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CHAPTER

The Origin of Mathematics 7

Polygons After Euclid

13.1 What We Missed in Book IV

In Prop. IV. 16 Euclid constructs a regular 15-gon by superimposing an equilateral triangle on a regular pentagon (Fig. 13.1).

Implicit in this solution is a general principle: If we are able to construct the regular r- and s-gon, and moreover, we know integers x, y such that $xr + ys = 1$, then we can construct the rs-gon as well. We need an arc of $\frac{1}{rs}$ of the full circle and x, y as above:

$$\frac{1}{rs} = \frac{xr + ys}{rs} = x\frac{1}{s} + y\frac{1}{r}.$$

Hence this combination gives us the desired arc. In the case of the 15-gon we had

$$\frac{2}{5} - \frac{1}{3} = \frac{1}{15}.$$

Since for any integers r, s with greatest common divisor $gcd(r, s) = 1$, we can use the Euclidean algorithm (VII.1,2) to find

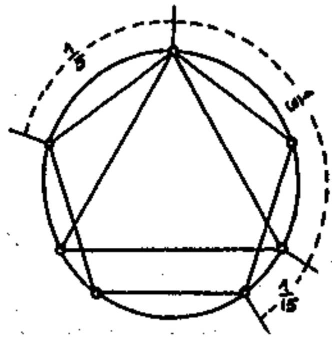

FIGURE 13.1

x, y with the desired properties, we are easily led to item (iii) listed below. This general formulation is, of course, modern. Euclid has no reason to formulate the general principle (iii) because he needs only the specific case he deals with. He generally refrains from hypothetical statements like the one above: "If $gcd(r, s) = 1$, then ..." if he does not have r, s with the desired property.

13.2 What Euclid Knew

With the above reservations we may say that Euclid (in principle) "knew" item (iii) below, and certainly he (or the pre-Euclidean author of Book IV) knew the trivial observations (i) and (iv):

(i) For all $n > 1$, the 2^n-gon is constructible.

(ii) The 3- and the 5-gons are constructible.

(iii) If the r- and s-gons are constructible and $gcd(r, s) = 1$, then the $r \cdot s$-gon is constructible.

(iv) If the n-gon is constructible and k divides n, then the k-gon is constructible.

Given (i)-(iv), the problem for general n is reduced to prime powers p^i for odd primes p.

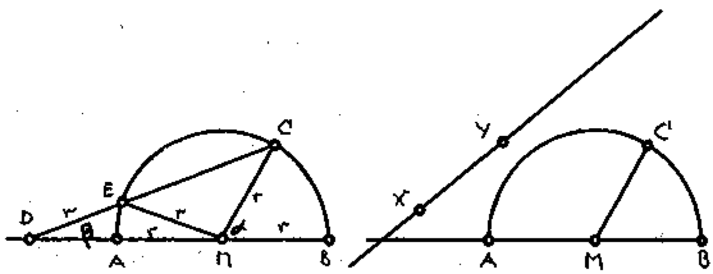

FIGURE 13.2

13.3 What Archimedes Did

By a certain variant of the *neusis* procedure, Archimedes was able to construct the regular 7-gon. By the ordinary *neusis*, he could construct the 9-gon. In fact, this *neusis* shows how to trisect an angle. (Knorr [1986], 185).

Look at Fig. 13.2 and proceed by analysis and synthesis. Let the circle with center M and radius r and the line DEC with distance $DE = r$ be given.

Then we find, for the angles α and β, using I.32 and I.5:

$$\begin{aligned}\alpha &= \angle MDE + \angle MCD \\ &= \angle MDE + \angle MEC \\ &= \angle MDE + 2\angle MDE \\ &= 3\beta.\end{aligned}$$

Synthesis. Let angle $\alpha < 90°$ be given. Mark distance $XY = r$ on the ruler and slide it into position such that X is on the (produced) line AB, Y is on the circle, and the (produced) line XY passes through C. The resulting angle β will be $\frac{1}{3}\alpha$.

