CHAPTER IV.

ARITHMETIC IN ARCHIMEDES.

Two of the treatises, the *Measurement of a circle* and the *Sand-reckoner*, are mostly arithmetical in content. Of the *Sand-reckoner* nothing need be said here, because the system for expressing numbers of any magnitude which it unfolds and applies cannot be better described than in the book itself; in the *Measurement of a circle*, however, which involves a great deal of manipulation of numbers of considerable size though expressible by means of the ordinary Greek notation for numerals, Archimedes merely gives the results of the various arithmetical operations, multiplication, extraction of the square root, etc., without setting out any of the operations themselves. Various interesting questions are accordingly involved, and, for the convenience of the reader, I shall first give a short account of the Greek system of numerals and of the methods by which other Greek mathematicians usually performed the various operations included under the general term λογιστική (the art of *calculating*), in order to lead up to an explanation (1) of the way in which Archimedes worked out approximations to the square roots of large numbers, (2) of his method of arriving at the two approximate values of $\sqrt{3}$ which he simply sets down without any hint as to how they were obtained*.

* In writing this chapter I have been under particular obligations to Hultsch's articles *Arithmetica* and *Archimedes* in Pauly-Wissowa's *Real-Encyclopädie*, II. 1, as well as to the same scholar's articles (1) *Die Näherungswerthe irrationaler Quadratwurzeln bei Archimedes* in the *Nachrichten von der kgl. Gesellschaft der Wissenschaften zu Göttingen* (1893), pp. 367 sqq., and (2) *Zur Kreismessung des Archimedes* in the *Zeitschrift für Math. u. Physik* (*Hist. litt. Abtheilung*) xxxix. (1894), pp. 121 sqq. and 161 sqq. I have also made use, in the earlier part of the chapter, of Nesselmann's work *Die Algebra der Griechen* and the histories of Cantor and Gow.

§ 1. Greek numeral system.

It is well known that the Greeks expressed all numbers from 1 to 999 by means of the letters of the alphabet reinforced by the addition of three other signs, according to the following scheme, in which however the accent on each letter might be replaced by a short horizontal stroke above it, as $\bar{\alpha}$.

αʹ, βʹ, γʹ, δʹ, εʹ, ϛʹ, ζʹ, ηʹ, θʹ are 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively.
ιʹ, κʹ, λʹ, μʹ, νʹ, ξʹ, οʹ, πʹ, ϟʹ „ 10, 20, 30, ... 90 „
ρʹ, σʹ, τʹ, υʹ, φʹ, χʹ, ψʹ, ωʹ, ϡʹ „ 100, 200, 300,......900 „

Intermediate numbers were expressed by simple juxtaposition (representing in this case addition), the largest number being placed on the left, the next largest following it, and so on in order. Thus the number 153 would be expressed by ρνγʹ or $\overline{\rho\nu\gamma}$. There was no sign for zero, and therefore 780 was ψπʹ, and 306 τϛʹ simply.

Thousands (χιλιάδες) were taken as units of a higher order, and 1,000, 2,000, ... up to 9,000 (spoken of as χίλιοι, δισχίλιοι, κ.τ.λ.) were represented by the same letters as the first nine natural numbers but with a small dash in front and below the line; thus e.g. ͵δʹ was 4,000, and, on the same principle of juxtaposition as before, 1,823 was expressed by ͵αωκγʹ or ͵$\overline{\alpha\omega\kappa\gamma}$, 1,007 by ͵αζʹ, and so on.

Above 9,999 came a *myriad* (μυριάς), and 10,000 and higher numbers were expressed by using the ordinary numerals with the substantive μυριάδες taken as a new denomination (though the words μύριοι, δισμύριοι, τρισμύριοι, κ.τ.λ. are also found, following the analogy of χίλιοι, δισχίλιοι and so on). Various abbreviations were used for the word μυριάς, the most common being M or Mυ; and, where this was used, the number of myriads, or the multiple of 10,000, was generally written over the abbreviation, though sometimes before it and even after it. Thus 349,450 was $\overset{\lambda\delta}{\mathrm{M}}$͵θυνʹ*.

Fractions (λεπτά) were written in a variety of ways. The most usual was to express the denominator by the ordinary numeral with two accents affixed. When the numerator was unity, and it was therefore simply a question of a symbol for a single word such as

* Diophantus denoted myriads followed by thousands by the ordinary signs for numbers of units, only separating them by a dot from the thousands. Thus for 3,069,000 he writes $\overline{\tau\varsigma}$. ͵$\bar{\theta}$, and $\overline{\lambda\gamma}$. ͵$\overline{\alpha\psi o\varsigma}$ for 331,776. Sometimes myriads were represented by the ordinary letters with two dots above, as $\ddot{\rho}$ = 100 myriads (1,000,000), and myriads of myriads with two pairs of dots, as $\overset{::}{\iota}$ for 10 myriad-myriads (1,000,000,000).

τρίτον, $\frac{1}{3}$, there was no need to express the numerator, and the symbol was *γ″*; similarly *ς″* $= \frac{1}{6}$, *ιε″* $= \frac{1}{15}$, and so on. When the numerator was not unity and a certain number of fourths, fifths, etc., had to be expressed, the ordinary numeral was used for the numerator; thus *θ′ ια″* $= \frac{9}{11}$, *ι′ οα″* $= \frac{10}{71}$. In Heron's *Geometry* the denominator was written twice in the latter class of fractions; thus $\frac{2}{5}$ (*δύο πέμπτα*) was *β′ ε″ ε″*, $\frac{23}{33}$ (*λεπτὰ τριακοστότριτα κγ′* or *εἰκοσιτρία τριακοστότριτα*) was *κγ′ λγ″ λγ″*. The sign for $\frac{1}{2}$, *ἥμισυ*, is in Archimedes, Diophantus and Eutocius ∟″, in Heron C or a sign similar to a capital **S***.

A favourite way of expressing fractions with numerators greater than unity was to separate them into component fractions with numerator unity, when juxtaposition as usual meant addition. Thus $\frac{3}{4}$ was written ∟″δ″ $= \frac{1}{2} + \frac{1}{4}$; $\frac{15}{16}$ was Cδ″η″ις″ $= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$; Eutocius writes ∟″ξδ″ or $\frac{1}{2} + \frac{1}{64}$ for $\frac{33}{64}$, and so on. Sometimes the same fraction was separated into several different sums; thus in Heron (p. 119, ed. Hultsch) $\frac{163}{224}$ is variously expressed as

(*a*) $\frac{1}{2} + \frac{1}{7} + \frac{1}{14} + \frac{1}{112} + \frac{1}{224}$,

(*b*) $\frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{112}$,

and (*c*) $\frac{1}{2} + \frac{1}{6} + \frac{1}{21} + \frac{1}{112} + \frac{1}{224}$.

Sexagesimal fractions. This system has to be mentioned because the only instances of the working out of some arithmetical operations which have been handed down to us are calculations expressed in terms of such fractions; and moreover they are of special interest as having much in common with the modern system of *decimal* fractions, with the difference of course that the submultiple is 60 instead of 10. The scheme of sexagesimal fractions was used by the Greeks in astronomical calculations and appears fully developed in the *σύνταξις* of Ptolemy. The circumference of a circle, and along with it the four right angles subtended by it at the centre, are divided into 360 parts (*τμήματα* or *μοῖραι*) or as we should say *degrees*, each *μοῖρα* into 60 parts called (*first*) *sixtieths*, (*πρῶτα*) *ἑξηκοστά*, or *minutes* (*λεπτά*), each of these again into *δεύτερα ἑξηκοστά* (*seconds*), and so on. A similar division of the radius of the circle into 60

* Diophantus has a general method of expressing fractions which is the exact reverse of modern practice; the denominator is written *above* the numerator, thus $\overset{\gamma}{\epsilon} = 5/3$, $\overset{\kappa\epsilon}{\kappa\alpha} = 21/25$, and $\overset{\alpha\,.\,\omega\iota\varsigma}{\rho\kappa\zeta\,.\,\phi\xi\eta} = 1{,}270{,}568/10{,}816$. Sometimes he writes down the numerator and then introduces the denominator with *ἐν μορίῳ* or *μορίου*, e.g. $\overline{\tau\varsigma}\,.\,\overline{{}_{,}\theta}\,\mu o \rho\,.\,\overline{\lambda\gamma}\,.\,\overline{{}_{,}\alpha\psi o\varsigma} = 3{,}069{,}000/331{,}776$.

parts (τμήματα) was also made, and these were each subdivided into sixtieths, and so on. Thus a convenient fractional system was available for general arithmetical calculations, expressed in units of any magnitude or character, so many of the fractions which we should represent by $\frac{1}{60}$, so many of those which we should write $(\frac{1}{60})^2$, $(\frac{1}{60})^3$, and so on to any extent. It is therefore not surprising that Ptolemy should say in one place "In general we shall use the method of numbers according to the sexagesimal manner because of the inconvenience of the [ordinary] fractions." For it is clear that the successive submultiples by 60 formed a sort of frame with fixed compartments into which any fractions whatever could be located, and it is easy to see that e.g. in additions and subtractions the sexagesimal fractions were almost as easy to work with as decimals are now, 60 units of one denomination being equal to one unit of the next higher denomination, and "carrying" and "borrowing" being no less simple than it is when the number of units of one denomination necessary to make one of the next higher is 10 instead of 60. In expressing the units of the circumference, *degrees*, μοῖραι or the symbol $\overset{\circ}{\mu}$ was generally used along with the ordinary numeral which had a stroke above it; *minutes*, *seconds*, etc. were expressed by one, two, etc. accents affixed to the numerals. Thus $\overset{\circ}{\mu}\,\overline{\beta} = 2°$, μοιρῶν $\overline{\mu\zeta}$ μβ′ μ″ = 47° 42′ 40″. Also where there was *no* unit in any particular denomination O was used, signifying οὐδεμία μοῖρα, οὐδὲν ἑξηκοστόν and the like; thus $\overline{O}$ α′ β″ O‴ = 0° 1′ 2″ 0‴. Similarly, for the units representing the divisions of the radius the word τμήματα or some equivalent was used, and the fractions were represented as before; thus τμημάτων $\overline{\xi\zeta}$ δ′ νε″ = 67 (units) 4′ 55″.

