Steve Chinn

MATHS LEARNING DIFFICULTIES, DYSLEXIA AND DYSCALCULIA

SECOND EDITION

Maths Learning Difficulties, Dyslexia and Dyscalculia

Second Edition

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Jessica Kingsley *Publishers*
London and Philadelphia

Contents

Chapter 6

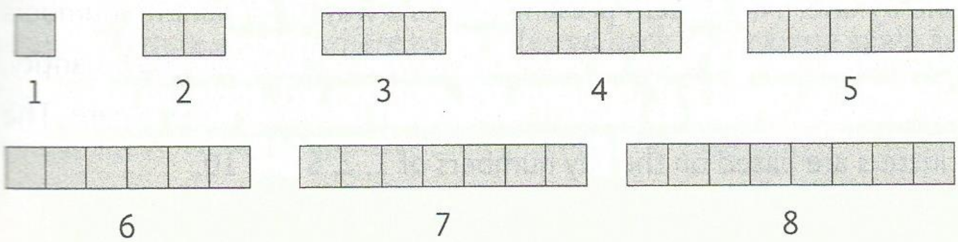
Early Number Experiences

Number (and operation) concepts are unequivocally fundamental to progress. Get the foundations secure, not just in long-term memory as facts, but as interlinking concepts.

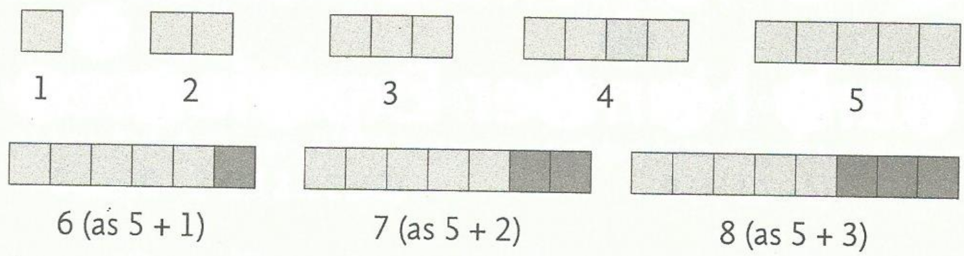
I sometimes visit bookshops to browse the books about numbers that are targeted at young children. Often, in fact, usually, they are more about design than content. Sadly, this observation applies to some books that target older learners, too, where the inclusion of drawings of cute animals in clustered groups blur the concept of number that they are supposed to illustrate. I realise that meerkats may not naturally cluster themselves in manageable and identifiable patterns in real life, but then the illustrations in maths books are rarely drawn to look like real life anyway.

I believe that two factors are influential here. One is the dominant power of the first learning experience (Buswell and Judd 1925) and the other is the negative influence of apparent inconsistencies in many of the aspects of learning. However, we must remember that consistency may take different manifestations for different children. For example, two ways could be used to represent numbers visually:

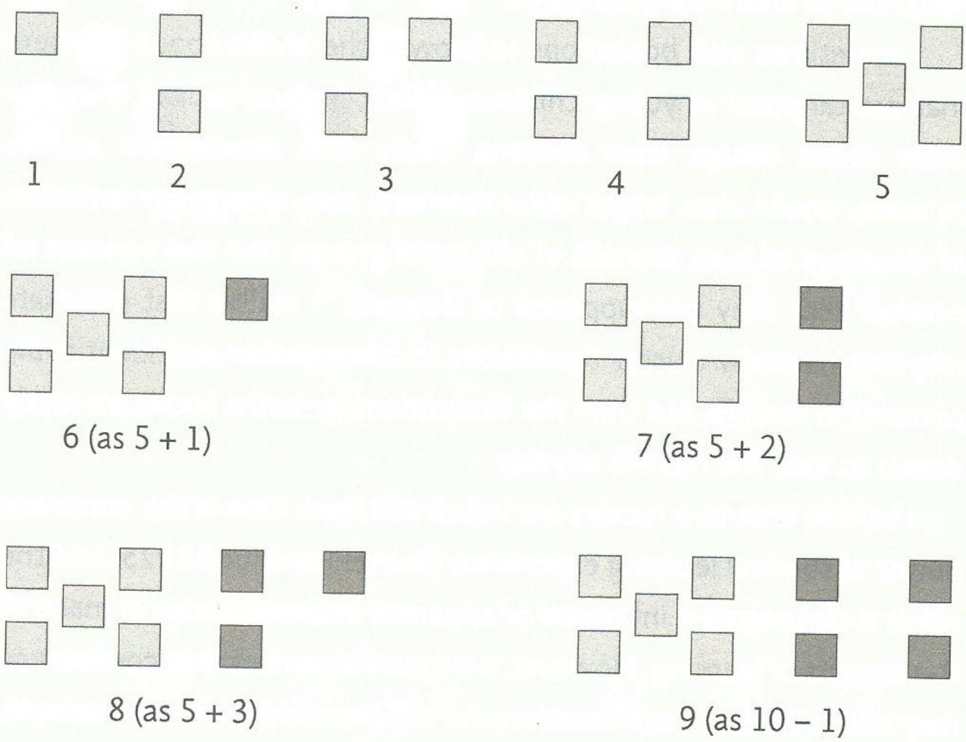
A linear arrangement:



Which could include a differentiation to show the contribution of 5:

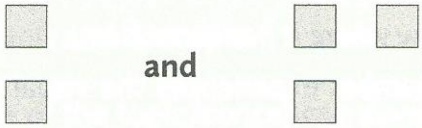


The second representation is as a pattern (see also Chapter 4):



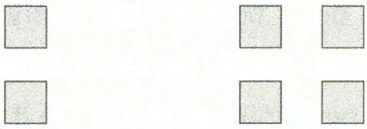
The linear presentation may suit the children who like to count, often one by one. The pattern presentation is a way of presenting numbers in clusters, using a form of subitising, a way of recognising a quantity, for example, 5 from its pattern rather than counting each square. The clusters are based on the key numbers of 1, 2, 5 and 10.

The squares may not be as attractive as the cartoons of worms or meerkats or ladybirds, but they can be used to show, in an uncluttered way, the relationships between numbers. For example, the patterns



can be used to show that 3 is more than 2, that 3 is bigger than 2, that 3 is one more than 2, that 3 is 2 + 1, that 2 is one less than 3, that 2 is 3 take away 1 (and the 'take away' is a literal interpretation if the square is actually taken away). Even for such basic relationships the vocabulary can be confusingly varied. I could have said, '1 is subtracted from 3' or '3 subtract 1' (thereby changing the order for the digits).

The two patterns



can be used to show that 4 is 'double 2', 'two times two', 'two plus two' (which is repeated addition and thus linked to multiplication, 2 × 2) or that 2 is half of 4 (showing division as the inverse of multiplication).

There are learning/teaching implications for this approach. The patterns, both the visual patterns and the symbol patterns, can be used at a range of levels. They are developmental. They take a basic concept and can be used to develop that into other concepts. This could be seen as sowing the seeds of maths development, setting down the roots for future growth, there to be referred back to time and again as the maths develops. For example, the question, 'Is it bigger or smaller?' encourages children to appraise their answers in a reasonably low-stress way, using an aspect of estimation. The relationships between 2 and 4 can set the foundations for repeated multiplication (×4 as ×2×2 and ×8 as ×2×2×2 or $\times 2^3$), for division and for the doubles facts.

For early maths it helps to know where the maths is going; for later maths it helps to know where it has come from.

For these early number experiences the visual representations are used to develop a sense of number and how numbers and operations inter-relate. This is using single-digit numbers, but that will progress to become two-digit numbers (and beyond).

Materials and manipulatives

As a teacher of physics, in the early days of my teaching career, it would have been bizarre not to use apparatus to set up experiments to demonstrate concepts, even at Advanced level. So, I find the use of materials to illustrate maths concepts quite a natural thing to do. I believe that every maths classroom should have a maths kit readily available, either in a cupboard or in a toolbox of the type you can buy in Do It Yourself stores, so that the contents are ready to use when explaining some maths process or concept that a pupil or pupils find confusing. Match the material to the concept and the vocabulary and, when possible, to the learner(s), using the materials and language alongside the maths symbols, so that the link is made between the 'bricks' and the 'numbers'.

There are many examples of materials being used throughout this book.

Before we move to the next development, two-digit numbers, I want to introduce a structure that will be used with each topic in the following chapters. The key learning factors will be considered for each topic. These are:

- vocabulary and language
- images, symbols and concepts
- the relevance of the topic/concept to developing maths skills
- meta-cognition (thinking about how and what you are thinking)
- things to do and practise.

Chapter 7

Two-Digit Numbers

Two-digit numbers start at 10 and end at 99.

Vocabulary and language

The English language does not offer consistency for the two-digit numbers, particularly from 10 to 19. This must be very confusing for young children when they meet them for the first time. The vocabulary does not support the pattern of the symbols.

The words for the first two-digit numbers after 10 are exceptions: eleven and twelve. The numbers that then take us towards 20, the teen numbers, suggest an order that is the reverse to what we write with digits, for example, we say 'fourteen' which hints at 4 and 10, but we write 14: 10 and 4. As well as the vocabulary, there is a very sophisticated concept here, the concept of place value.

Images, symbols and concepts

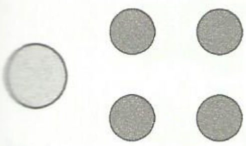
The symbols for fourteen are 14, a 1 and a 4 in a specific order. Change the order and 14 becomes 41, forty-one. There are two places where we could place the digits:

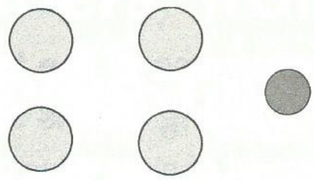
____ ____ which can be 1 4 or 4 1

If we write 1 in the first place and 4 in the second place then that 14 is fourteen, 1 ten and 4 ones.

If we write 4 in the first place and 1 in the second place then that 41 is forty-one, 4 tens and 1 one.

