$

$$r_3 = \sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{4}+\frac{2\cdot 2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\cdot 2\pi}{3}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{19}{12}\pi + i\sin\frac{19}{12}\pi\right)$$



Riešenie kvadratickej rovnice: $ax^2+bx+c=0$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right) =$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right) = a\left(\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right)$$

$$= a\left(x+\frac{b}{2a}+\frac{\sqrt{D}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{D}}{2a}\right) = 0$$

$$\Leftrightarrow \quad x = \frac{-b\pm\sqrt{D}}{2a}$$

Ak povolíme komplexnú odmocninu z diskriminantu, môžeme tak nájsť riešenia kvadratickej rovnice aj v prípade $D<0$.

Navyše tento postup funguje aj pre komplexné koeficienty a, b, c.

príklad 9 Vyriešte rovnice:

a) $x^2-4x+6=0$ b) $x^2-(3+2i)x+5+i=0$

a) $D = 16-4\cdot 6 = -8$

$$x_{1,2} = \frac{4\pm\sqrt{-8}}{2} = \frac{4\pm i2\sqrt{2}}{2} = 2\pm i\sqrt{2}$$

b) $D = (3+2i)^2-4(5+i) =$

$$= 9+12i+4i^2-20-4i = -15+8i$$

Potrebujeme spočítať $\sqrt{D}$ => 2 spôsoby

A) pomocou goniometrického tvaru:

$$|-15+8i| = \sqrt{(-15)^2+8^2} = \sqrt{289} = 17$$

$$-15+8i = 17\left(-\frac{15}{17}+\frac{8}{17}i\right) = 17\left(\cos\arccos\left(-\frac{15}{17}\right)+i\sin\arcsin\left(\frac{8}{17}\right)\right)$$

$$\sqrt{-15+8i} = \{r_1, r_2\}$$

$$r_1 = \sqrt{17}\left(\cos\frac{\arccos\left(-\frac{15}{17}\right)}{2}+i\sin\frac{\arcsin\left(\frac{8}{17}\right)}{2}\right)$$

...

B) pomocou algebraického tvaru

$\sqrt{-15+8i} = u + vi, \quad u, v \in \mathbb{R}$

$-15+8i = (u+vi)^2 = u^2 + 2uvi + v^2 i^2$

$= u^2 - v^2 + 2uvi$

$\Rightarrow$ I. $-15 = u^2 - v^2$ $\Rightarrow$ I. $-15 = (u+v)(u-v)$

II. $8 = 2uv$ II. $4 = uv$

$\Rightarrow$ $u = 1$, $v = 4$ alebo $u = -1$, $v = -4$

$\sqrt{-15+8i} = \{1+4i, -1-4i\}$

je jedno, ktoré zoberieme

späť k príkladu:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3+2i \pm (1+4i)}{2}$$

$$= \begin{cases} \dfrac{4+6i}{2} = 2+3i \\[2ex] \dfrac{2-2i}{2} = 1-i \end{cases}$$

úloha na body:

Vypočítajte a výsledky zobrazte: $\sqrt[3]{i}$.

Vyriešte rovnicu $x^2-2x+5=0$.

Nájdite algebraický tvar komplexného čísla $(1-i)^6$.

$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$r_1 = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$

$r_2 = \cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)$

$r_3 = \cos\left(\frac{9}{6}\pi\right) + i\sin\left(\frac{9}{6}\pi\right) = \cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)$

$\frac{20}{2} + \frac{1}{2}$

$6 \cdot 7 = \frac{42}{4} = \frac{21}{2}$

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$= \sqrt{2}\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right)$

$(1-i)^6 = 8\left(\cos\frac{42}{4}\pi + i\sin\frac{42}{4}\pi\right) =$

$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 8i$