§ 2. Addition and Subtraction.

There is no doubt that, in writing down numbers for these purposes, the several powers of 10 were kept separate in a manner corresponding practically to our system of numerals, and the hundreds, thousands, etc., were written in separate vertical rows. The following would therefore be a typical form of a sum in addition;

͵αυκδ′	=	1424
ρ γ′		103
$\overset{\alpha}{M}$͵βσπα′		12281
$\overset{\gamma}{M}$ λ′		30030
$\overset{\delta}{M}$͵γωλη′		43838

and the mental part of the work would be the same for the Greek as for us.

Similarly a subtraction would be represented as follows:

$\overset{\theta}{\text{M}}$,γχλϛ′	= 93636
$\overset{\beta}{\text{M}}$,γυ θ′	23409
$\overset{\zeta}{\text{M}}$ σκζ′	70227

§ 3. Multiplication.

A number of instances are given in Eutocius' commentary on the *Measurement of a circle*, and the similarity to our procedure is just as marked as in the above cases of addition and subtraction. The multiplicand is written first, and below it the multiplier preceded by ἐπί (= "into"). Then the highest power of 10 in the multiplier is taken and multiplied into the terms containing the separate multiples of the successive powers of 10, beginning with the highest and descending to the lowest; after which the next highest power of 10 in the multiplier is multiplied into the various denominations in the multiplicand in the same order. The same procedure is followed where either or both of the numbers to be multiplied contain fractions. Two instances from Eutocius are appended from which the whole procedure will be understood.

(1)

ψπ′	780	
ἐπὶ ψπ′	× 780	
$\overset{\mu\theta}{\text{M}}\overset{\epsilon}{\text{M}}$,ϛ′	490000	56000
$\overset{\epsilon}{\text{M}}$,ϛ,ϛυ′	56000	6400
ὁμοῦ $\overset{\xi}{\text{M}}$,ηυ′	*sum* 608400	

(2)

,γιγ′ ∟″δ″	$3013\frac{1}{2}\,\frac{1}{4}$ [$=3013\frac{3}{4}$]				
ἐπὶ ,γιγ′ ∟″δ″	× $3013\frac{1}{2}\,\frac{1}{4}$				
$\overset{\lambda}{\text{M}}\overset{\gamma}{\text{M}}$,θ,αφψν′	9,000,000	30,000	9,000	1500	750
$\overset{\gamma}{\text{M}}$ρλε′β′ ∟″	30,000	100	30	5	$2\frac{1}{2}$
,θλθ′α′ ∟″ ∟″δ″	9,000	30	9	$1\frac{1}{2}$	$\frac{1}{2}+\frac{1}{4}$
,αφ′ε′α′ ∟″δ″η″	1,500	5	$1\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
ψν′β′ ∟″ ∟″δ″η″ιϛ″	750	$2\frac{1}{2}$	$\frac{1}{2}+\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

[ὁμοῦ] $\overset{\lambda\eta}{\text{M}}$,βχπθ′ιϛ″ [$9{,}041{,}250 + 30{,}137\frac{1}{2} + 9{,}041\frac{1}{4} + 1506 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 753 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$]

$= 9{,}082{,}689\frac{1}{16}$.

One instance of a similar multiplication of numbers involving fractions may be given from Heron (pp. 80, 81). It is only one of many, and, for brevity, the Greek notation will be omitted. Heron has to find the product of $4\frac{33}{64}$ and $7\frac{62}{64}$, and proceeds as follows:

$$4 \,.\, 7 = 28,$$

$$4 \,.\, \tfrac{62}{64} = \tfrac{248}{64},$$

$$\tfrac{33}{64} \,.\, 7 = \tfrac{231}{64},$$

$$\tfrac{33}{64} \,.\, \tfrac{62}{64} = \tfrac{2046}{64} \,.\, \tfrac{1}{64} = \tfrac{31}{64} + \tfrac{62}{64} \,.\, \tfrac{1}{64}.$$

The result is accordingly

$$28 + \tfrac{510}{64} + \tfrac{62}{64} \,.\, \tfrac{1}{64} = 28 + 7 + \tfrac{62}{64} + \tfrac{62}{64} \,.\, \tfrac{1}{64}$$

$$= 35 + \tfrac{62}{64} + \tfrac{62}{64} \,.\, \tfrac{1}{64}.$$

The multiplication of 37° 4′ 55″ (in the sexagesimal system) by itself is performed by Theon of Alexandria in his commentary on Ptolemy's σύνταξις in an exactly similar manner.

§ 4. Division.

The operation of dividing by a number of one digit only was easy for the Greeks as for us, and what we call "long division" was with them performed, *mutatis mutandis*, in the same way as now with the help of multiplication and subtraction. Suppose, for instance, that the operation in the first case of multiplication given above had to be reversed and that $\overset{\xi}{\mathrm{M}}$,ηυ′ (608,400) had to be divided by ψπ′ (780). The terms involving the different powers of 10 would be mentally kept separate as in addition and subtraction, and the first question would be, how many times will 7 hundreds go into 60 myriads, due allowance being made for the fact that the 7 hundreds have 80 behind them and that 780 is not far short of 8 hundreds? The answer is 7 hundreds or ψ′, and this multiplied by the divisor ψπ′ (780) would give $\overset{\nu\delta}{\mathrm{M}}$,ϛ′ (546,000) which, subtracted from $\overset{\xi}{\mathrm{M}}$,ηυ′ (608,400), leaves the remainder $\overset{ϛ}{\mathrm{M}}$,βυ′ (62,400). This remainder has then to be divided by 780 or a number approaching 8 hundreds, and 8 tens or π′ would have to be tried. In the particular case the result would then be complete, the quotient being ψπ′ (780), and there being no remainder, since π′ (80) multiplied by ψπ′ (780) gives the exact figure $\overset{ϛ}{\mathrm{M}}$,βυ′ (62,400).

An actual case of long division where the dividend and divisor contain sexagesimal fractions is described by Theon. The problem is to divide 1515 20′ 15″ by 25 12′ 10″, and Theon's account of the process comes to this.

Divisor		Dividend			Quotient
25 12′ 10″		1515	20′	15″	First term 60
	25 . 60 =	1500			
	Remainder	15 =	900′		
	Sum		920′		
	12′ . 60 =		720′		
	Remainder		200′		
	10″ . 60 =		10′		
	Remainder		190′		Second term 7′
	25 . 7′ =		175′		
			15′ =	900″	
	Sum			915″	
	12′ . 7′			84″	
	Remainder			831″	
	10″ . 7′			1″ 10‴	
	Remainder			829″ 50‴	Third term 33″
	25 . 33″			825″	
	Remainder			4″ 50‴ = 290‴	
	12′ . 33″			396‴	
	(too great by)			106‴	

Thus the quotient is something less than 60 7′ 33″. It will be observed that the difference between this operation of Theon's and that followed in dividing $\overset{\xi}{\mathrm{M}}$͵ηυ′ (608,400) by ψπ′ (780) as above is that Theon makes *three* subtractions for one term of the quotient, whereas the remainder was arrived at in the other case after *one* subtraction. The result is that, though Theon's method is quite clear, it is longer, and moreover makes it less easy to foresee what will be the proper figure to try in the quotient, so that more time would be apt to be lost in making unsuccessful trials.

§ 5. Extraction of the square root.

