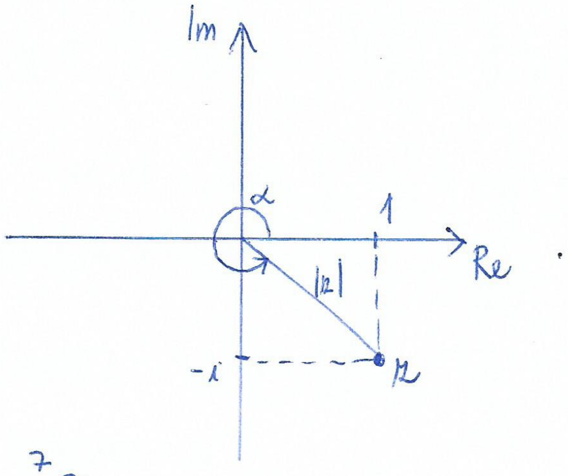
Goniometrický tvar komplexního čísla

1. Zobrazte v Gaussově rovině komplexní číslo z a vyjádřete ho v goniometrickém tvaru.

a) $z = 1 - i$



$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

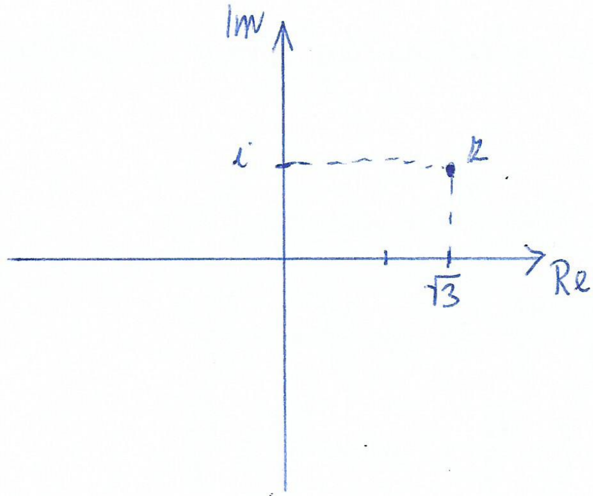
$\cos\alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha_0 = \frac{\pi}{4}$

$\alpha = 2\pi - \alpha_0 = \frac{7}{4}\pi$

$\sin\alpha = -\frac{1}{\sqrt{2}} \Rightarrow \alpha_0 = -\frac{\pi}{4} \Rightarrow \alpha = \frac{7}{4}\pi$

$$z = \sqrt{2} \cdot \left(\cos\frac{7}{4}\pi - i \cdot \sin\frac{7}{4}\pi\right)$$

b) $z = \sqrt{3} + i$



$|z| = \sqrt{3+1} = 2$

$\cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi_0 = \varphi = \frac{\pi}{6}$

$\sin\varphi = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{6}$

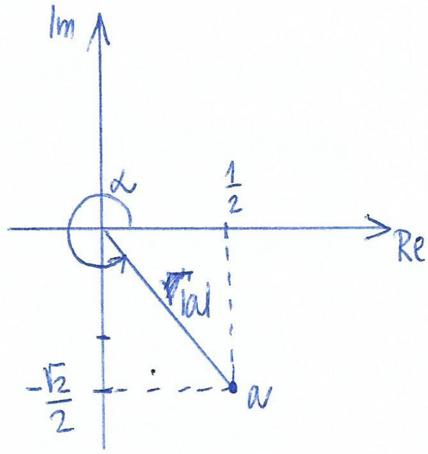
$$z = 2 \cdot \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

Význačné hodnoty funkcí sin a cos

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

V ostatních kvadrantech využíváme těchto hodnot.

2. Vyjádřete v goniometrickém tvaru komplexní číslo $\frac{1}{2} - \frac{\sqrt{2}}{2} i$



$$a = \frac{1}{2} - \frac{\sqrt{2}}{2} i$$

$$|a| = \sqrt{\frac{1}{4} + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{2}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\cos\alpha = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \alpha_0 = 54°44'$$