Application: Trisect an angle α of 60° to find $\beta = 20°$. With 2β at the center of a circle construct the regular 9-gon.

13.4 What Gauss Proved

Carl Friedrich Gauss (30 April 1777–23 Feb. 1855) started his scientific diary on 30 March 1796 with the following entry:

> Foundations on which the division of the circle rests, that is its geometrical divisibility in seventeen parts, and so on.

As this short note reveals, the teenager Gauss had not only found the construction of the 17-gon, but also the general principles behind it. He explicitly states in his first publication, in April 1796, that the 17-gon is only a special case of his investigations. He had already begun working on his great work *Arithmetical Investigations (Disquisitiones arithmeticae)*, which appeared in 1801 and in which he proved the general theorem (article 365):

> For an odd prime number p, the regular p^i-gon is constructible by ruler and compass if and only if $i = 1$ and p is a prime of the form $p = 2^{2^k} + 1$, i.e., if p is a so-called Fermat prime.

The reduction of the problem to prime powers as above is done by Gauss in art. 336.

Not all Fermat numbers $F_k = 2^{2^k} + 1$ are prime. Except for the first few values $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$, which are prime, all others with $k < 24$ are known to be composite. As long as no general result about the Fermat numbers is obtained, the question about the constructibility of regular polygons remains open.

Note. We have seen how to construct a regular 9-gon with a marked ruler. Gauss's theorem shows that the 9-gon is not constructible by ruler and compass. Hence the use of a marked ruler is definitely a stronger method than the ordinary ones in geometry. (For more and precise information, see Hartshorne [2000], Section 30.)

13.5 How Gauss Did It

For this part we assume that the reader is familiar with the elementary properties of the complex numbers. This new tool stands the test by solving an old problem. We will present Gauss's method only for the pentagon and in a somewhat modernized version. For the 17-gon and a complete treatment, see Hartshorne, chapter 6. In the complex plane, the regular n-gon is represented by the nth roots of unity, that is, the solutions of the equation

$$z^n - 1 = 0.$$

These solutions are

$$\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

and its powers $\zeta^2, \ldots, \zeta^{n-1}, \zeta^n = \zeta^0 = 1$. We will investigate only the case $n = 5$, that is, the regular pentagon, from a new vantage point (Fig. 13.3).

The representation

$$\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

will not help us. We have to use the algebraic equation

$$0 = z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1).$$

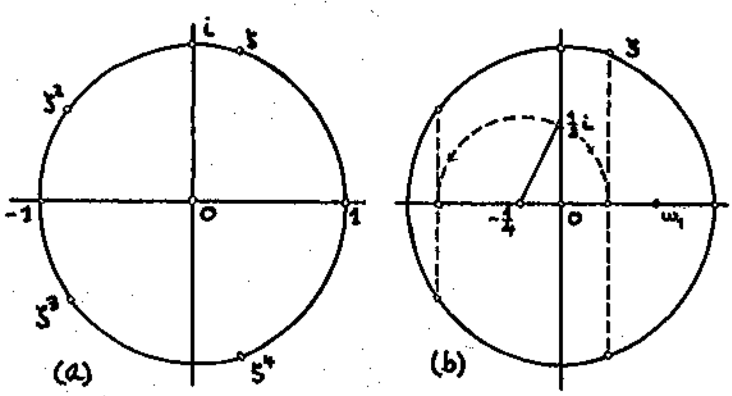

FIGURE 13.3

The first factor represents the solution $\zeta^0 = 1$ and is of no further interest. It just fixes the position of the pentagon in the circle. Because all our solutions will be different from zero, we may divide the second factor by z^2 and obtain the new equation

$$\begin{aligned} 0 &= z^2 + z + 1 + z^{-1} + z^{-2} \\ &= z^2 + 2 + z^{-2} + z + z^{-1} - 1 \\ &= (z + z^{-1})^2 + (z + z^{-1}) - 1. \end{aligned}$$