If we use coins as our kinaesthetic/visual images, then fourteen, 14, is one 10p coin and four 1p coins and forty-one, 41, is four 10p coins and one 1p coin:





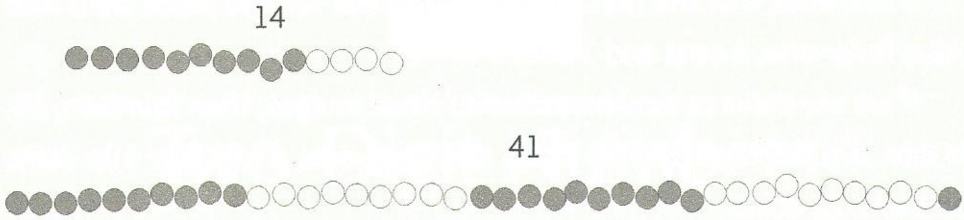
So, the *place* where we write the digits (relative to each other) in the number determines the *value* they represent. In this example, 1 can represent 1 ten or it can represent 1 one. 4 can represent 4 ones or 4 tens. This is the logic of *place value.*

One key reason for this being a big problem is that this first experience that children have of place value as a concept, a very vital and pervasive concept, is not supported by the vocabulary.

Things improve in the twenties, thirties, forties and on. For example, we say forty-five and write 45. Sadly, we don't say 4 tens and 5. We learn to understand that the '-ty' is a distortion of 'ten'.

Another problem for some children is discriminating (aurally) the sounds of 'thirteen' and 'thirty', 'fourteen' and 'forty', and even 'twenty' and 'twelve', but if teachers and parents are aware of the potential for confusion then they can guard against it, possibly by using visuals or materials (such as coins or base 10 materials/Dienes blocks).

Bead strings and coins are good materials for supporting the concept of place value as are base 10 blocks. The effective strategy is to show the materials alongside the numbers and to connect these and to talk the learner through the relevance of the illustration.



The relevance of the topic/concept to developing maths skills

The concept of place value underpins much of the arithmetic part of mathematics. For example, addition and subtraction, multiplying and dividing by tens, hundreds, thousands and so on, and decimals. As well as developing a flexible understanding of numbers, for example 'seeing' that 46 is 30 + 16 and that 9 is one less than 10, learners need to understand the role of zero in place value. Roman numerals do not include a zero. Zero was introduced when the Hindu-Arabic system of number entered the UK some seven hundred years ago.

Two of the major goals for teachers are to develop a clear understanding of place value in their pupils and to wean them off counting in ones.

Chapter 8

Moving from One-Digit to Two-Digit Numbers, from Two-Digit to Three-Digit Numbers...and Back

The process of 'crossing' the tens, for example, from 9 to 10 (and the hundreds, for example, from 99 to 100, and so on for the thousands, etc.), is one of the fundamental concepts of maths. It is a very important part of place value.

Vocabulary and language

'Crossing' the tens means crossing from 10 ones to 1 ten. This is sometimes known as 'trading' or as 'carrying' or as 're-naming' when doing addition problems.

When the 'crossing' is going back, it is crossing from 1 ten to 10 ones. This can be called 're-grouping' or 'decomposing' or 'borrowing' or 'trading'. (My preferred word for both crossings is 'trading'.) This is used in some subtraction problems.

Images, symbols and concepts

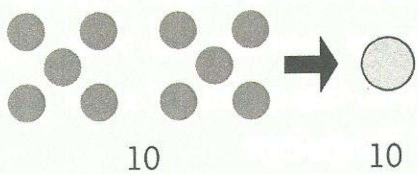
The images used below are coins. Note that the symbols, the digits, are written next to the images. You must link the visual to the symbol.

Remember coins are not proportional in size to the values they represent.

THE TASK

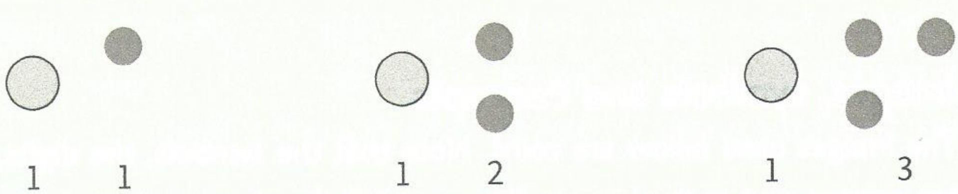
When we count up to, and then past 10, we move from a one-digit number to a two-digit number where one digit represents tens and the other represents ones according to the place they hold in the number.

Let's start at 8 and count up to 12 and let's look at how it works by using 1p and 10p coins:

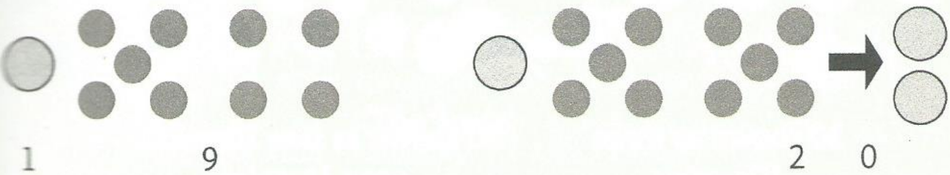


When we reach ten, we can exchange to ten 1p coins for one 10p coin. So, as we cross the tens there is an exchange (or trade). Sometimes in maths this is called re-naming or re-grouping.

Now as we count onwards, 11, 12, 13, the 10p coin represents, of course, the ten, and the 1p coins represent the ones.

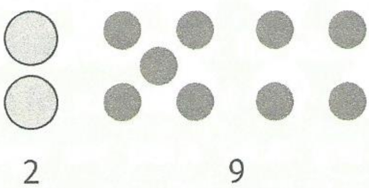


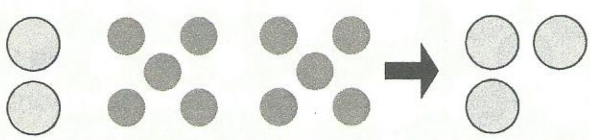
Another exchange happens as we cross the tens and count from 19 to 20:



The zero, 0, in 20 tells us that there are zero ones. It also keeps the tens-digit in the correct place, where the tens-digits should be. If the zero, 0, was not there then we would have just 2.

The next exchange, the next time we cross the tens, is from 29 to 30:



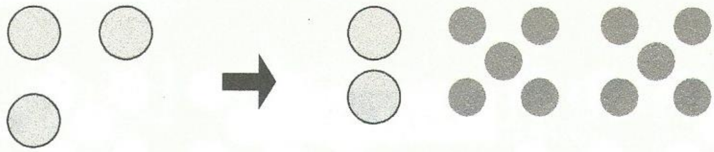


30, shown as 2 tens and 10 ones 20 + 10

30 as 3 tens

When we count backwards the exchange is reversed; it is the opposite procedure. We change one 10p coin for ten 1p coins. For example, it helps to 'see' or visualise 30 as three 10p coins and then as two 10p coins plus ten 1p coins, that is as 20 + 10:

$30 = 3 \times 10$ $30 = 2 \times 10 + 10 \times 1$

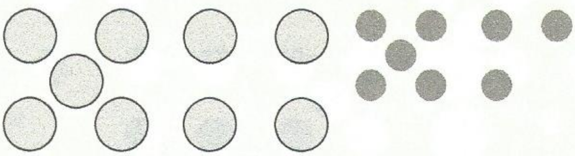


30 shown as three 10p coins

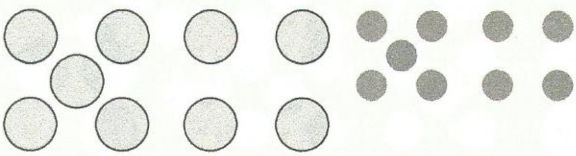
30 shown as two 10p coins + ten 1p coins

Then we can count back in ones and take away one 1p coin each time.

The same concept applies when we move from two-digit to three-digit numbers. So, for the sequence, 98, 99, 100, 101 we have:

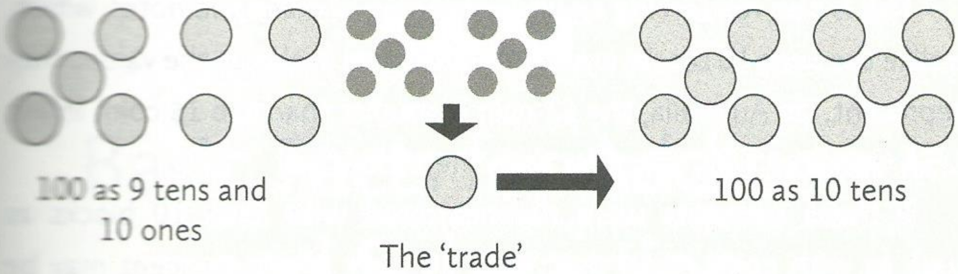


9 8



9 9

Now we get a double cross (!). The next one we add takes the 9 ones to 10 ones, which crosses the tens and we trade the 10 ones (1p coins) for 1 ten (a 10p coin). That makes ten 10p coins so we trade again, but this time we trade ten 10p coins for one £1 coin (100p).



These visual images should be demonstrated first with the coins and backed up with discussion and diagnostic questioning to ensure that the concept is understood.

The relevance of the topic/concept to developing maths skills

Counting on builds the foundation for addition and counting back builds the foundation for subtraction.

The concepts here are place value and 'trading'. Place value underpins our number system. Trading will be a key concept when we move onto addition and subtraction. It will be used again in other topics.

All the demonstrations above can and should be demonstrated with base 10 blocks (which can be bought online from a range of suppliers).

Meta-cognition (thinking about how and what you are thinking)

The learner should be encouraged to articulate their perceptions of these demonstrations. Diagnostic questioning may well help this, for example, 'Can you explain what is happening with the coins when you show me the sequence 38, 39, 40, 41?'

Can they now demonstrate 998, 999, 1000 (a £10 note)?

This is also reinforcing an understanding of coins and notes, which, unlike the base 10 blocks are not proportional in size to the values they represent. (In Australia, the $2 coin is smaller than the 1$ coin! In the USA the dime, 10c, is smaller than the nickel, 5c!)

If these concepts were demonstrated with base 10 blocks as well as with coins, then that 'base 10' part of the concept may be demonstrated more directly since the sizes of these materials are in proportion to the values they represent.

Things to do and practise

Practise crossing the tens, both ways, for a range of examples, such as 49 to 50 and 50 to 49, and 79 to 80 and 80 to 79. Do this with coins and symbols, base 10 blocks and symbols and then only symbols.

Practise counting on in tens, starting from numbers such as 17 (27, 37, 47...), 14 (24, 34, 44, 54...), 35 (45, 55, 65, 75...), or 48 (58, 68, 78...). Do a similar practice exercise counting backwards.

Do similar exercises for 197 to 202, 499 to 503 and 789 to 801.

Can the learner extend the concept to 999 to 1001?