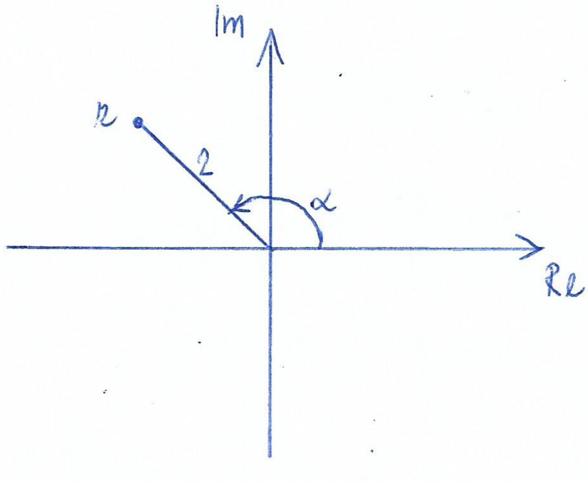
$$\alpha = 2\pi - \alpha_0 = 305°15'$$

$$\sin\alpha = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \alpha_0 = -54°44'$$

$$\alpha = 305°15'$$

$$\underline{\underline{a = \frac{\sqrt{3}}{2} \cdot (\cos 305°15' + i \sin 305°15')}}$$

3. Převeďte na algebraický tvar komplexní čísla v goniometrickém tvaru.



a) $z = 2 \cdot \left(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi\right)$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \Rightarrow \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \Rightarrow \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

$$z = 2 \cdot \left(-\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) = \underline{\underline{-\sqrt{2} + i\sqrt{2}}}$$

b) $a = \cos\frac{26}{4}\pi + i \sin\frac{26}{4}\pi$

$$\frac{26}{4}\pi = 3 \cdot 2\pi + \frac{2}{4}\pi = 3 \cdot 2\pi + \frac{1}{2}\pi$$

$$\Rightarrow a = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = 0 + i \cdot 1 = \underline{\underline{i}}$$

4. Určete početně ~~i graficky~~ součin komplexních čísel

$$z_1 = -\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}, \quad z_2 = 1 - i$$

jejich převedením na goniometrický tvar.

$$|z_1| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\cos\varphi_1 = \frac{-\frac{1}{2}}{1} = -\frac{1}{2} \Rightarrow \varphi_1 = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$$ (jsme ve 2. kvadrantu)

$$\sin\varphi_1 = \frac{\frac{\sqrt{3}}{2}}{1} \Rightarrow \varphi_1 = \frac{2\pi}{3}$$

$$z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$|z_2| = \sqrt{2}$$

$$\cos\varphi_2 = \frac{1}{\sqrt{2}} \Rightarrow \varphi_2 = 2\pi - \frac{\pi}{4}$$ (jsme ve 4. kvadrantu)

$$\sin\varphi_2 = \frac{-1}{\sqrt{2}} \Rightarrow \varphi_2 = \frac{7}{4}\pi$$

$$z_2 = \sqrt{2}\cdot\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right)$$

$$z_1\cdot z_2 = |z_1|\cdot|z_2|\cdot\left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right)$$

$$z_1\cdot z_2 = 1\cdot\sqrt{2}\cdot\left(\cos\left(\frac{2\pi}{3}+\frac{7\pi}{4}\right) + i\sin\left(\frac{2\pi}{3}+\frac{7\pi}{4}\right)\right) =$$

$$= \sqrt{2}\cdot\left(\cos\frac{29\pi}{12} + i\sin\frac{29\pi}{12}\right) =$$

$$= \sqrt{2}\cdot\left(\cos 2\frac{5}{12}\pi + i\sin 2\frac{5}{12}\pi\right) =$$

$$= \underline{\underline{\sqrt{2}\cdot\cos\left(\frac{5}{12}\pi + i\sin\frac{5}{12}\pi\right)}}$$

5. Vypočtěte mocniny komplexních čísel jejich převedením na goniometrický tvar.

a) $(\sqrt{3}+i)^3$

$|z| = 2$

$\cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$ (1. kvadrant)

$z = 2\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Moivreova věta: $\forall \varphi \in \mathbb{R}, \forall n \in \mathbb{N}: (\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$

$\Rightarrow z^n = |z|^n\cdot(\cos n\varphi + i\sin n\varphi)$

$z^3 = 2^3\cdot\left(\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}\right) = 8\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \underline{\underline{8i}}$

b) $(-1+i)^4$

$|z| = \sqrt{2}$

$\cos\varphi = \frac{-1}{\sqrt{2}} \Rightarrow \varphi = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$ (2. kvadrant)

$\sin\varphi = \frac{1}{\sqrt{2}} \Rightarrow \varphi = \frac{3}{4}\pi$

$z = \sqrt{2}\cdot\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

$z^4 = \sqrt{2}^4\cdot(\cos 3\pi + i\sin 3\pi) = 4\cdot(\cos\pi + i\sin\pi) =$

$= 4\cdot(-1+0) = \underline{\underline{-4}}$