We are now in a position to see how the operation of extracting the square root would be likely to be attacked. First, as in the case of division, the given whole number whose square root is required would be separated, so to speak, into compartments each containing such and such a number of units and of the separate powers of 10. Thus there would be so many units, so many tens, so many hundreds, etc., and it would have to be borne in mind that the squares of numbers from 1 to 9 would lie between 1 and 99, the squares of numbers from 10 to 90 between 100 and 9900, and so on. Then the first term of the square root would be some number of tens or hundreds or thousands, and so on, and would have to be found in much the same way as the first term of a quotient in a "long division," by trial if necessary. If A is the number whose square root is required, while a represents the first term or denomination of the square root and x the next term or denomination still to be found, it would be necessary to use the identity $(a+x)^2 = a^2 + 2ax + x^2$ and to find x so that $2ax + x^2$ might be somewhat less than the remainder $A - a^2$. Thus by trial the highest possible value of x satisfying the condition would be easily found. If that value were b, the further quantity $2ab + b^2$ would have to be subtracted from the first remainder $A - a^2$, and from the second remainder thus left a third term or denomination of the square root would have to be derived, and so on. That this was the actual procedure adopted is clear from a simple case given by Theon in his commentary on the σύνταξις. Here the square root of 144 is in question, and it is obtained by means of Eucl. II. 4. The highest possible denomination (i.e. power of 10) in the square root is 10; 10^2 subtracted from 144 leaves 44, and this must contain not only twice the product of 10 and the next term of the square root but also the square of that next term itself. Now, since 2 . 10 itself produces 20, the division of 44 by 20 suggests 2 as the next term of the square root; and this turns out to be the exact figure required, since

$$2 \cdot 20 + 2^2 = 44.$$

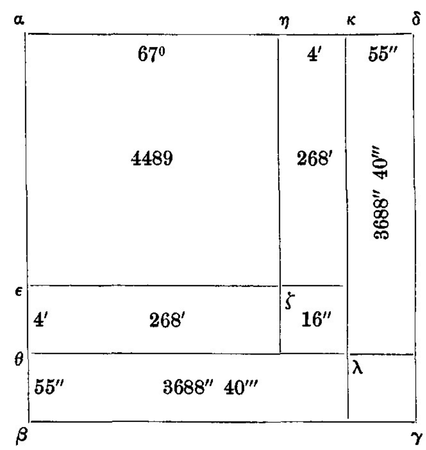
The same procedure is illustrated by Theon's explanation of Ptolemy's method of extracting square roots according to the sexagesimal system of fractions. The problem is to find approximately the square root of 4500 μοῖραι or *degrees*, and a geometrical figure is used which makes clear the essentially Euclidean basis of the whole method. Nesselmann gives a complete reproduction of the passage of Theon, but the following purely arithmetical representation of its purport will probably be found clearer, when looked at side by side with the figure.

Ptolemy has first found the integral part of $\sqrt{4500}$ to be 67.

Now $67^2 = 4489$, so that the remainder is 11. Suppose now that the rest of the square root is expressed by means of the usual sexagesimal fractions, and that we may therefore put

$$\sqrt{4500} = \sqrt{67^2 + 11} = 67 + \frac{x}{60} + \frac{y}{60^2},$$

where x, y are yet to be found. Thus x must be such that $\frac{2 \cdot 67x}{60}$ is somewhat less than 11, or x must be somewhat less than $\frac{11 \cdot 60}{2 \cdot 67}$ or $\frac{330}{67}$, which is at the same time greater than 4. On trial, it turns out that 4 will satisfy the conditions of the problem, namely that $\left(67 + \frac{4}{60}\right)^2$ must be less than 4500, so that a remainder will be left by means of which y may be found.



Now $11 - \frac{2 \cdot 67 \cdot 4}{60} - \left(\frac{4}{60}\right)^2$ is the remainder, and this is equal to

$$\frac{11 \cdot 60^2 - 2 \cdot 67 \cdot 4 \cdot 60 - 16}{60^2} = \frac{7424}{60^2}.$$

Thus we must suppose that $2\left(67 + \frac{4}{60}\right)\frac{y}{60^2}$ approximates to $\frac{7424}{60^2}$, or that $8048y$ is approximately equal to $7424 \cdot 60$.

Therefore y is approximately equal to 55. We have then to subtract

$$2\left(67+\frac{4}{60}\right)\frac{55}{60^2}+\left(\frac{55}{60^2}\right)^2, \text{ or } \frac{442640}{60^3}+\frac{3025}{60^4},$$

from the remainder $\frac{7424}{60^2}$ above found.

The subtraction of $\frac{442640}{60^3}$ from $\frac{7424}{60^2}$ gives $\frac{2800}{60^3}$, or $\frac{46}{60^2}+\frac{40}{60^3}$; but Theon does not go further and subtract the remaining $\frac{3025}{60^4}$, instead of which he merely remarks that the square of $\frac{55}{60^2}$ approximates to $\frac{46}{60^2}+\frac{40}{60^3}$. As a matter of fact, if we deduct the $\frac{3025}{60^4}$ from $\frac{2800}{60^3}$, so as to obtain the correct remainder, it is found to be $\frac{164975}{60^4}$.

To show the power of this method of extracting square roots by means of sexagesimal fractions, it is only necessary to mention that Ptolemy gives $\frac{103}{60}+\frac{55}{60^2}+\frac{23}{60^3}$ as an approximation to $\sqrt{3}$, which approximation is equivalent to 1·7320509 in the ordinary decimal notation and is therefore correct to 6 places.

But it is now time to pass to the question how Archimedes obtained the two approximations to the value of $\sqrt{3}$ which he assumes in the *Measurement of a circle.* In dealing with this subject I shall follow the historical method of explanation adopted by Hultsch, in preference to any of the mostly *a priori* theories which the ingenuity of a multitude of writers has devised at different times.

§ 6. Early investigations of surds or incommensurables.

From a passage in Proclus' commentary on Eucl. I.* we learn that it was Pythagoras who discovered the *theory of irrationals* (ἡ τῶν ἀλόγων πραγματεία). Further Plato says (*Theaetetus* 147 D), "On square roots this Theodorus [of Cyrene] wrote a work in

* p. 65 (ed. Friedlein).

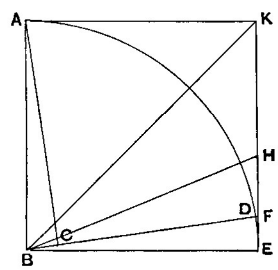
which he proved to us, with reference to those of 3 or 5 [square] feet. that they are incommensurable in length with the side of one square foot, and proceeded similarly to select, one by one, each [of the other incommensurable roots] as far as the root of 17 square feet, beyond which for some reason he did not go." The reason why $\sqrt{2}$ is not mentioned as an incommensurable square root must be, as Cantor says, that it was before known to be such. We may therefore conclude that it was the square root of 2 which was geometrically constructed by Pythagoras and proved to be incommensurable with the side of a square in which it represented the diagonal. A clue to the method by which Pythagoras investigated the value of $\sqrt{2}$ is found by Cantor and Hultsch in the famous passage of Plato (*Rep.* VIII. 546 B, C) about the 'geometrical' or 'nuptial' number. Thus, when Plato contrasts the ῥητή and ἄρρητος διάμετρος τῆς πεμπάδος, he is referring to the diagonal of a square whose side contains five units of length; the ἄρρητος διάμετρος, or the irrational diagonal, is then $\sqrt{50}$ itself, and the nearest rational number is $\sqrt{50-1}$, which is the ῥητὴ διάμετρος. We have herein the explanation of the way in which Pythagoras must have made the first and most readily comprehensible approximation to $\sqrt{2}$; he must have taken, instead of 2, an improper fraction equal to it but such that the denominator was a square in any case, while the numerator was as near as possible to a complete square. Thus Pythagoras chose $\frac{50}{25}$, and the first approximation to $\sqrt{2}$ was accordingly $\frac{7}{5}$, it being moreover obvious that $\sqrt{2} > \frac{7}{5}$. Again, Pythagoras cannot have been unaware of the truth of the proposition, proved in Eucl. II. 4, that $(a+b)^2 = a^2 + 2ab + b^2$, where a, b are any two straight lines, for this proposition depends solely upon propositions in Book I. which precede the Pythagorean proposition I. 47 and which, as the basis of I. 47, must necessarily have been in substance known to its author. A slightly different geometrical proof would give the formula $(a-b)^2 = a^2 - 2ab + b^2$, which must have been equally well known to Pythagoras. It could not therefore have escaped the discoverer of the first approximation $\sqrt{50-1}$ for $\sqrt{50}$ that the use of the formula with the *positive* sign would give a much nearer approximation, viz. $7 + \frac{1}{14}$, which is only greater than $\sqrt{50}$ to the extent of $\left(\frac{1}{14}\right)^2$. Thus we may properly assign to Pythagoras the discovery of the fact represented by

$$7\frac{1}{14} > \sqrt{50} > 7.$$

The consequential result that $\sqrt{2} > \frac{1}{5}\sqrt{50-1}$ is used by Aristarchus of Samos in the 7th proposition of his work *On the size and distances of the sun and moon**.