Ask the learner to point to the tens-digit in numbers such as 2961, 641 and 1007 and then do similar exercises to point out hundreds and thousands digits.

Encourage the learner to articulate their thinking as they carry out the tasks (meta-cognition again).

Chapter 9

Basic Facts for Addition and Subtraction

These are the 'facts' for the addition of any one-digit number or 10 to another one-digit number or 10. That is, from 0 + 0 to 10 + 10. They are probably known as 'basic' facts because, if students know them, they can work out all other whole number additions.

There is an equivalent collection of basic facts for subtraction, from 20 – 10 to 0 – 0.

One of the most useful things about these basic facts is that they can be used to work out a fact that may have been forgotten, or to check it, if the learner is not 100 per cent sure of the answer (and preferable to using finger counting). These basic facts interlink. For example, if you know that 10 + 7 is 17, and you understand that 9 + 7 will give you an answer that is smaller and smaller by 1, so then, from 10 + 7 = 17 you can get 9 + 7 = 16, without counting 7 onto 9, or indeed retrieving the fact from long-term memory. From 10 + 7 you can extrapolate to 100 + 70, 1000 + 700 and so on (and thus re-visit place value).

So, if pupils can use number skills to work out more facts and answers from the ones that they do remember, then it is worth considering which facts are the most useful to memorise.

Vocabulary and language

There are several words that are used for the addition symbol + and the subtraction symbol –. Learners need to be familiar with the words used for these symbols.

So, for example, 7 + 6 can be said as:

7 plus 6
7 add 6
7 and 6
7 and 6 more
7 more than 6
the total for 6 and 7 is

and 13 – 6 can be said as:

13 minus 6
13 take away 6
What is the difference between 13 and 6?
13 subtract 6
What is 6 less than 13?

Images, symbols and concepts

The images and/or materials that could be used to illustrate these facts include counters, coins, Cuisenaire rods, base 10 blocks and number lines. Creative minds could find many other resources, such as sweets or rulers.

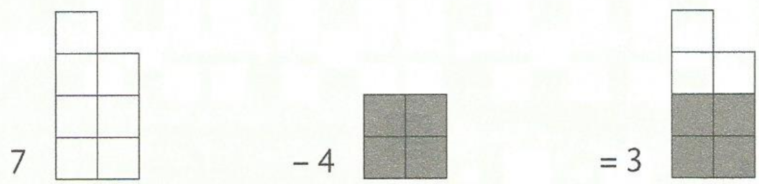
The concepts here are that:

addition and subtraction are reverse 'operations' so,
the two operations are linked and thus,
the number facts involved interlink, e.g. 7 + 6 = 13
13 – 7 = 6.

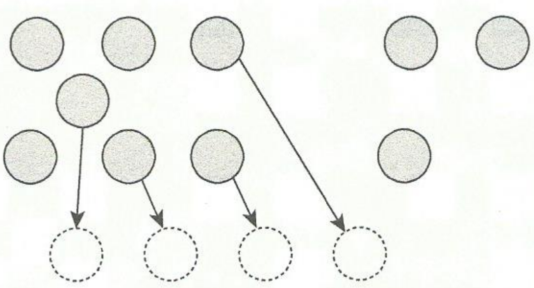
There can be an interaction between the vocabulary used and the manipulatives that are used. For example, subtraction can be implied with the words:

'What is the difference between 7 and 4?' or 'From 7 take away 4'.

The first can be modelled with Stern blocks, which can be compared to show the 'difference'.



The second can be modelled with counters.



THE TASK

So, which are the most useful facts to learn? Part of the way we find the answer to that question is by asking a second question, 'Which facts are most useful for working out other facts and doing other maths?'

Three of the most useful sets of facts are:

the doubles, e.g. 7 + 7

the number bonds for 10, e.g. 6 + 4   2 + 8

and the 10 plus a single digit, e.g. 10 + 3

Chapter 10

Addition and Subtraction

As the topics and strands of maths develop, learners should move further and further away from counting in ones. The 'basic facts' or 'number bonds' are part of that process. The previous chapter looked at how these facts could be retrieved by methods that supported memory and sense of number. In this chapter the processes of addition and subtraction are extended to all numbers. Whilst using finger counting to work out 6 + 8 can be effective, it can be inefficient and prone to error, but using a similar strategy is not going to be feasible for examples such as 364 + 877 or 2009 – 743. Even if the counting strategy is used for the steps in a procedure, it will slow the procedure down, thus putting more demand on working memory and leading to errors.

The strategies that were explained in the last chapter use chunks, for example 7 is chunked as 5 + 2 rather than seven ones. As with all the themes in this book, this idea will be developed and used as the maths develops. The principle is that using familiar, automatic chunks reduces the load on working memory.

Vocabulary and language

The vocabulary and the language used when addition and subtraction are put into word problems would require a chapter of its own (Chinn 2017a). In this chapter addition and subtraction are only presented as number problems. Later, in Chapter 20, we will look at word problems built around all four operations.

The vocabulary used in this chapter will centre on the key concepts, that is, place value and crossing the places (another example of an alternate phrase, used here, is 'bridging the tens'). A variety of words have been used at different times in the history of maths in schools for the process of 'trading' 10 ones for 1 ten and vice-versa, as explained in the previous chapter. For example, changing 1 ten to 10 ones has been called 'decomposition' and 're-grouping' and 'renaming'. There was also a variation on this which was called 'borrow and pay-back'. Teachers should make clear which vocabulary they are going to use and model it with materials and/or visual images so that the learner understands. So, if the teacher says, 'I am going to decompose', it is best if the learner knows that this refers to a specific procedure in maths. It is worth bearing in mind that a previous teacher may have preferred, for example, 'rename'. The pupil needs to know that the teacher is referring to the same procedure, but is using a different word.

Addition is what is known as 'commutative'. This means that it does not matter in which order you add numbers, the total will be the same, for example, 721 + 52 = 52 + 721 = 773.

This is not true for subtraction, for example 600 – 10 does not give the same answer as 10 – 600.

Different word order can lead to confusion. For example, '70 take away 50' presents the 70 and 50 in the order for the symbols used in the calculation, 70 – 50. But 'Take 50 away from 70' does not.

Images, symbols and concepts

The key prerequisite concepts are place value and trading. As ever, maths is developmental. The role of zero may also require specific attention and instruction. As with any topic in maths, if the prerequisite concepts are not understood, then the grasp on the new topic may well be insecure.

Since trading and place value are key concepts it is likely that base 10 blocks and coins will create effective kinaesthetic and visual images.

It is useful to remember that addition and subtraction are complementary operations. Adding is about putting numbers together. Subtraction is about splitting up numbers.

$5 + 2 = 7$ $\quad\quad$ $7 - 2 = 5$

Algebra often expresses ideas succinctly, and it generalises and shows patterns, which makes it such a shame that people shy away from it. In this example:

Addition is:	$A + B = C$	e.g.	$25 + 63 = 88$
Subtraction is:	$C - B = A$	e.g.	$88 - 63 = 25$

The relevance of the topic/concept to developing maths skills

Adding and subtracting are important skills for life, particularly for money (though maybe less so these days with touch cards). They are also prerequisite skills for multiplication and division. Another maths example is calculating a mean or average which also requires accurate addition skills.

THE TASK

Written procedures for adding and subtracting usually follow the sequence of place value, that is, ones, tens, hundreds, thousands and onwards. The process moves from right to left on the page, small place values to large place values. This is opposite to our normal writing and reading direction. Early maths so often appears to be inconsistent.

The first examples shown below do not require any bridging (or crossing) of place values. The procedures can be modelled with base 10 blocks, but only symbols are used here.

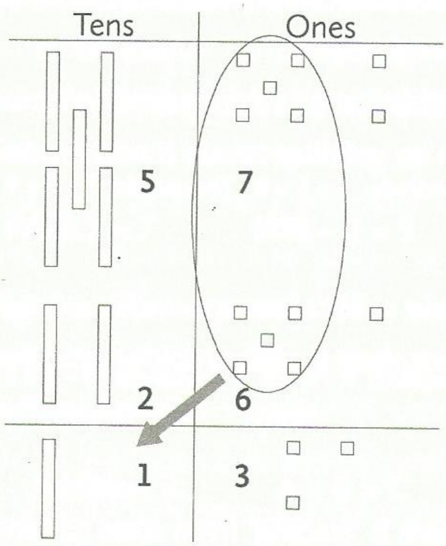
7432				
+2516	Ones: 2 + 6 = 8	Tens: 3 + 1 = 4	Hundreds: 4 + 5 = 9	Thousands: 7 + 2 = 9
9948	(8)	(40)	(900)	(9000)

9948				
-7432	Ones: 8 - 2 = 6	Tens: 4 - 3 = 1	Hundreds: 9 - 4 = 5	Thousands: 9 - 7 = 2
2516	(6)	(10)	(500)	(2000)

For the second group of examples, there is 'trading' from ones to tens for addition and from tens to ones for subtraction. The examples are modelled with images of base 10 blocks. Apart from the 'trading' these examples are the same as the first examples. The ones are added, then the tens are added.

57		57		
+26		+26	First add the ones:	7 + 6 = 13
	7 + 6 =	13		
	50 + 20 =	+70	Then add the tens:	50 + 20 = 70
		83	Now add the two sub-additions:	70 + 13 = 83

With base 10 blocks:

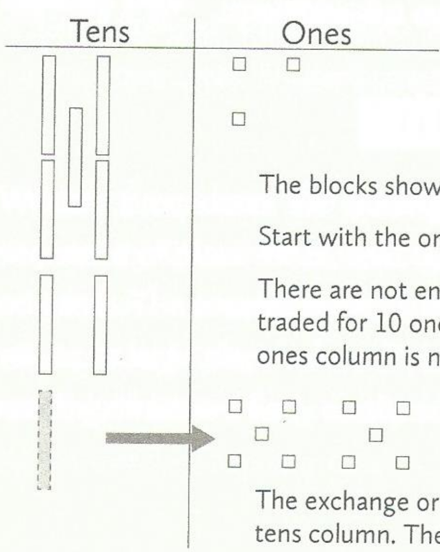


First add the ones: 7 + 6 = 13
Trade ten one cubes for a ten block

Now add the tens, that is 5 tens + 2 tens plus the 'traded' ten, 5 + 2 + 1 = 8 tens.

The final answer is 83.