With this little trick we have obtained a quadratic equation for $w = z + z^{-1}$:

$$(*) \qquad \begin{aligned} w^2 + w - 1 &= 0, \\ \left(w + \frac{1}{2}\right)^2 &= \frac{5}{4}, \\ w_{1,2} + \frac{1}{2} &= \pm\frac{\sqrt{5}}{2}. \end{aligned}$$

For a geometric interpretation of this, we have to observe that

$$\zeta^{-1} = \zeta^4 = \overline{\zeta} \quad \text{and} \quad \zeta^{-2} = \zeta^3 = \overline{\zeta^2}.$$

Our two solutions for w then amount to

$$\begin{aligned} w_1 &= \zeta + \zeta^{-1} = 2 \cdot \text{ real part of } \zeta, \\ w_2 &= \zeta^2 + \zeta^{-2} = 2 \cdot \text{ real part of } \zeta^2. \end{aligned}$$

This gives us something constructible:

$$\begin{aligned} \text{real part of } \zeta &= -\frac{1}{4} + \frac{\sqrt{5}}{4}, \\ \text{real part of } \zeta^2 &= -\frac{1}{4} - \frac{\sqrt{5}}{4}. \end{aligned}$$

Figure 13.3 (b) shows how it is done by using the right triangle with vertices 0, $\frac{1}{2}i$, $-\frac{1}{4}$. For the final determination of $\zeta = x + iy$ we know x and have to solve a second quadratic equation,

$$x^2 + y^2 = 1$$

for y. The result is

$$y = \frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}}, \quad \text{for } \zeta,$$
$$y = \frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \quad \text{for } \zeta^2.$$

This gives us

$$f = \frac{1}{2}\sqrt{10-2\sqrt{5}}$$

for the side f of a regular pentagon inscribed in the unit circle and $d = \frac{1}{2}\sqrt{10+2\sqrt{5}}$ for its diagonal.

13.6 The Moral of the Story

Note (i). Our quadratic equation (∗) for w is similar to the one we found for the diagonal of a pentagon with side $f = 1$. There we had

$$d^2 - d = 1.$$

If we keep $f = 1$ and let $d = f + w = 1 + w$, the geometric equation becomes

$$(w+1)^2 - (w+1) = 1, \tag{13.1}$$
$$w^2 + w = 1, \tag{13.2}$$

the same as above. There are many different constructions of the pentagon, but closer inspection will always turn up this same quadratic equation (or a close relative as above). This equation is what one calls the **abstract essence** of the problem. Abstraction makes clear what the real nucleus of the problem is, apart from all different disguises.

Note (ii). With what was called "a little trick" we reduced the solution of the biquadratic equation for ζ to the successive solution of two ordinary quadratic equations. The trick is in fact Gauss's general method: Find the roots different from 1 for the equation $z^n - 1 = 0$ by solving quadratic equations successively, which can be done by ruler and compass. This new method, based on the new tool of complex numbers, is the unifying idea for the construction of regular polygons. Euclid had to deal with each case separately. Gauss, by abstraction, found the general solution. Generalization and abstraction made the problem accessible and the solution transparent.

13.7 What Plotinus Has to Say About All This

The neo-Platonic philosopher Plotinus (~ 200–270 C.E.) wrote a treatise about beauty. He has found the right words for what we have seen in a particularly significant case:

> But where the Ideal Form has entered, it has grouped and coordinated what from a diversity of parts was to become a unity: it has rallied confusion into cooperation: it has made the sum one harmonious coherence: for the Idea is a unity and what it moulds must come to unity as far as multiplicity may.
>
> And on what has thus been compacted to unity, Beauty enthrones itself, giving itself to the parts as to the sum: when it lights on some natural unity, a thing of like parts, then it gives itself to that whole. Thus, for an illustration, there is the beauty, conferred by craftsmanship, of all a house with all its parts, and the beauty which some natural quality may give to a single stone. (Plotinus, *First Ennead* VI, "On Beauty" 2, p. 22, cf. Plotinus [1952])