With reference to the investigations of the values of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}, \ldots\ldots \sqrt{17}$ by Theodorus, it is pretty certain that $\sqrt{3}$ was geometrically represented by him, in the same way as it appears

* Part of the proof of this proposition was a sort of foretaste of the first part of Prop. 3 of Archimedes' *Measurement of a circle*, and the substance of it is accordingly appended as reproduced by Hultsch.



$ABEK$ is a square, KB a diagonal, $\angle HBE = \frac{1}{2}\angle KBE$, $\angle FBE = 3°$, and AC is perpendicular to BF so that the triangles ACB, BEF are similar.

Aristarchus seeks to prove that

$$AB : BC > 18 : 1.$$

If R denote a right angle, the angles KBE, HBE, FBE are respectively $\frac{30}{60}R$, $\frac{15}{60}R$, $\frac{2}{60}R$.

Then $\quad HE : FE > \angle HBE : \angle FBE.$

[This is assumed as a known lemma by Aristarchus as well as Archimedes.]

Therefore $\quad HE : FE > 15 : 2 \ldots\ldots\ldots\ldots (\alpha).$

Now, by construction, $\quad BK^2 = 2BE^2.$

Also [Eucl. VI. 3] $\quad BK : BE = KH : HE$;

whence $\quad KH = \sqrt{2}\,HE.$

And, since $\quad \sqrt{2} > \sqrt{\frac{50-1}{25}},$

$$KH : HE > 7 : 5,$$

so that $\quad KE : EH > 12 : 5 \ldots\ldots\ldots\ldots (\beta).$

From (α) and (β), *ex aequali*,

$$KE : FE > 18 : 1.$$

Therefore, since $\quad BF > BE$ (or KE),

$$BF : FE > 18 : 1,$$

so that, by similar triangles,

$$AB : BC > 18 : 1.$$

afterwards in Archimedes, as the perpendicular from an angular point of an equilateral triangle on the opposite side. It would thus be readily comparable with the side of the "1 square foot" mentioned by Plato. The fact also that it is the side of three square *feet* (τρίπους δύναμις) which was proved to be incommensurable suggests that there was some special reason in Theodorus' proof for specifying *feet*, instead of units of length simply; and the explanation is probably that Theodorus subdivided the sides of his triangles in the same way as the Greek foot was divided into halves, fourths, eighths and sixteenths. Presumably therefore, exactly as Pythagoras had approximated to $\sqrt{2}$ by putting $\frac{50}{25}$ for 2, Theodorus started from the identity $3 = \frac{48}{16}$. It would then be clear that

$$\sqrt{3} < \sqrt{\frac{48+1}{16}}, \text{ i.e. } \frac{7}{4}.$$

To investigate $\sqrt{48}$ further, Theodorus would put it in the form $\sqrt{49-1}$, as Pythagoras put $\sqrt{50}$ into the form $\sqrt{49+1}$, and the result would be

$$\sqrt{48}\,(=\sqrt{49-1}) < 7 - \frac{1}{14}.$$

We know of no further investigations into incommensurable square roots until we come to Archimedes.

§ 7. Archimedes' approximations to $\sqrt{3}$.

Seeing that Aristarchus of Samos was still content to use the first and very rough approximation to $\sqrt{2}$ discovered by Pythagoras, it is all the more astounding that Aristarchus' younger contemporary Archimedes should all at once, without a word of explanation, give out that

$$\frac{1351}{780} > \sqrt{3} > \frac{265}{153},$$

as he does in the *Measurement of a circle*.

In order to lead up to the explanation of the probable steps by which Archimedes obtained these approximations, Hultsch adopts the same method of *analysis* as was used by the Greek geometers in solving problems, the method, that is, of supposing the problem solved and following out the necessary consequences. To compare the two fractions $\frac{265}{153}$ and $\frac{1351}{780}$, we first divide both denominators into their smallest factors, and we obtain

$$780 = 2 . 2 . 3 . 5 . 13,$$
$$153 = 3 . 3 . 17.$$

We observe also that $2 . 2 . 13 = 52$, while $3 . 17 = 51$, and we may therefore show the relations between the numbers thus,

$$780 = 3 . 5 . 52,$$
$$153 = 3 . 51.$$

For convenience of comparison we multiply the numerator and denominator of $\frac{265}{153}$ by 5; the two original fractions are then

$$\frac{1351}{15 . 52} \text{ and } \frac{1325}{15 . 51},$$

so that we can put Archimedes' assumption in the form

$$\frac{1351}{52} > 15\sqrt{3} > \frac{1325}{51},$$

and this is seen to be equivalent to

$$26 - \frac{1}{52} > 15\sqrt{3} > 26 - \frac{1}{51}.$$

Now $26 - \frac{1}{52} = \sqrt{26^2 - 1 + \left(\frac{1}{52}\right)^2}$, and the latter expression is an approximation to $\sqrt{26^2 - 1}$.

We have then $$26 - \frac{1}{52} > \sqrt{26^2 - 1}.$$

As $26 - \frac{1}{52}$ was compared with $15\sqrt{3}$, and we want an approximation to $\sqrt{3}$ itself, we divide by 15 and so obtain

$$\frac{1}{15}\left(26 - \frac{1}{52}\right) > \frac{1}{15}\sqrt{26^2 - 1}.$$

But $\frac{1}{15}\sqrt{26^2 - 1} = \sqrt{\frac{676 - 1}{225}} = \sqrt{\frac{675}{225}} = \sqrt{3}$, and it follows that

$$\frac{1}{15}\left(26 - \frac{1}{52}\right) > \sqrt{3}.$$

The lower limit for $\sqrt{3}$ was given by

$$\sqrt{3} > \frac{1}{15}\left(26 - \frac{1}{51}\right),$$

and a glance at this suggests that it may have been arrived at by simply substituting $(52-1)$ for 52.

Now as a matter of fact the following proposition is true. *If* $a^2 \pm b$ *is a whole number which is not a square, while* a^2 *is the nearest square number (above or below the first number, as the case may be), then*

$$a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}.$$

Hultsch proves this pair of inequalities in a series of propositions formulated after the Greek manner, and there can be little doubt that Archimedes had discovered and proved the same results in substance, if not in the same form. The following circumstances confirm the probability of this assumption.

(1) Certain approximations given by Heron show that he knew and frequently used the formula

$$\sqrt{a^2 \pm b} \backsim a \pm \frac{b}{2a},$$

(where the sign $\backsim$ denotes "is approximately equal to").

Thus he gives

$$\sqrt{50} \backsim 7 + \frac{1}{14},$$

$$\sqrt{63} \backsim 8 - \frac{1}{16},$$

$$\sqrt{75} \backsim 8 + \frac{11}{16}.$$

(2) The formula $\sqrt{a^2 + b} \backsim a + \frac{b}{2a+1}$ is used by the Arabian Alkarkhī (11th century) who drew from Greek sources (Cantor, p. 719 sq.).

It can therefore hardly be accidental that the formula

$$a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}$$

gives us what we want in order to obtain the two Archimedean approximations to $\sqrt{3}$, and that in direct connexion with one another*.

* Most of the *a priori* theories as to the origin of the approximations are open to the serious objection that, as a rule, they give series of approximate values in which the two now in question do not follow consecutively, but are separated by others which do not appear in Archimedes. Hultsch's explanation is much preferable as being free from this objection. But it is fair to say that the actual formula used by Hultsch appears in Hunrath's solution of the puzzle

We are now in a position to work out the synthesis as follows. From the geometrical representation of $\sqrt{3}$ as the perpendicular from an angle of an equilateral triangle on the opposite side we obtain $\sqrt{2^2-1}=\sqrt{3}$ and, as a first approximation,

$$2-\frac{1}{4}>\sqrt{3}.$$

Using our formula we can transform this at once into

$$\sqrt{3}>2-\frac{1}{4-1}, \text{ or } 2-\frac{1}{3}.$$

Archimedes would then square $\left(2-\frac{1}{3}\right)$, or $\frac{5}{3}$, and would obtain $\frac{25}{9}$, which he would compare with 3, or $\frac{27}{9}$; i.e. he would put $\sqrt{3}=\sqrt{\frac{25+2}{9}}$ and would obtain

$$\frac{1}{3}\left(5+\frac{1}{5}\right)>\sqrt{3}, \text{ i.e. } \frac{26}{15}>\sqrt{3}.$$

To obtain a still nearer approximation, he would proceed in the same manner and compare $\left(\frac{26}{15}\right)^2$, or $\frac{676}{225}$, with 3, or $\frac{675}{225}$, whence it would appear that

$$\sqrt{3}=\sqrt{\frac{26^2-1}{225}},$$

and therefore that

$$\frac{1}{15}\left(26-\frac{1}{52}\right)>\sqrt{3},$$

that is,

$$\frac{1351}{780}>\sqrt{3}.$$

The application of the formula would then give the result

$$\sqrt{3}>\frac{1}{15}\left(26-\frac{1}{52-1}\right),$$

that is,

$$\sqrt{3}>\frac{1326-1}{15 \,.\, 51}, \text{ or } \frac{265}{153}.$$

The complete result would therefore be

$$\frac{1351}{780}>\sqrt{3}>\frac{265}{153}.$$

(*Die Berechnung irrationaler Quadratwurzeln vor der Herrschaft der Decimalbrüche*, Kiel, 1884, p. 21; cf. *Ueber das Ausziehen der Quadratwurzel bei Griechen und Indern*, Hadersleben, 1883), and the same formula is implicitly used in one of the solutions suggested by Tannery (*Sur la mesure du cercle d'Archimède* in *Mémoires de la société des sciences physiques et naturelles de Bordeaux*, 2ᵉ série, IV. (1882), p. 313–337).