Now look at subtraction. It is about reversing the process. (Note that some learners find reversing a process quite difficult, so the steps need to be presented comprehensively and clearly.) The 'trading' is illustrated with base 10 blocks.



83
−26

The blocks show 8 tens and 3 ones.

Start with the ones, 3 – 6

There are not enough ones for the subtraction, so, 1 ten is traded for 10 ones, making 13 units. The subtraction in the ones column is now ready 13 – 6 = 7

The exchange or trade of 1 ten for 10 ones leaves 7 tens in the tens column. The subtraction is then 7 – 2 = 5 (70 – 50)

The answer is 57, which, of course, tallies with the numbers used in the corresponding addition problem.

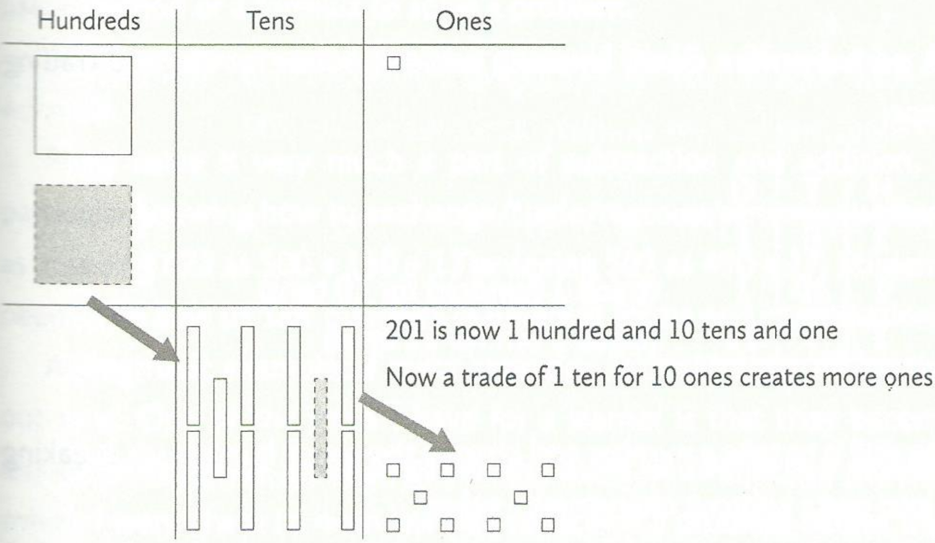
Subtracting when zero is involved

The subtraction problems that often cause most difficulties are those that involve a zero, for example 304 – 67. However, the logic remains the same...there will have to be some trading to create more ones for the subtraction in the ones column. The confusion arises because there are no tens in the tens column for use for trading; trading will need to start at the first available place, in this case, the hundreds column.

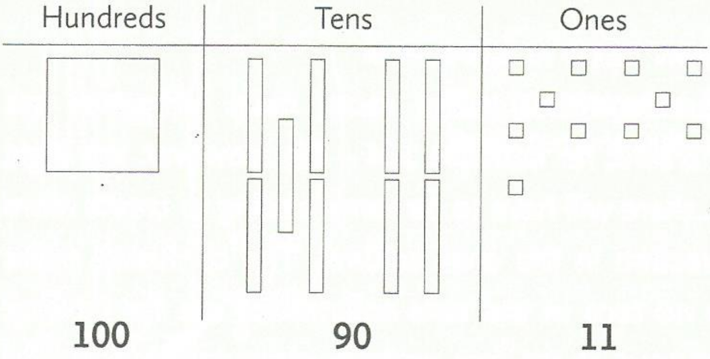
If this is understood as the objective, to get some ones into the ones column, then the logic is to use the knowledge of the place value system to do this.

The first practice could be confined to trading to get more ones:

For 201, trading makes it 1 hundred plus 9 tens, plus 11 ones = 100 + 90 + 11



The 2 hundreds and 1 one has become 1 hundred, 9 tens and 11 ones and the subtraction can begin:



$$\begin{array}{r} 100 \\ 90 \\ +11 \\ \hline 201 \end{array}$$

This procedure might be written without the 'scaffolding'/support of place value columns as:

9 11

1 10

2 0 1

Or, perhaps more clearly by using two extra lines for the two trading steps:

H	T	O	
2	0	1	
1	10	1	(trade 1 hundred for 10 tens)
1	9	11	(trade 1 ten for 10 ones)

The skill is to take a number and change the way it is made up, breaking it into parts without changing its value.

This skill can be used for trading in additions and subtractions, but remember it was also used for basic facts such as 7 + 7, which can be changed to 5 + 2 + 5 + 2 = 5 + 5 + 2 + 2 = 10 + 4 = 14.

Maths can progress by taking an idea or skill and extending it to new areas and problems. This is much better for memory and understanding than seeing each extension of an idea as something completely new, instead of a development of previous understanding.

Meta-cognition (thinking about how and what you are thinking)

As number sense develops, students can start to estimate answers before starting to compute and appraise answers after computation. The level of sophistication used for this can be refined as confidence grows. For example, for 57 + 26, the first attempt could be 50 + 20 = 70. The next attempt could be to consider the ones, which will add to make more than 10 (since both are more than 5), taking the estimate to 80.

Each problem should be overviewed to see if there are alternative and better ways to tackle it. As is stated above, it can be very helpful to understand how ideas develop and inter-relate: to know *why* you can use a method or a formula to solve a problem.

An alternative method for 57 + 26 could be 55 + 25 + 2 + 1 = 80 + 3 = 83.

Place value is very much a part of subtraction and addition, in particular knowing how to exchange/trade/regroup/rename numbers, as in seeing 673 as 500 + 160 + 13 or seeing 507 as 490 + 17. Students need to be able to 'see' numbers in different forms and combinations.

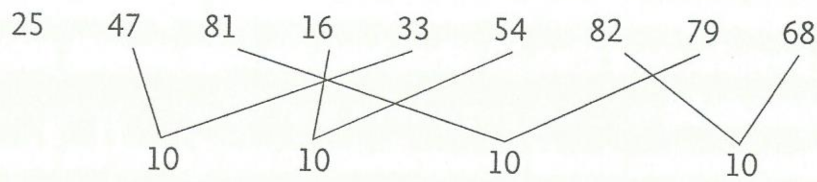
A pragmatic example of meta-cognition is the strategy of 'casting out tens'. One of the goals I have for a learner is that they become less of an impulsive problem solver and more of a reflective problem solver. I want them to overview and reflect on a problem before they attempt the solution. I want them to think about how they will think about the problem. 'Casting out tens' can be used an early experience of this strategy.

Here is an example. Add the numbers:

25 47 81 16 33 54 82 79 68

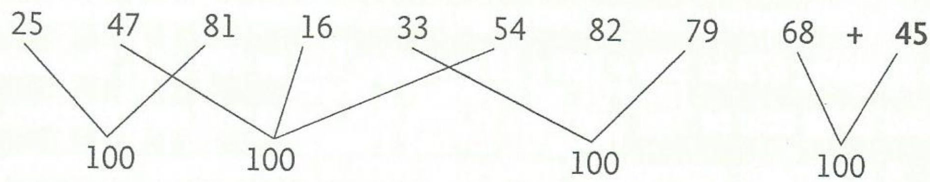
The addition could be done sequentially, adding the ones digits in the order they are given, then doing the same for the tens digits. (This could also be considered as an inchworm strategy.) This requires several addition steps, each of which may be a challenge (and a potential source of error) if the pupil is not secure in the retrieval of number facts.

If the numbers are overviewed, then it is usually possible to find number combinations (number bonds) for 10, for example:



Only the 5 (from 25) has not been paired. The total for the ones is 40 + 5 = **45**.

Now look at the tens digits (with the 45 from the ones total included) and find the number combinations for 100.



Only the 80 (from 82) has not been combined. The total for the tens is 400 + 80 = 480, making the total 485.

The cognitive strategy is to look at the numbers before starting to add them, to see if there is another way to add, using facts in an efficient and user-friendly way. In a sense this strategy is a variation on the chunking strategy.

A very basic estimation of the total can be obtained with an overview of the numbers. There are 9 two-digit numbers with a spread of values (not all near 100, not all near 10), so take an average number value of 50 and multiply it by 9 giving an estimate of 450. This can prevent 'big' errors such as the place value error, 45 + 44 = 89 (instead of 45 + 440).

Things to do and practise

Practise trading, crossing tens and hundreds with materials such as base 10 blocks or coins, but also with symbols (the digits and numbers). This practising need not involve actual adding or subtracting; just rename numbers, for example 72 as 60 + 12 (as the French language does anyway!).

Chapter 11

Basic Facts for Multiplication and Division

The times table facts are probably a child's first experience of persistent failure. Luckily this is not true for every child, but I suspect that it is a very significant percentage, particularly among those who are dyslexic. It should be noted that even when children do commit this body of facts to memory, it does not guarantee that they will understand the principles that underlie them, nor that they will remember them when they are older, when there is less practising and less recall.

It is also likely to be a child's first experience of the consequences of a maths belief, implicitly inferred or explicitly stated. That belief is that children, all children, can and should learn the times tables. A secondary belief is that maths will be very difficult if these facts are not retrievable, quickly, from memory.

These somewhat simplistic beliefs are not correct for every child and neither are they reasonable. The basis for my counter-beliefs is that there is an alternative and it is a mathematical alternative with cognitive benefits. And they are attainable. But the belief in the efficacy of 'learning by heart' is a very entrenched belief, often coupled with a sense of superiority.

For many people, the problematic facts lie in the bottom right-hand corner of the multiplication square. It seems like a small problem... unless you can't do it.

×	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											
9											
10											

Vocabulary and language

Once again, the vocabulary is not helpful in communicating the concept and once again there is no consistency. Let's start with, 'What is multiplication?' and the standard reply, 'Repeated addition'. Adding a column of different numbers could be interpreted as 'repeated addition'. In this specific case, multiplication, it means, 'Repeated addition of the same number'.

An example would be 6 + 6 + 6 + 6 + 6 + 6 + 6 = 7 × 6.

Then there is a range of vocabulary to infer multiplication, including, 'times', 'lots of', 'product' and nothing at all, as in 'What are six sevens?' That is a question where you need to know the maths (lack of) language code.

If the definition of multiplication involves 'repeated addition' then it would follow that the inverse operation, division, should be 'repeated subtraction of the same number'. The two definitions emphasise the link between the two operations.

There is a smaller range of vocabulary for division, including 'share' and 'How many...in...?', 'What is 25 over 5?', 'per' as in 'percentage... divided by 100' and nothing as in 3/5 or $\frac{3}{5}$

But there is, as with subtraction, an issue with the order of words. For example, 460 divided by 5 gives the numbers and the operation word in the correct order for keying into a calculator or for writing in symbols, 460 ÷ 5.

The order of the wording for 'How many tens in 500?' does not work for a calculator, but it might help for the 'bus stop' presentation:

$10\overline{)500}$

Images, symbols and concepts

The symbols used to infer multiplication and subtraction are more varied than the solitary + and – used for addition and subtraction. Again, consistency is not there to reassure the learner.

For multiplication we can use:

- × as in 5 × 7
- indices as in 8^2 which means 8 × 8
- brackets as in 5 (4 + 3) which means 5 × 4 + 5 × 3
- nothing at all, as in algebra, where ab means a × b.

For division we can use:

- $\div$ as in $10 \div 2$
- the 'bus stop' layout for the division procedure $2\overline{)10}$
- a negative index as in 25^{-2}
- a stroke as in 5/10
- a line with one number above another number as in $\frac{5}{7}$

A reminder: all four operations are interlinked, a concept that can be used constructively to provide alternative ways to solve problems.

The relevance of the topic/concept to developing maths skills

If the definition of multiplication is taken as repeated addition of the same number then aspects of algebra follow, for example:

$$2 + 2 + 2 + 2 = 4 \times 2$$
$$5 + 5 + 5 + 5 = 4 \times 5$$
$$8 + 8 + 8 + 8 = 4 \times 8$$
$$a + a + a + a = 4 \times a = 4a$$

If these repeated additions are collected into chunks, then the process of 'long' multiplication follows, a process that depends on partial products, for example:

$$7+7+7+7+7+7+7+7+7+7+7+7 = 10 \times 7 + 2 \times 7 = (10 + 2) \times 7$$

The links between multiplication and division are relevant in many areas of maths (see also the next chapter). For example, it helps to be able to relate these three equations and to understand their relationship:

$$6 \times 3 = 18$$

$$6 = 18 \div 3 \quad \text{or} \quad \frac{18}{3}$$

$$3 = 18 \div 6 \quad \text{or} \quad \frac{18}{6}$$

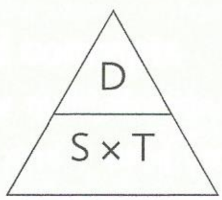
This is relevant to many equations/formulas such as:

$$\text{Distance} = \text{Speed} \times \text{Time} \qquad D = S \times T$$

$$\text{or} \quad \text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{or} \quad \text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

Such three component equations are sometimes presented in a triangle as an aide-memoire for the inter-relationships. It's good to be able to understand why.



(There is also a clue in the units used for speed 'miles per hour' or 'kilometres per hour'. Miles and kilometres are for distance and hour is for time, so speed is equal to distance per time, and 'per' means divide.)

Meta-cognition (thinking about how and what you are thinking)

Remembering the basic multiplication and division facts is beneficial, but knowing how they inter-relate – how facts can be combined to find new facts (often via partial products) – means that the benefits will be far greater.

In the equation above, S = D × T, it may help to relate it to a basic fact which is known comfortably and accurately, such as 2 × 5 = 10. The two other forms then follow and verify themselves:

$$5 = \frac{10}{2} \quad \text{and} \quad 2 = \frac{10}{5}$$

Children need to be taught how to understand the concepts of multiplication and division, how to think around the ways they can be used in other applications, such as percentages, area, algebra and fractions. They need to realise that in an equation like xy = constant, then as x gets bigger, y gets smaller. This can lead to understanding sequences such as:

$$\begin{aligned} 12 \div 12 &= 1 \\ 12 \div 6 &= 2 \\ 12 \div 3 &= 4 \\ 12 \div 2 &= 6 \\ 12 \div 1 &= 12 \\ 12 \div \tfrac{1}{2} &= 24 \end{aligned}$$

and thus, to the realisation that, in certain situations, division can make the answer bigger (than the divided number).

Although the culture and beliefs around maths seem to be that basic facts are always retrieved in one step, there is a strong case for arguing that two steps are a very viable and mathematically alternative way of thinking about some of these 'facts'. Meta-cognition comes from knowing how partial products work for multiplication and for division. This knowledge leads us to alternative ways to multiply and divide.

THE TASK

The task is to be able to access all the basic times and division facts, by recall or by strategy, as swiftly as possible (without raising anxiety).

One possibility is to use mnemonics. I confess that I am not a great fan of the extensive use of mnemonics. The odd one here and there, like 'Richard Of York Gave Battle In Vain' for the colours of the rainbow 'Red, Orange, Yellow, Green, Indigo, Violet' is great, but to create a whole book of them dedicated to mnemonics for times table facts is making *War and Peace* out of a nursery rhyme. However, if it works, it works. But remember that recall on its own isn't cognition.

A powerful rote-learning strategy is the self-voice echo strategy of Dr Colin Lane, where the facts or information to be learned are recorded, *in the learner's own voice* on a PC, with a matching visual on the screen. That fact is then repeated back time and again (well, it is rote learning), preferably through headphones. My own research into this (Lane and Chinn 1986) showed that when it works, it works dramatically and with long-lasting retention. But it doesn't work for everyone, which is a valuable lesson for anyone who thinks they may have found 'the' cure for learning difficulties.

A method based on meta-cognition relates back to some early maths concepts.

First: let's consider the definition and understanding of what this collection of facts is:

It is a collection of repeated additions, relating to the vocabulary, 'lots of'. For example

6 × 8 is	8 + 8 + 8 + 8 + 8 + 8	6 lots of 8	and
7 × 7 is	7 + 7 + 7 + 7 + 7 + 7 + 7	7 lots of 7	

Second: you can cluster these additions into chunks rather than add on one number at a time. The 'chunks' that help are likely to be the ones using the 'easy' numbers: 1, 2, 5 and 10. In terms of cognitive style, this will appeal more to grasshopper than inchworms.

The chunks, for example, for 6 × 8 are 5 × 8 and 1 × 8 which are 40 and 8. These are 'partial products'. (A product is the outcome of a multiplication. Partial products can be combined to make the complete product.) So, then the 40 and the 8 can be added to make 48, the product for 6 × 8.

$$(8 + 8 + 8 + 8 + 8) + 8$$

For 7 × 7 the process can also be done as two partial products:

$$(7 + 7 + 7 + 7 + 7) + (7 + 7)$$

$5 \times 7 = 35$ $\quad 2 \times 7 = 14$ $\quad 35 + 14 = 49$

Partial products can be combined by adding, or extended by multiplying, for example, 4 × 7 can be accessed via 2 lots of 2 × 7:

$$4 \times 7 = 2 \times 7 \times 2 = 2 \times 14 = 28$$

This strategy re-defines what makes a basic fact 'basic'. The **key** basic facts are the minimum number of facts that you need to work out other facts efficiently. They are the 1×, 2×, 5× and 10× facts for a number. For example, for the 8 times table, the core basic facts are:

$1 \times 8 = 8$
$2 \times 8 = 16$
$5 \times 8 = 40$
$10 \times 8 = 80$

These four partial products can be combined to access all the other 8× table facts, and some extra facts, too.

$1 \times 8 = 8$

$2 \times 8 = 16$

$3 \times 8 = 24$ $3 \times 8 = 2 \times 8 + 1 \times 8 = 16 + 8 = 24$

$4 \times 8 = 32$ $4 \times 8 = 2 \times 8 + 2 \times 8$ OR $2 \times (2 \times 8) = 16 + 16 = 32$

$5 \times 8 = 40$

$6 \times 8 = 48$ $6 \times 8 = 5 \times 8 + 1 \times 8 = 40 + 8 = 48$

$7 \times 8 = 56$ $7 \times 8 = 5 \times 8 + 2 \times 8 = 40 + 16 = 56$

$8 \times 8 = 64$ $8 \times 8 = 2 \times 2 \times 2 \times 8 = 2(2 \times 16) = 2 \times 32 = 64$

$9 \times 8 = 72$ $9 \times 8 = 10 \times 8 - 1 \times 8 = 80 - 8 = 72$

$10 \times 8 = 80$

Extra facts:

$11 \times 8 = 88$ $11 \times 8 = 10 \times 8 + 1 \times 8 = 80 + 8 = 88$

$12 \times 8 = 96$ $12 \times 8 = 10 \times 8 + 2 \times 8 = 80 + 16 = 96$

$15 \times 8 = 120$ $15 \times 8 = 10 \times 8 + 5 \times 8 = 80 + 40 = 120$

$19 \times 8 = 152$ $19 \times 8 = 20 \times 8 - 1 \times 8 = 2 \times 10 \times 8 - 1 \times 8 = 160 - 8 = 152$

The principle that underpins this strategy is the principle that underpins 'long' multiplication.

Division facts are the complementary or reverse facts to the multiplication facts, for example, $6 \times 8 = 48$ is a multiplication fact. Two division facts are complementary, for example, $48 \div 6 = 8$ and $48 \div 8 = 6$. These two division facts could be written as $6 \times ? = 48$ and $8 \times ? = 48$, thus interlinking division and multiplication. If the multiplication facts are known, then the ability to visualise them in this form will give the division facts.

The answers to division questions and facts can be accessed from the key basic facts. Where multiplication is repeated addition of the same numbers, division is repeated subtraction of the same numbers. For example, to access $72 \div 8$, subtract partial products:

The two partial products that will be used in this example are:

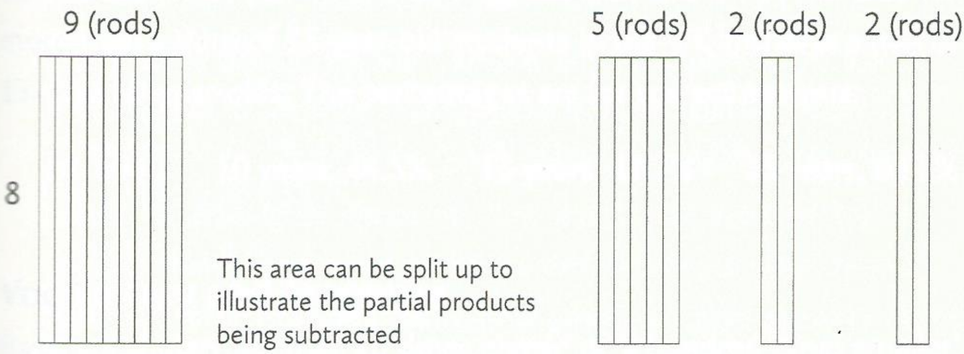
$2 \times 8 = 16$ and $5 \times 8 = 40$

Start by subtracting 40	(**5** × 8)	72 − 40 = 32
then 16	(**2** × 8)	32 − 16 = 16
and 16	(**2** × 8)	16 − 16 = 0

9 lots of 8 have been subtracted so $72 \div 8 = 9$

Partial products can be demonstrated with Cuisenaire rods. Take the examples, $9 \times 8 = 72$ and $72 \div 8 = 9$.