Thus Archimedes probably passed from the first approximation $\frac{7}{4}$ to $\frac{5}{3}$, from $\frac{5}{3}$ to $\frac{26}{15}$, and from $\frac{26}{15}$ directly to $\frac{1351}{780}$, the closest approximation of all, from which again he derived the less close approximation $\frac{265}{153}$. The reason why he did not proceed to a still nearer approximation than $\frac{1351}{780}$ is probably that the squaring of this fraction would have brought in numbers much too large to be conveniently used in the rest of his calculations. A similar reason will account for his having started from $\frac{5}{3}$ instead of $\frac{7}{4}$; if he had used the latter, he would first have obtained, by the same method, $\sqrt{3} = \sqrt{\frac{49-1}{16}}$, and thence $\frac{7-\frac{1}{14}}{4} > \sqrt{3}$, or $\frac{97}{56} > \sqrt{3}$; the squaring of $\frac{97}{56}$ would have given $\sqrt{3} = \frac{\sqrt{97^2-1}}{56}$, and the corresponding approximation would have given $\frac{18817}{56 \,.\, 194}$, where again the numbers are inconveniently large for his purpose.

§ 8. Approximations to the square roots of large numbers.

Archimedes gives in the *Measurement of a circle* the following approximate values:

(1) $3013\frac{3}{4} > \sqrt{9082321}$,

(2) $1838\frac{9}{11} > \sqrt{3380929}$,

(3) $1009\frac{1}{6} > \sqrt{1018405}$,

(4) $2017\frac{1}{4} > \sqrt{4069284\frac{1}{36}}$,

(5) $591\frac{1}{8} < \sqrt{349450}$,

(6) $1172\frac{1}{8} < \sqrt{1373943\frac{33}{64}}$,

(7) $2339\frac{1}{4} < \sqrt{5472132\frac{1}{16}}$.

There is no doubt that in obtaining the integral portion of the square root of these numbers Archimedes used the method based on the Euclidean theorem $(a+b)^2 = a^2 + 2ab + b^2$ which has already been exemplified in the instance given above from Theon, where an approximation to $\sqrt{4500}$ is found in sexagesimal fractions. The method does not substantially differ from that now followed; but whereas, to take the first case, $\sqrt{9082321}$, we can at once see what will be the number of digits in the square root by marking off pairs of digits in the given number, beginning from the end, the absence of a sign for 0 in Greek made the number of digits in the square root less easy to ascertain because, as written in Greek, the number $\overset{\text{ϡ}\eta}{\mathrm{M}}{}_{,}\beta\tau\kappa\alpha'$ only contains six signs representing digits instead of seven. Even in the Greek notation however it would not be difficult to see that, of the denominations, units, tens, hundreds, etc. in the square root, the units would correspond to $\kappa\alpha'$ in the original number, the tens to ${}_{,}\beta\tau$, the hundreds to $\overset{\eta}{\mathrm{M}}$, and the thousands to $\overset{\text{ϡ}}{\mathrm{M}}$. Thus it would be clear that the square root of 9082321 must be of the form

$$1000x + 100y + 10z + w,$$

where x, y, z, w can only have one or other of the values $0, 1, 2, \ldots 9$. Supposing then that x is found, the remainder $N - (1000x)^2$, where N is the given number, must next contain $2 . 1000x . 100y$ and $(100y)^2$, then $2(1000x + 100y) . 10z$ and $(10z)^2$, after which the remainder must contain two more numbers similarly formed.

In the particular case (1) clearly $x = 3$. The subtraction of $(3000)^2$ leaves 82321, which must contain $2 . 3000 . 100y$. But, even if y is as small as 1, this product would be 600,000, which is greater than 82321. Hence there is no digit representing *hundreds* in the square root. To find z, we know that 82321 must contain

$$2 . 3000 . 10z + (10z)^2,$$

and z has to be obtained by dividing 82321 by 60,000. Therefore $z = 1$. Again, to find w, we know that the remainder

$$(82321 - 2 . 3000 . 10 - 10^2),$$

or 22221, must contain $2 . 3010w + w^2$, and dividing 22221 by $2 . 3010$ we see that $w = 3$. Thus 3013 is the integral portion of the square root, and the remainder is $22221 - (2 . 3010 . 3 + 3^2)$, or 4152.

The conditions of the proposition now require that the approximate value to be taken for the square root must not be less than the real value, and therefore the fractional part to be added to 3013 must be if anything too great. Now it is easy to see that the fraction to be added is greater than $\frac{1}{2}$ because $2 . 3013 . \frac{1}{2} + \left(\frac{1}{2}\right)^2$ is less than the remainder 4152. Suppose then that the number required (which is nearer to 3014 than to 3013) is $3014 - \frac{p}{q}$, and $\frac{p}{q}$ has to be if anything too small.

$$\text{Now} \quad (3014)^2 = (3013)^2 + 2 . 3013 + 1 = (3013)^2 + 6027$$
$$= 9082321 - 4152 + 6027,$$

whence
$$9082321 = (3014)^2 - 1875.$$

By applying Archimedes' formula $\sqrt{a^2 \pm b} < a \pm \frac{b}{2a}$, we obtain

$$3014 - \frac{1875}{2 . 3014} > \sqrt{9082321}.$$

The required value $\frac{p}{q}$ has therefore to be not greater than $\frac{1875}{6028}$. It remains to be explained why Archimedes put for $\frac{p}{q}$ the value $\frac{1}{4}$ which is equal to $\frac{1507}{6028}$. In the first place, he evidently preferred fractions with unity for numerator and some power of 2 for denominator because they contributed to ease in working, e.g. when two such fractions, being equal to each other, had to be added. (The exceptions, the fractions $\frac{9}{11}$ and $\frac{1}{6}$, are to be explained by exceptional circumstances presently to be mentioned.) Further, in the particular case, it must be remembered that in the subsequent work 2911 had to be added to $3014 - \frac{p}{q}$ and the sum divided by 780, or $2 . 2 . 3 . 5 . 13$. It would obviously lead to simplification if a factor could be divided out, e.g. the best for the purpose, 13. Now, dividing 2911 + 3014, or 5925, by 13, we obtain the quotient 455, and a remainder 10, so that $10 - \frac{p}{q}$ remains to be divided by 13. Therefore $\frac{p}{q}$ has to be so chosen that $10q - p$ is divisible by 13, while $\frac{p}{q}$ approximates to, but is not greater than, $\frac{1875}{6028}$. The solution $p = 1$, $q = 4$ would therefore be natural and easy.

(2) $\sqrt{3380929}$.

The usual process for extraction of the square root gave as the integral part of it 1838, and as the remainder 2685. As before, it was easy to see that the exact root was nearer to 1839 than to 1838, and that

$$\sqrt{3380929} = 1838^2 + 2685 = 1839^2 - 2 \,.\, 1838 - 1 + 2685$$
$$= 1839^2 - 992.$$

The Archimedean formula then gave

$$1839 - \frac{992}{2 \,.\, 1839} > \sqrt{3380929}.$$

It could not have escaped Archimedes that $\frac{1}{4}$ was a near approximation to $\frac{992}{3678}$ or $\frac{1984}{7356}$, since $\frac{1}{4} = \frac{1839}{7356}$; and $\frac{1}{4}$ would have satisfied the necessary condition that the fraction to be taken must be less than the real value. Thus it is clear that, in taking $\frac{2}{11}$ as the approximate value of the fraction, Archimedes had in view the simplification of the subsequent work by the elimination of a factor. If the fraction be denoted by $\frac{p}{q}$, the sum of $1839 - \frac{p}{q}$ and 1823, or $3662 - \frac{p}{q}$, had to be divided by 240, i.e. by $6 \,.\, 40$. Division of 3662 by 40 gave 22 as remainder, and then p, q had to be so chosen that $22 - \frac{p}{q}$ was conveniently divisible by 40, while $\frac{p}{q}$ was less than but approximately equal to $\frac{992}{3678}$. The solution $p = 2$, $q = 11$ was easily seen to satisfy the conditions.

(3) $\sqrt{1018405}$.

The usual procedure gave $1018405 = 1009^2 + 324$ and the approximation

$$1009 \frac{324}{2018} > \sqrt{1018405}.$$

It was here necessary that the fraction to replace $\frac{324}{2018}$ should be greater but approximately equal to it, and $\frac{1}{6}$ satisfied the conditions, while the subsequent work did not require any change in it.

(4) $\sqrt{4069284\frac{1}{36}}$.