First, for $72 \div 8$ use 9 'eight rods'. (The area this creates represents the 72.)



This image relates to the written presentation for division:

$$8 \overline{)72}^{\,9}$$

To demonstrate 9 × 8, the rods can be put together as three partial products, 5 × 8 and 2 × 8 and 2 × 8.

'5 lots of 8, plus 2 lots of 8 plus 2 lots of 8.'

Things to do and practise

Practise and learn the key basic facts described above. Practise combining them to make other facts.

Write the key facts on cards and play memory games with them, for example, after spreading a number of (8× or another number) cards face down on a table ask questions such as, 'Find two cards that can make 56.' (It will be a 5 × 8 card and a 2 × 8 card.)

Use the rods to demonstrate how areas are constructed and split up (de-constructed) using partial products.

Chapter 13

Multiplication and Division

Once the problem of the basic facts has been addressed, the next step is to be able to compute longer/harder problems for multiplication and division. The data collected to create the norm-referenced maths test in my book on diagnosing maths difficulties (Chinn 2017b) revealed low levels of performance in these tasks. For example, the percentages from this large (1783) sample of UK school children who achieved the *correct* answers for the questions were very low:

For $9\overline{)927}$

10 years, 14.5%   13 years, 31.4%   15 years, 46.8%

For $\begin{array}{r} 541 \\ \times 203 \\ \hline \end{array}$

10 years, 14.1%   13 years, 15.2%   15 years, 38.2%

Poor abilities with such questions obviously stretch beyond the special needs population. *This could suggest that methods that are introduced for pupils with specific learning difficulties will help many more children as well.*

An example of the developmental nature of maths is that the strategies used to extend the key basic facts to other basic facts were based on partial products and these are now used in long multiplication. This concept will be developed further in this chapter.

The four operations are interlinked, a concept that can be used constructively to provide alternative ways to solve problems. For example, it is possible to obtain an answer to a subtraction problem by adding on. For example, to compute 100 – 63, start with 63 and add 7 to make 70, then add 30 to make 100. 30 + 7 have been added, so the answer is 37.

It is possible to divide by thinking of multiplication. For example, 42 ÷ 7 = ? could be written in a different order as ? × 7 = 42, becoming, 'What do I multiply 7 by to get 42?', but to do this there must be an understanding of that link between multiplication and division. Another example of this link is that 'long' multiplication is carried out by adding together partial products and 'long' division is carried out by subtracting partial products.

Vocabulary and language

Multiplication, like addition, is commutative. That means that the order in which you multiply numbers does not change the final answer, for example:

$6 \times 5 = 5 \times 6$ or $7 \times 8 \times 9 = 9 \times 8 \times 7$

This is not true for division, where the sequence of numbers and symbols affects the answer, for example:

$30 \div 6 = 5$ but $30 \div 5 = 6$ and $30 \div 6 = 5$

30 ÷ 6 can be stated in words as 'thirty divided by six' where the order of the words matches the symbols. It could also be stated as, 'How many sixes in thirty?' where the order of the numbers is now reversed. There may also be an interpretation issue. 'How many sixes in thirty?' gives no clue as to the process that might lead to an answer. The learner needs to know the mathematical meaning of the vocabulary.

One of the fascinating errors for division I have encountered is:

$$\begin{array}{r} 01 \\ 5\overline{)35} \end{array} \qquad \begin{array}{r} 02 \\ 4\overline{)28} \end{array}$$

A guess as to the method used is, 'How many fives in 3? There aren't any, so 0. How many fives in 5? There's 1.' And 'How many fours in 2? There aren't any, so 0. How many fours in 8? 2.' Many of the errors encountered in the standardising data were probably down to part remembered procedures or mis-applications.

Images, symbols and concepts

There are some concepts and skills that are prerequisites for these tasks and this might be one reason for their apparent difficulty. The prerequisites have been forgotten. When a maths procedure becomes more complex and demands more steps, it obviously becomes more likely to create failure. If the learner is relying entirely on memorising the steps without any understanding, then only perfect recall will lead to accuracy. Just one mistake in that recall is all it takes to fail.

The key concepts (and prerequisite skills) are place value and the ability to understand and carry out multiplication and division by powers of 10. The methods for multiplication inevitably require an ability to add and subtract accurately. An ability to access the key basic facts is required as is an ability to organise work on the page. For the organisation problem it may be beneficial to provide (appropriately sized) squared paper for some pupils. The use, and understanding, of partial products which was introduced for accessing some basic facts is also a prerequisite skill.

The visual image that fits these procedures and the concepts of multiplication and division is a rectangle (or square). It is area that illustrates the steps and the outcome. The area could be a simple drawing, or base 10 blocks or square counters or Cuisenaire rods.

The relevance of the topic/concept to developing maths skills

Multiplication and division are used widely in topics in maths. For example, the basic relationships of A × B = C and A = C ÷ B are used for topics such as:

- area = height × width
- sine = opposite ÷ hypotenuse
- force = mass × acceleration
- euros = pounds × exchange rate
- speed = distance ÷ time
- pounds = 2.2 × kilograms
- circumference = π × d.

THE TASK

As stated above, the analysis of the results used to standardise my 15-minute maths screener test suggests that multiplication and division skills are not well developed in the UK. One hypothesis is that the methods advocated are too reliant on recall and application of procedures which are not backed by understanding. Methods that make heavy demands on memory and do not offer a rationale for 'Why am I doing this?' do not suit the needs of many learners who experience learning difficulties. And we need to remember that meta-cognition is not the exclusive preserve of the more able and thus not patronise our learners.

Multiplication and division are essentially about partial products. If a multiplication or a division is perceived as being too difficult to compute in one step, then two or more steps can be used. The steps use partial products. For example, for 67 × 21:

Step 1	67 × **1** =	67	(67 is the first partial product)
Step 2:	67 × **20** =	1340	(1340 is the second partial product)
Step 3: Add the partial products		1407	

Division follows the same pattern, but with subtraction, for example, 1407 ÷ 21

Step 1:	Subtract the partial product, **20** x 67	1407
		−1340
		67
Step 2:	Subtract the partial product, **1** x 67	− 67
		0
Step 3:	Add **20** + **1** to give 21 as the answer	

Meta-cognition (thinking about how and what you are thinking)

The concept used for basic facts is developed further for more complex multiplication and division problems. Adding and subtracting are also still present and so, if those skills are insecure, then multiplication and division will be insecure. Understanding is so important for securing the method in long-term memory. Learning how to think about the procedures and understand them, rather than relying solely on memory, is the meta-cognition. Being flexible and responsive to each different problem is also meta-cognition. The inter-relationship between multiplication and division enables students to move from the classic physics equation, F = ma, to a = F/m and to think about the answer rather than simply writing down a number. For example, thinking about a = F/m will tell the learner that a bigger mass (m) will experience a smaller acceleration (a) for a given force (F).

The link between multiplication and division could be visualised in the layout used for the traditional procedure for division. The sides of the frame

could be interpreted as two sides of a rectangle. So, for an area problem:

71		71		width
6	426	6	426	height	area

The three versions of the relationship are:

area	= width × height	426 = 71 × 6
width	= area ÷ height	71 = 426 ÷ 6
height	= area ÷ width	6 = 426 ÷ 71

A further opportunity for meta-cognition is the breaking down and building up of numbers, for example, by using place value, 68 can be interpreted as 60 + 8, but by using key basic facts, 68 can, alternatively, be seen as 70 – 2. The first 'breakdown' results in two partial products, but the student will have to recall 6× and 8× multiplication facts. The second 'breakdown' results in more partial products, so 68 = 50 + 20 – 2, but all are the product of a key basic multiplication fact and more likely to be retrieved. The partial products can also be used as estimates and thus reduce the chance of 'big' errors. Examples of both strategies are given below:

716 × 68

Using place value to create partial products:

$$\begin{array}{r} 716 \\ \times\ 68 \\ \hline \end{array}$$

5728	(716 × 8)	
42960	(716 × 60)	This step is often done without the correct place value.
48688		

Using key numbers to create partial products:

716		
× 68		
35800	(50 × 716)	This answer can be used as a very approximate estimate.
14320	(20 × 716)	
50120	(70 × 716)	This answer can be used as a closer estimate.

Now we use 68 = 70 – 2 and subtract 2 × 716

50120	(70 × 716)
–1432	(2 × 716)
48688	

There are more steps, but the method circumvents the barrier that occurs when 'harder' multiplication facts cannot be retrieved. This example also provides a very rough estimate (50 × 716 = 35800) and a close estimate (70 × 716 = 50120). The method uses the same concept of partial products as the 'traditional' method. It is about making a procedure accessible, rather than shorter. And for an extra check, some partial products are related by place value. In this example 2 × 716 = 1432 and 20 × 716 = 14320.

The same method can be applied to division. Division is likely to cause more difficulty and often will generate the 'no attempt' strategy. Consider the first step in the traditional method:

$68 \overline{)48688}$

'How many 68s in 486?' That is challenging for any student, but for those with a poor retrieval of basic facts, even more so.

The method used for multiplication, based on partial products from key basic facts, can be adapted to circumvent that 'getting started' barrier:

Step 1. Set up a table for the partial products of 68:

$1 \times 68 = 68$
$10 \times 68 = 680$
$100 \times 68 = 6800$
$2 \times 68 = 136$
$20 \times 68 = 1360$
$200 \times 68 = 13600$
$5 \times 68 = 340$ (for a check, compare to 10 × 68...it should be half of 680)
$50 \times 68 = 3400$
$500 \times 68 = 34000$

(Setting up such tables is good revision for inter-relating partial products, such as 2 × 68 and 20 × 68 or 5 × 68 and 50 × 68.)

Step 2. Subtract the partial products of 68 from 48688:

48688	
− 34000	**500** × 68
14688	
− 13600	**200** × 68
1088	
− 680	**10** × 68
408	
− 340	**5** × 68
68	
− 68	**1** × 68
716 × 68	

Again, the procedure is longer, but it may make an 'impossible' task into a possible one and there are built in checks by inter-relating the partial products.

Things to do and practise

Practise breaking down numbers by place value and by key numbers.

Practise doing long multiplications by both methods to investigate which works best for the learner and the particular example they are working on.

Practise multiplication involving powers of 10, for example × 5, × 50, × 500, × 5000. Compare the answers. Look for patterns.

Try pre-calculation estimates and post-calculation appraisals.

Looking at prerequisites and their impact on learners

As maths topics develop, they demand more prerequisite skills. For example, it is difficult to understand number bonds that create a two-digit answer without understanding place value. 'Long' multiplication, or any multiplication where partial products are used, requires the ability to add. Consequently, division requires an ability to subtract.

Multiplying and dividing by 10 and powers of 10 requires an understanding of place value. Later, multiplying by a fraction that is less than 1 challenges the belief, based on previous experience, that multiplying always results in a bigger answer.

Check if the prerequisite concepts and skills are there before embarking on teaching a new topic.

Intervention is often about knowing how far back to go to before you begin. That 'going back' will often be much further back than you might initially think. But 'going back' may also be brief, just enough to refresh memory.

Note: The 'grid method' was advocated for multiplication some years ago in England. The evidence from my standardisation data is that it is not working for too many pupils, but, see the next chapter for an illustration and explanation of the grid method. It could be another example of a blind application of a procedure resulting in errors.

Chapter 14

The Development of Multiplication

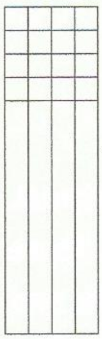
The Area Model

One of the themes in this book is that mathematics is developmental. The concepts develop, the facts develop and the skills develop. There are consequences. New topics often have prerequisite skills and concepts that need to be understood first. This leads to a compelling case for tracking back when planning an intervention: tracking back to find insecurities and tracking back to find the point where the learner is secure. The gaps in learning for students may often be in fundamental concepts. Place value is a frequent example of this, especially when zeros are involved.

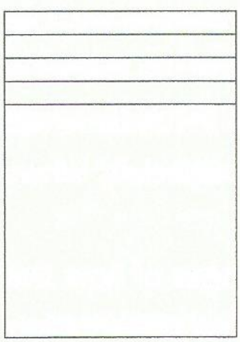
The sequence of diagrams below acts as visual images of how the concept of multiplication develops. It is based on the visual presentation of multiplication as area. The diagrams represent base 10 blocks.

Stage 1. 1 × 14. As 1 ten and 4 ones.

Stage 2. 2 × 14. As 2 × 10 plus 2 × 4
= 20 + 8 = 28

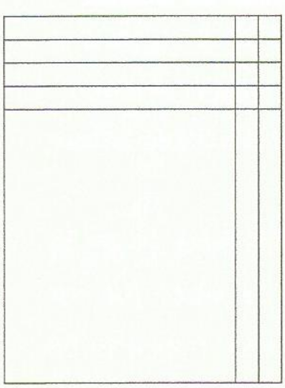


Stage 3. 4 × 14. As 4 × 10 plus 4 × 4 OR
as 2 × 2 × 14 = 56
(The area is twice that of 2 × 14)



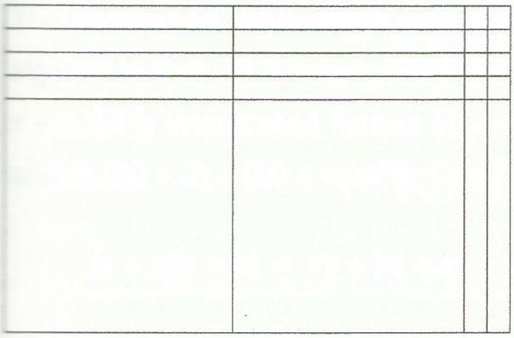
Stage 4. 10 × 14. As 10 × 10 plus 10 × 4
= 100 + 40 = 140

Another illustration of place value and ×10.
The 4 ones become 4 tens and
the ten becomes 1 hundred.

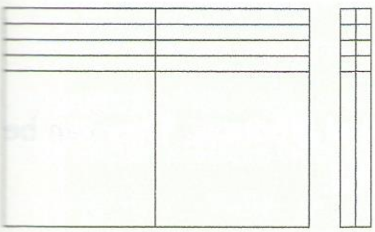


Stage 5. 12 × 14. As 10 × 14 plus 2 × 14
= 140 + 28 = 168

This is an example of partial products.
This example relates back to strategies used for basic facts, such as 7 × 6 = 5 × 6 + 2 × 6.
It relates onwards to 'bigger' multiplication problems.



Stage 6. 22 × 14 = 308



Stage 6a. 22 × 14
The image shows 20 × 14 plus 2 × 14.
20 × 14 is 10 × bigger than 2 × 14.
Also, it is 2 × bigger than 10 × 14.
Again, it is about place value and inter-relating numbers.

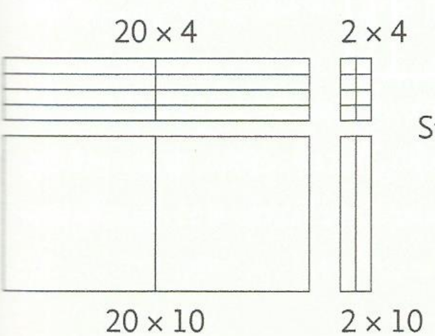
(The area model supports discussions about relative number values. These discussions should be encouraged.)

The next visual image shows how the 'grid method' works.

The multiplication 22 × 14 is broken down into 4 partial products as illustrated by 4 areas above.

The 4 areas are

20 × 10 = 200
20 × 4 = 80
2 × 10 = 20
2 × 4 = 8
308



Stage 6b. 22 × 14 is shown as
20 × 4 plus 2 × 4
plus 20 × 10 plus 2 × 10

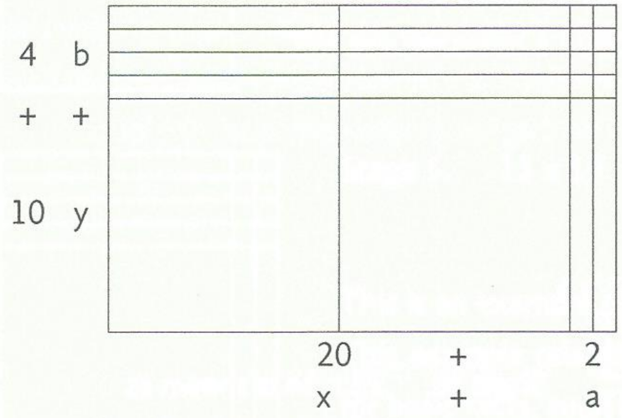
This leads to the grid method which uses digits only.

	20	2	
10	**200**	**20**	220
4	**80**	**8**	88
	280	28	308

The four numbers in bold are the four partial products. They can be added across or down to give the answer, 308.

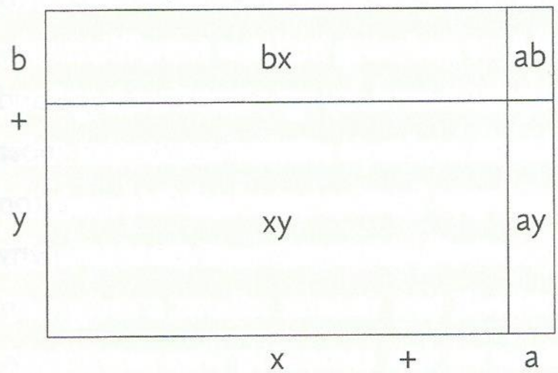
Stage 7. Algebra

Algebra follows the same rules as used for numbers. Algebra often acts as a generalisation for all numbers. This step to the next stage may seem to some to be a challenging one, but it is closely linked to and developed from the stages that preceded it. Compare it to stages 6a and 6b above.



The illustration is the same as used for 22 × 14. The connection to algebra is that 22 is interpreted first as 20 + 2 and then in letters as x + a. 14 is interpreted first as 10 + 4 and then in letters as y + b. The area is (20 + 2) × (10 + 4) or (x + a)(y + b). Letters replace numbers.

$$(x + a)(y + b) = xy + bx + ay + ab$$



bx ab xy and ay and bx and ab are all partial products.

This visual is a good way of seeing how to work out (x + a)(y + b) = xy + bx + ay + ab) rather than as a mnemonic such as FOIL which tells to multiply the First (xy), the Outers (ab), the Inners (ay) and the Last (ab) letters.

Chapter 15

Fractions

In another of my surveys of teachers, and, by now another large and international sample, I ask which topics in maths cause the most problems for pupils. Not surprisingly, fractions are included as a top topic for causing difficulties. There will be a reason – or reasons – why this is such a universal problem.

Vocabulary and language

The fractions that are met most frequently, which are also the fractions that are used to introduce the topic, have unhelpful vocabulary. There are inconsistencies. The most frequently encountered fraction, a half, is a vocabulary exception to the naming rule for 1/5, 1/6, 1/7 onwards and in the way that the word 'twelve' gives no information about how to write the number, the word 'half' does not give any information about how to write the fraction.

'Third' and 'quarter' are also outside the pattern. The alternative for 'quarter', that is 'fourth', does fit the subsequent pattern. Another confusion may come from the use of all the words in the pattern fifth, sixth, seventh...to denote position in an order of things. (For example, 'I came fifth in that race.')

When multiplying by fractions, children meet the word 'of' which is used here to mean multiply. In everyday life, for example, when we say, 'Can I have three of those sweets, please?' 'of' does get interpreted as 'multiply'.

Language can be helpful in some examples of work with fractions. 'One fifth plus one fifth' is logically 'two fifths' and this helps when explaining how to add fractions that have the same name (or 'denominator'). One fifth plus one sixth is, linguistically, not possible to answer. This may support the argument that fractions sometimes have to be 're-named', the word we used for subtraction and addition, but with a different method attached to it for fractions.