The usual process gave $4069284\frac{1}{36} = 2017^2 + 995\frac{1}{36}$; it followed that

$$2017 + \frac{36 . 995 + 1}{36 . 2 . 2017} > \sqrt{4069284\tfrac{1}{36}},$$

and $2017\frac{1}{4}$ was an obvious value to take as an approximation somewhat greater than the left side of the inequality.

(5) $\sqrt{349450}$.

In the case of this and the two following roots an approximation had to be obtained which was *less*, instead of greater, than the true value. Thus Archimedes had to use the second part of the formula

$$a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}.$$

In the particular case of $\sqrt{349450}$ the integral part of the root is 591, and the remainder is 169. This gave the result

$$591 + \frac{169}{2 . 591} > \sqrt{349450} > 591 + \frac{169}{2 . 591 + 1},$$

and since $169 = 13^2$, while $2 . 591 + 1 = 7 . 13^2$, it resulted without further calculation that

$$\sqrt{349450} > 591\tfrac{1}{7}.$$

Why then did Archimedes take, instead of this approximation, another which was not so close, viz. $591\frac{1}{8}$? The answer which the subsequent working and the other approximations in the first part of the proof suggest is that he preferred, for convenience of calculation, to use for his approximations fractions of the form $\frac{1}{2^n}$ only. But he could not have failed to see that to take the nearest fraction of this form, $\frac{1}{8}$, instead of $\frac{1}{7}$ might conceivably affect his final result and make it less near the truth than it need be. As a matter of fact, as Hultsch shows, it does not affect the result to take $591\frac{1}{7}$ and to work onwards from that figure. Hence we must suppose that Archimedes had satisfied himself, by taking $591\frac{1}{7}$ and proceeding on that basis for some distance, that he would not be introducing any appreciable error in taking the more convenient though less accurate approximation $591\frac{1}{8}$.

(6) $\sqrt{1373943\frac{33}{64}}$.

In this case the integral portion of the root is 1172, and the remainder $359\frac{33}{64}$. Thus, if R denote the root,

$$R > 1172 + \frac{359\frac{33}{64}}{2.1172+1}$$

$$> 1172 + \frac{359}{2.1172+1}, \textit{ a fortiori.}$$

Now $2.1172 + 1 = 2345$; the fraction accordingly becomes $\frac{359}{2345}$, and $\frac{1}{7}\left(=\frac{359}{2513}\right)$ satisfies the necessary conditions, viz. that it must be approximately equal to, but not greater than, the given fraction. Here again Archimedes would have taken $1172\frac{1}{7}$ as the approximate value but that, for the same reason as in the last case, $1172\frac{1}{8}$ was more convenient.

(7) $\sqrt{5472132\frac{1}{16}}$.

The integral portion of the root is here 2339, and the remainder $1211\frac{1}{16}$, so that, if R is the exact root,

$$R > 2339 + \frac{1211\frac{1}{16}}{2.2339+1}$$

$$> 2339\tfrac{1}{4}, \textit{ a fortiori.}$$

A few words may be added concerning Archimedes' ultimate reduction of the inequalities

$$3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} > \pi > 3 + \frac{284\frac{1}{4}}{2017\frac{1}{4}}$$

to the simpler result $3\frac{1}{7} > \pi > 3\frac{10}{71}$.

As a matter of fact $\frac{1}{7} = \frac{667\frac{1}{2}}{4672\frac{1}{2}}$, so that in the first fraction it was only necessary to make the small change of diminishing the denominator by 1 in order to obtain the simple $3\frac{1}{7}$.

As regards the *lower* limit for π, we see that $\frac{284\frac{1}{4}}{2017\frac{1}{4}} = \frac{1137}{8069}$; and Hultsch ingeniously suggests the method of trying the effect of increasing the denominator of the latter fraction by 1. This produces $\frac{1137}{8070}$ or $\frac{379}{2690}$; and, if we divide 2690 by 379, the quotient is between 7 and 8, so that

$$\frac{1}{7} > \frac{379}{2690} > \frac{1}{8}.$$

Now it is a known proposition (proved in Pappus VII. p. 689) that, if $\frac{a}{b} > \frac{c}{d}$, then

$$\frac{a}{b} > \frac{a+c}{b+d}.$$

Similarly it may be proved that

$$\frac{a+c}{b+d} > \frac{c}{d}.$$

It follows in the above case that

$$\frac{379}{2690} > \frac{379+1}{2690+8} > \frac{1}{8},$$

which exactly gives

$$\frac{10}{71} > \frac{1}{8},$$

and $\frac{10}{71}$ is very much nearer to $\frac{379}{2690}$ than $\frac{1}{8}$ is.

Note on alternative hypotheses with regard to the approximations to $\sqrt{3}$.

For a description and examination of all the various theories put forward, up to the year 1882, for the purpose of explaining Archimedes' approximations to $\sqrt{3}$ the reader is referred to the exhaustive paper by Dr Siegmund Günther, entitled *Die quadratischen Irrationalitäten der Alten und deren Entwickelungsmethoden* (Leipzig, 1882). The same author gives further references in his *Abriss der Geschichte der Mathematik und der Naturwissenschaften im Altertum* forming an Appendix to Vol. v. Pt. 1 of Iwan von Müller's *Handbuch der klassischen Altertums-wissenschaft* (München, 1894).

Günther groups the different hypotheses under three general heads:

(1) those which amount to a more or less disguised use of the method of continued fractions and under which are included the solutions of De Lagny, Mollweide, Hauber, Buzengeiger, Zeuthen, P. Tannery (first solution), Heilermann;

(2) those which give the approximations in the form of a series of fractions such as $a + \frac{1}{q_1} + \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} + \ldots$; under this class come the solutions of Radicke, v. Pessl, Rodet (with reference to the Çulvasūtras), Tannery (second solution);

(3) those which locate the incommensurable surd between a greater and lesser limit and then proceed to draw the limits closer and closer. This class includes the solutions of Oppermann, Alexejeff, Schönborn, Hunrath, though the first two are also connected by Günther with the method of continued fractions.

Of the methods so distinguished by Günther only those need be here referred to which can, more or less, claim to rest on a historical basis in the sense of representing applications or extensions of principles laid down in the works of Greek mathematicians other than Archimedes which have come down to us. Most of these quasi-historical solutions connect themselves with the system of *side-* and *diagonal-numbers* (πλευρικοὶ and διαμετρικοὶ ἀριθμοί) explained by Theon of Smyrna (c. 130 A.D.) in a work which was intended to give so much of the principles of mathematics as was necessary for the study of the works of Plato.

The *side-* and *diagonal-numbers* are formed as follows. We start with two units, and (*a*) from the sum of them, (*b*) from the sum of twice the first unit and once the second, we form two new numbers; thus

$$1 \cdot 1 + 1 = 2, \qquad 2 \cdot 1 + 1 = 3.$$

Of these numbers the first is a *side-* and the second a *diagonal*-number respectively, or (as we may say)

$$a_2 = 2, \qquad d_2 = 3.$$

In the same way as these numbers were formed from $a_1 = 1$, $d_1 = 1$, successive pairs of numbers are formed from a_2, d_2, and so on, in accordance with the formula

$$a_{n+1} = a_n + d_n, \qquad d_{n+1} = 2a_n + d_n,$$

whence we have

$$a_3 = 1 \cdot 2 + 3 = 5, \qquad d_3 = 2 \cdot 2 + 3 = 7,$$
$$a_4 = 1 \cdot 5 + 7 = 12, \qquad d_4 = 2 \cdot 5 + 7 = 17,$$

and so on.

Theon states, with reference to these numbers, the general proposition which we should express by the equation

$$d_n^2 = 2a_n^2 \pm 1.$$

The proof (no doubt omitted because it was well-known) is simple. For we have

$$\begin{aligned} d_n^2 - 2a_n^2 &= (2a_{n-1} + d_{n-1})^2 - 2(a_{n-1} + d_{n-1})^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2) \\ &= +(d_{n-2}^2 - 2a_{n-2}^2), \text{ and so on,} \end{aligned}$$

while $d_1^2 - 2a_1^2 = -1$; whence the proposition is established.

Cantor has pointed out that any one familiar with the truth of this proposition could not have failed to observe that, as the numbers were successively formed, the value of d_n^2/a_n^2 would approach more and more nearly to 2, and consequently the successive fractions d_n/a_n would give nearer and nearer approximations to the value of $\sqrt{2}$, or in other words that

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots\ldots$$

are successive approximations to $\sqrt{2}$. It is to be observed that the third of these approximations, $\frac{7}{5}$, is the Pythagorean approximation which appears to be hinted at by Plato, while the above scheme of Theon, amounting to a method of finding all the solutions in positive integers of the indeterminate equation

$$2x^2 - y^2 = \pm 1,$$

and given in a work designedly introductory to the study of Plato, distinctly suggests, as Tannery has pointed out, the probability that even in Plato's lifetime the systematic investigation of the said equation had already begnn in the Academy. In this connexion Proclus' commentary on Eucl. I. 47 is interesting. It is there explained that in isosceles right-angled triangles "it is not possible to find numbers corresponding to the sides; for there is no square number which is double of a square except in the sense of *approximately* double, e.g. 7^2 is double of 5^2 less 1." When it is remembered that Theon's process has for its object the finding of any number of squares differing only by unity from double the squares of another series of numbers respectively, and that the sides of the two sets of squares are called *diagonal*- and *side*-numbers respectively, the conclusion becomes almost irresistible that Plato had such a system in mind when he spoke of ῥητὴ διάμετρος (*rational* diagonal) as compared with ἄρρητος διάμετρος (*irrational* diagonal) τῆς πεμπάδος (cf. p. lxxviii above).

One supposition then is that, following a similar line to that by which successive approximations to $\sqrt{2}$ could be obtained from the successive solutions, in rational numbers, of the indeterminate equations $2x^2 - y^2 = \pm 1$, Archimedes set himself the task of finding all the solutions, in rational numbers, of the two indeterminate equations bearing a similar relation to $\sqrt{3}$, viz.

$$x^2 - 3y^2 = 1,$$
$$x^2 - 3y^2 = -2.$$

Zeuthen appears to have been the first to connect, *eo nomine*, the ancient approximations to $\sqrt{3}$ with the solution of these equations, which are also made by Tannery the basis of his first method. But, in substance, the same method had been used as early as 1723 by De Lagny, whose hypothesis will be, for purposes of comparison, described after Tannery's which it so exactly anticipated.

Zeuthen's solution.

After recalling the fact that, even before Euclid's time, the solution of the indeterminate equation $x^2 + y^2 = z^2$ by means of the substitutions

$$x = mn, \qquad y = \frac{m^2 - n^2}{2}, \qquad z = \frac{m^2 + n^2}{2}$$

was well known, Zeuthen concludes that there could have been no difficulty in deducing from Eucl. II. 5 the identity

$$3(mn)^2 + \left(\frac{m^2 - 3n^2}{2}\right)^2 = \left(\frac{m^2 + 3n^2}{2}\right)^2,$$

from which, by multiplying up, it was easy to obtain the formula

$$3(2mn)^2 + (m^2 - 3n^2)^2 = (m^2 + 3n^2)^2.$$

If therefore one solution $m^2 - 3n^2 = 1$ was known, a second could at once be found by putting

$$x = m^2 + 3n^2, \qquad y = 2mn.$$

Now obviously the equation

$$m^2 - 3n^2 = 1$$

is satisfied by the values $m = 2$, $n = 1$; hence the next solution of the equation

$$x^2 - 3y^2 = 1$$

is $$x_1 = 2^2 + 3 . 1 = 7, \qquad y_1 = 2 . 2 . 1 = 4;$$

and, proceeding in like manner, we have any number of solutions as

$$x_2 = 7^2 + 3 . 4^2 = 97, \qquad y_2 = 2 . 7 . 4 = 56,$$

$$x_3 = 97^2 + 3 . 56^2 = 18817, \qquad y_3 = 2 . 97 . 56 = 10864,$$

and so on.

Next, addressing himself to the other equation

$$x^2 - 3y^2 = -2,$$

Zeuthen uses the identity

$$(m + 3n)^2 - 3(m + n)^2 = -2(m^2 - 3n^2).$$

Thus, if we know one solution of the equation $m^2 - 3n^2 = 1$, we can proceed to substitute

$$x = m + 3n, \qquad y = m + n.$$

Suppose $m = 2$, $n = 1$, as before; we then have

$$x_1 = 5, \qquad y_1 = 3.$$

If we put $x_2 = x_1 + 3y_1 = 14$, $y_2 = x_1 + y_1 = 8$, we obtain

$$\frac{x_2}{y_2} = \frac{14}{8} = \frac{7}{4}$$

(and $m = 7$, $n = 4$ is seen to be a solution of $m^2 - 3n^2 = 1$).

Starting again from x_2, y_2, we have

$$x_3 = 38, \qquad y_3 = 22,$$

and $$\frac{x_3}{y_3} = \frac{19}{11}$$

($m = 19$, $n = 11$ being a solution of the equation $m^2 - 3n^2 = -2$);

$$x_4 = 104, \qquad y_4 = 60,$$

whence $$\frac{x_4}{y_4} = \frac{26}{15}$$

(and $m=26$, $n=15$ satisfies $m^2-3n^2=1$),

$$x_5=284, \quad y_5=164,$$

or

$$\frac{x_5}{y_5}=\frac{71}{41}.$$

Similarly $\frac{x_6}{y_6}=\frac{97}{56}$, $\frac{x_7}{y_7}=\frac{265}{153}$, and so on.

This method gives all the successive approximations to $\sqrt{3}$, taking account as it does of both the equations

$$x^2-3y^2=1,$$
$$x^2-3y^2=-2.$$

Tannery's first solution.

Tannery asks himself the question how Diophantus would have set about solving the two indeterminate equations. He takes the first equation in the generalised form

$$x^2-ay^2=1,$$

and then, assuming one solution (p, q) of the equation to be known, he supposes

$$p_1=mx-p, \quad q_1=x+q.$$

Then $$p_1{}^2-aq_1{}^2\equiv m^2x^2-2mpx+p^2-ax^2-2aqx-aq^2=1,$$

whence, since $p^2-aq^2=1$, by hypothesis,

$$x=2\,.\,\frac{mp+aq}{m^2-a},$$

so that $$p_1=\frac{(m^2+a)p+2amq}{m^2-a}, \quad q_1=\frac{2mp+(m^2+a)q}{m^2-a},$$

and $p_1{}^2-aq_1{}^2=1$.

The values of p_1, q_1 so found are rational but not necessarily integral; if integral solutions are wanted, we have only to put

$$p_1=(u^2+av^2)p+2auvq, \quad q_1=2puv+(u^2+av^2)q,$$

where (u, v) is another integral solution of $x^2-ay^2=1$.

Generally, if (p, q) be a known solution of the equation

$$x^2-ay^2=r,$$

suppose $p_1=\alpha p+\beta q$, $q_1=\gamma p+\delta q$, and "il suffit pour déterminer $\alpha, \beta, \gamma, \delta$ de connaître les trois groupes de solutions les plus simples et de résoudre deux couples d'équations du premier degré à deux inconnues." Thus (1) for the equation

$$x^2-3y^2=1,$$

the first three solutions are

$$(p=1, q=0), \quad (p=2, q=1), \quad (p=7, q=4),$$

whence $$\left.\begin{matrix}2=\alpha\\1=\gamma\end{matrix}\right\} \text{ and } \left.\begin{matrix}7=2\alpha+\beta\\4=2\gamma+\delta\end{matrix}\right\},$$

so that $$\alpha=2, \beta=3, \gamma=1, \delta=2,$$

and it follows that the fourth solution is given by

$$p = 2\,.\,7 + 3\,.\,4 = 26,$$
$$q = 1\,.\,7 + 2\,.\,4 = 15\,;$$

(2) for the equation $x^2 - 3y^2 = -2$,

the first three solutions being (1, 1), (5, 3), (19, 11), we have

$$\left.\begin{matrix} 5 = \alpha + \beta \\ 3 = \gamma + \delta \end{matrix}\right\} \text{ and } \left.\begin{matrix} 19 = 5\alpha + 3\beta \\ 11 = 5\gamma + 3\delta \end{matrix}\right\},$$

whence $\alpha = 2$, $\beta = 3$, $\gamma = 1$, $\delta = 2$, and the next solution is given by

$$p = 2\,.\,19 + 3\,.\,11 = 71,$$
$$q = 1\,.\,19 + 2\,.\,11 = 41,$$

and so on.

Therefore, by using the two indeterminate equations and proceeding as shown, all the successive approximations to $\sqrt{3}$ can be found.

Of the two methods of dealing with the equations it will be seen that Tannery's has the advantage, as compared with Zeuthen's, that it can be applied to the solution of *any* equation of the form $x^2 - ay^2 = r$.

De Lagny's method.

The argument is this. If $\sqrt{3}$ could be exactly expressed by an improper fraction, that fraction would fall between 1 and 2, and the square of its numerator would be three times the square of its denominator. Since this is impossible, two numbers have to be sought such that the square of the greater differs as little as possible from 3 times the square of the smaller, though it may be either greater or less. De Lagny then evolved the following successive relations,

$$2^2 = 3\,.\,1^2 + 1, \quad 5^2 = 3\,.\,3^2 - 2, \quad 7^2 = 3\,.\,4^2 + 1, \quad 19^2 = 3\,.\,11^2 - 2,$$
$$26^2 = 3\,.\,15^2 + 1, \quad 71^2 = 3\,.\,41^2 - 2, \text{ etc.}$$

From these relations were derived a series of fractions greater than $\sqrt{3}$, viz. $\frac{2}{1}, \frac{7}{4}, \frac{26}{15}$, etc., and another series of fractions less than $\sqrt{3}$, viz. $\frac{5}{3}, \frac{19}{11}, \frac{71}{41}$, etc. The law of formation was found in each case to be that, if $\frac{p}{q}$ was one fraction in the series and $\frac{p'}{q'}$ the next, then

$$\frac{p'}{q'} = \frac{2p + 3q}{p + 2q}.$$

This led to the results

$$\frac{2}{1} > \frac{7}{4} > \frac{26}{15} > \frac{97}{56} > \frac{362}{209} > \frac{1351}{780} \ldots > \sqrt{3},$$

and

$$\frac{5}{3} < \frac{19}{11} < \frac{71}{41} < \frac{265}{153} < \frac{989}{571} < \frac{3691}{2131} \ldots < \sqrt{3};$$

while the law of formation of the successive approximations in each series is precisely that obtained by Tannery as the result of treating the two indeterminate equations by the Diophantine method.

Heilermann's method.

This method needs to be mentioned because it also depends upon a generalisation of the system of *side-* and *diagonal-*numbers given by Theon of Smyrna.

Theon's rule of formation was

$$S_n = S_{n-1} + D_{n-1}, \quad D_n = 2S_{n-1} + D_{n-1};$$

and Heilermann simply substitutes for 2 in the second relation any arbitrary number a, developing the following scheme,

$$\begin{array}{ll} S_1 = S_0 + D_0, & D_1 = aS_0 + D_0, \\ S_2 = S_1 + D_1, & D_2 = aS_1 + D_1, \\ S_3 = S_2 + D_2, & D_3 = aS_2 + D_2, \\ \vdots & \vdots \\ S_n = S_{n-1} + D_{n-1}, & D_n = aS_{n-1} + D_{n-1}. \end{array}$$

It follows that

$$aS_n^2 = aS_{n-1}^2 + 2aS_{n-1}D_{n-1} + aD_{n-1}^2,$$

$$D_n^2 = a^2S_{n-1}^2 + 2aS_{n-1}D_{n-1} + D_{n-1}^2.$$

By subtraction, $D_n^2 - aS_n^2 = (1-a)(D_{n-1}^2 - aS_{n-1}^2)$

$$= (1-a)^2(D_{n-2}^2 - aS_{n-2}^2), \text{ similarly,}$$

$$= \ldots\ldots\ldots$$

$$= (1-a)^n (D_0^2 - aS_0^2).$$

This corresponds to the most general form of the "Pellian" equation

$$x^2 - ay^2 = (\text{const.}).$$

If now we put $D_0 = S_0 = 1$, we have

$$\frac{D_n^2}{S_n^2} = a + \frac{(1-a)^{n+1}}{S_n^2},$$

from which it appears that, where the fraction on the right-hand side approaches zero as n increases, $\frac{D_n}{S_n}$ is an approximate value for $\sqrt{a}$.

Clearly in the case where $a = 3$, $D_0 = 2$, $S_0 = 1$ we have

$$\frac{D_0}{S_0} = \frac{2}{1}, \ \frac{D_1}{S_1} = \frac{5}{3}, \ \frac{D_2}{S_2} = \frac{14}{8} = \frac{7}{4}, \ \frac{D_3}{S_3} = \frac{19}{11}, \ \frac{D_4}{S_4} = \frac{52}{30} = \frac{26}{15},$$

$$\frac{D_5}{S_5} = \frac{71}{41}, \ \frac{D_6}{S_6} = \frac{194}{112} = \frac{97}{56}, \ \frac{D_7}{S_7} = \frac{265}{153},$$

and so on.

But the method is, as shown by Heilermann, more rapid if it is used to find, not $\sqrt{a}$, but $b\sqrt{a}$, where b is so chosen as to make b^2a (which takes the place of a) somewhat near to unity. Thus suppose $a=\frac{27}{25}$, so that $\sqrt{a}=\frac{3}{5}\sqrt{3}$, and we then have (putting $D_0=S_0=1$)

$$S_1=2,\quad D_1=\frac{52}{25},\ \text{and}\ \sqrt{3}\backsim\frac{5}{3}\cdot\frac{26}{25},\ \text{or}\ \frac{26}{15},$$

$$S_2=\frac{102}{25},\quad D_2=\frac{54+52}{25}=\frac{106}{25},\ \text{and}\ \sqrt{3}\backsim\frac{5}{3}\cdot\frac{106}{102},\ \text{or}\ \frac{265}{153},$$

$$S_3=\frac{208}{25},\quad D_3=\frac{102\,.\,27}{25\,.\,25}+\frac{106}{25}=\frac{5404}{25\,.\,25},$$

and
$$\sqrt{3}\backsim\frac{5404}{25\,.\,208}\cdot\frac{5}{3},\ \text{or}\ \frac{1351}{780}.$$

This is one of the very few instances of success in bringing out the two Archimedean approximations in immediate sequence without any foreign values intervening. No other methods appear to connect the two values in this direct way except those of Hunrath and Hultsch depending on the formula

$$a\pm\frac{b}{2a}>\sqrt{a^2\pm b}>a\pm\frac{b}{2a\pm1}.$$

We now pass to the second class of solutions which develops the approximations in the form of the sum of a series of fractions, and under this head comes

Tannery's second method.

This may be exhibited by means of its application (1) to the case of the square root of a large number, e.g. $\sqrt{349450}$ or $\sqrt{571^2+23409}$, the first of the kind appearing in Archimedes, (2) to the case of $\sqrt{3}$.

(1) Using the formula

$$\sqrt{a^2+b}\backsim a+\frac{b}{2a},$$

we try the effect of putting for $\sqrt{571^2+23409}$ the expression

$$571+\frac{23409}{1142}.$$

It turns out that this gives correctly the integral part of the root, and we now suppose the root to be

$$571+20+\frac{1}{m}.$$

Squaring and regarding $\frac{1}{m^2}$ as negligible, we have

$$571^2+400+22840+\frac{1142}{m}+\frac{40}{m}=571^2+23409,$$

whence $$\frac{1182}{m}=169,$$

and $$\frac{1}{m}=\frac{169}{1182}>\frac{1}{7},$$

so that $$\sqrt{349450}>591\frac{1}{7}.$$

(2) Bearing in mind that
$$\sqrt{a^2+b}\backsim a+\frac{b}{2a+1},$$

we have $$\sqrt{3}=\sqrt{1^2+2}\backsim 1+\frac{2}{2.1+1}$$
$$\backsim 1+\frac{2}{3}, \text{ or } \frac{5}{3}.$$

Assuming then that $\sqrt{3}=\left(\frac{5}{3}+\frac{1}{m}\right)$, squaring and neglecting $\frac{1}{m^2}$, we obtain
$$\frac{25}{9}+\frac{10}{3m}=3,$$

whence $m=15$, and we get as the second approximation
$$\frac{5}{3}+\frac{1}{15}, \text{ or } \frac{26}{15}.$$

We have now $$26^2-3.15^2=1,$$

and can proceed to find other approximations by means of Tannery's first method.

Or we can also put $$\left(1+\frac{2}{3}+\frac{1}{15}+\frac{1}{n}\right)^2=3,$$

and, neglecting $\frac{1}{n^2}$, we get
$$\frac{26^2}{15^2}+\frac{52}{15n}=3,$$

whence $n=-15.52=-780$, and
$$\sqrt{3}\backsim\left(1+\frac{2}{3}+\frac{1}{15}-\frac{1}{780}=\frac{1351}{780}\right).$$

It is however to be observed that this method only connects $\frac{1351}{780}$ with $\frac{26}{15}$ and not with the intermediate approximation $\frac{265}{153}$, to obtain which Tannery implicitly uses a particular case of the formula of Hunrath and Hultsch.

Rodet's method was apparently invented to explain the approximation in the Çulvasūtras*
$$\sqrt{2}\backsim 1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34};$$

* See Cantor, *Vorlesungen über Gesch. d. Math.* p. 600 sq.

but, given the approximation $\frac{4}{3}$, the other two successive approximations indicated by the formula can be obtained by the method of squaring just described* without such elaborate work as that of Rodet, which, when applied to $\sqrt{3}$, only gives the same results as the simpler method.

Lastly, with reference to the third class of solutions, it may be mentioned

(1) that Oppermann used the formula

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b},$$

which gave successively

$$\frac{2}{1} > \sqrt{3} > \frac{3}{2},$$

$$\frac{7}{4} > \sqrt{3} > \frac{12}{7},$$

$$\frac{97}{56} > \sqrt{3} > \frac{168}{97},$$

but only led to one of the Archimedean approximations, and that by combining the last two ratios, thus

$$\frac{97+168}{56+97} = \frac{265}{153},$$

(2) that Schönborn came somewhat near to the formula successfully used by Hunrath and Hultsch when he proved† that

$$a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm \sqrt{b}}.$$

* Cantor had already pointed this out in his first edition of 1880.

† *Zeitschrift für Math. u. Physik* (*Hist. litt. Abtheilung*) XXVIII. (1883), p. 169 sq.