Fractions challenge the consistency of previous experience of what happens when the operations multiply and divide are used. The outcomes of examples such as $\frac{1}{4} \times 8$ and $12 \div \frac{1}{2}$ are the opposite to the whole number examples previously encountered. Multiplication now gives a smaller answer and division a bigger answer. One of the ways these new outcomes can be explained is by using appropriate vocabulary and language. For division, the vocabulary that supports the concept is, 'How many halves in 12?' and 'How many halves make 12?' (an example of repeated addition), rather than 'What is 12 divided by a half?' For multiplication, 'What is a quarter of 8?' rather than, 'What is 8 times a quarter?' or 'What is 8 divided by 4?'

Images, symbols and concepts

Images used to teach fractions in schools often include pizzas. In fact, I wonder why a whole generation of children has not been put off eating pizza. Luckily most parents are unlikely to make this treat a test of skill with fractions: 'If Dave has 2/7 of the pizza, Lisa has 1/5 and Callum has 4/9, how much is left for Dad?'

Cuisenaire rods can be a useful manipulative for introducing and demonstrating fractions, but squares of plain paper, for folding or cutting up and thus dividing, can also provide a good demonstration of how fractions work. If a pedantic approach is taken to the topic, then pizzas (and cakes and apples) are not an acceptable image for fractions.

Fractions must be precise. Dividing a pizza in three pieces and for those pieces to be precisely equal is close to impossible, especially if there are lots of toppings. The same caution applies to half an apple.

Symbolic, abstract representation of information is often more of a problem for dyslexic and dyscalculic pupils so we need to use visual images or kinaesthetic experiences alongside the symbols. The kinaesthetic bit might make pizzas seem a good idea, but this is not mathematically accurate.

Fractions will challenge the learner's need for consistency and the security of knowing about multi-digit numbers and place value. There are other inconsistencies in work with fractions. For example, the question, 'What is one fifth plus one fifth?' has the logical (and correct) answer of 'Two fifths'. If this same problem is presented in symbols, the answer that is often given is two tenths, which is incorrect:

$$\frac{1}{5} + \frac{1}{5} = \frac{2}{10} = \text{wrong!}$$

There are some new rules to learn before fractions can be added. The logic of the + sign, to a non-mathematician, should be that it applies to both the top and the bottom numbers in the fraction. This is not the correct maths logic. However, seemingly inconsistently, the logic returns for multiplication where the × sign operates on top and bottom numbers:

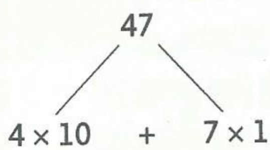
$$\frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$$

These inconsistencies are a serious barrier for many children. For a fraction item in my 15 minutes maths test (Chinn 2017b):

$$\frac{2}{5} + \frac{3}{8}$$

the percentage of *correct* answers for 13- and 14-year-old students were respectively 24.6 per cent and 39.1 per cent. It is sometimes very interesting (and often depressing) to know how the general population of students perform in maths.

The problem with the symbol code for fractions is that it hides a division sign. Ordinary two-digit numbers could be viewed as hiding a multiplication sign and an addition sign, for example:



Fractions hide the division sign with the line that separates the two numbers (top and bottom).

Of course, a division has a different outcome to a multiplication, which is why fractions obey different rules, one of which is that the fractions in the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$$

are getting smaller, but if the pupil focuses on the digits 1, 2, 3, 4, 5, previous experience has that sequence as getting bigger. Consistency is challenged unless the fractions are explained so that this misconception is addressed.

Adding fractions is more complicated than adding whole numbers or decimals, but there are some ideas in the next section.

The relevance of the topic/concept to developing maths skills

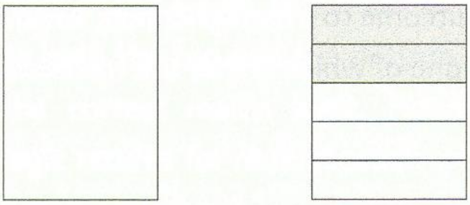
It is quite hard to make a strong case for the relevance of fractions in everyday life other than those in common usage: half, quarter, third and tenths. However, there is some transfer to the concept of proportion.

Meta-cognition (thinking about how and what you are thinking)

It is vital to think about the difference in the concept of a fraction to the concept of a whole number. The two digits or numbers involved in a fraction have different roles. The bottom number is about how many parts the whole has been divided into, for example for

$$\frac{2}{5}$$

the bottom digit tells us that the whole has been divided into 5 equal parts and the top digit tells us that we have 2 of them.



Another difference is in the idea of 'equivalent' fractions. A fraction can be expressed in more than one combination of numbers. The most common example is the half, 1/2, 1 out of 2. We meet a half as:

a half of an hour is 30 out of 60 minutes 30/60

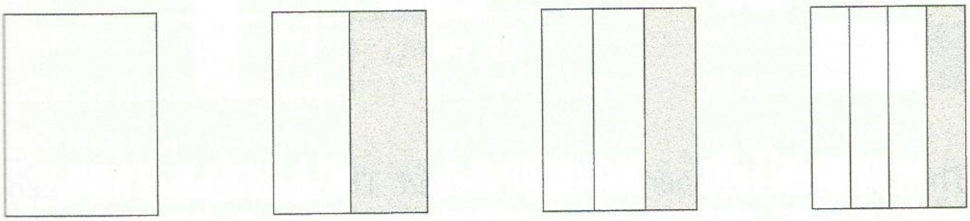
half a £1 is 50 pence out of 100 pence, 50/100

half a kilometre is 500 metres out of 1000 metres, 500/1000 half a day is 12 hours out of 24 hours, 12/24

half a year is 6 months out of 12 months, 6/12.

Each of these fractions share a common characteristic. The top number is half of the bottom number, hence all these fractions are a half.

A simple, low cost manipulative for illustrating concepts and procedures with fractions is a sheet of paper. By folding the paper, fractions are created:



1 $\quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4}$

The images are simple (and not cluttered by pizza toppings). Discussions can introduce the vocabulary and concept of fractions, for example, 'This is 1 half. There are 2 halves made from the whole sheet of paper. There are 4 quarters. To make a quarter the half is halved again (divided by 2 again). 2 quarters can be added to make a half. A quarter and a third cannot be added (yet) as they are different sizes. They are not the same.'

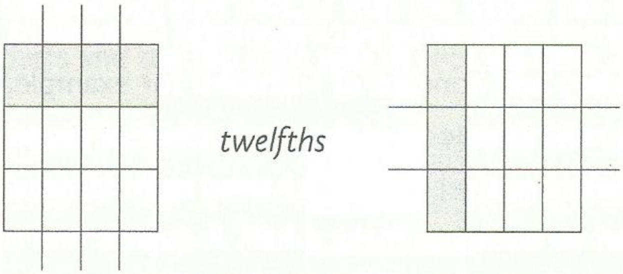
The model (the paper sheet) has the main characteristic required of a material/manipulative, in that it directly relates to the concept, the symbols and the procedures, plus it is not a cluttered image.

The procedure for adding fractions that have different names, different denominators, is to rename them so that they do have the same name. In the case of 1/3 + 1/4 this requires both fractions to be renamed so that they do have the same name. It is about creating two equivalent fractions before the adding (or subtracting) can take place.

Equivalent means that the new fraction keeps the same value, but that the denominator and numerator are different to those in the original fraction, for example, 1/2 and 50/100.

Paper-folding can be used to model the procedure. The demonstration shows that, when both fractions have been changed to have the same name, it is possible to add them.

The first piece of paper has been 'thirded'. The second piece has been 'quartered'. So, that means that to make the two papers the same, the thirded paper is quartered and the quartered paper is thirded.

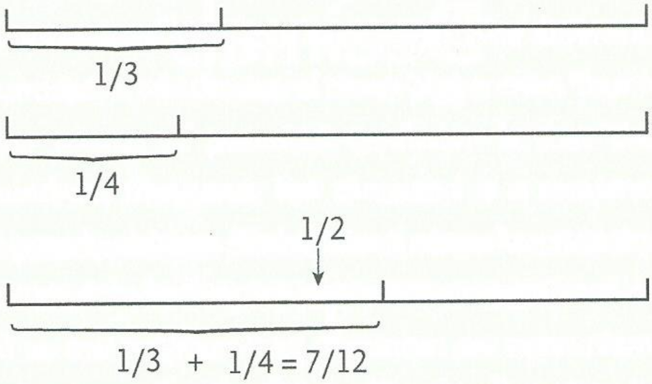


Both whole papers are now showing 12/12. The 1/3 has become 4/12 and the ¼ has become 3/12. The two fractions now have the same name (twelfths) and can be added:

$$\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

The explanation that should accompany this demonstration has to link the visual images to the symbols and steps in the standard procedure.

Empty number lines can be used to estimate answers, for example, 1/3 + 1/4:



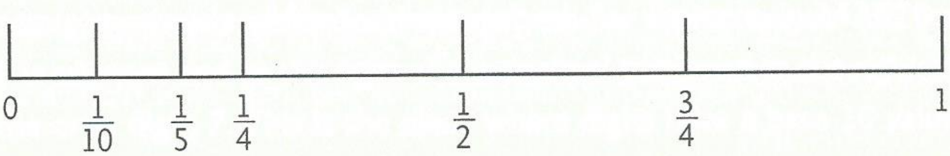
The answer is about a half, but it is more than a half. The estimate is used to prevent (or reveal) big mistakes. In this example, half is a useful comparison value. So, the estimate here would be 'about a half' or 'just more than a half'.

Things to do and practise

Practise renaming fractions to make equivalent fractions. Use the vocabulary alongside the symbols for mutual clarification. Perhaps use, for example, 'two fifths' alongside 'two out of five'.

Fold paper squares and relate the outcomes to the mathematical operations and symbols and write the fractions on the squares.

Use a number line (as below) for estimating answers. This skill is considered by a large USA research project to be a key indication that fractions are understood. Linking fractions to decimals can further support this skill.



Chapter 16

Decimals

Our early experience of maths is with whole numbers, but as we grow up we meet part-numbers, numbers that are part of a whole, for example, a half. The main ways we represent part-numbers are as fractions, as decimals and as percentages. This chapter is about decimals and the way they can be understood as a development of place value.

The most common example of decimals that we meet is money, specifically pence. When we write a price, for example £16.85, the .85 is a decimal which is also, in this example, 85 pence. The 85 pence is a part (85/100) of a pound. It is not a whole pound.

In the USA, the equivalent of our pence is a cent. The word cent gives the clue as to its value. 'Cent' infers 100. There are 100 cents in a US dollar. The word 'pence' doesn't give us that clue, but there are 100 pence in a pound. This can also be understood as: 1 pence is one hundredth of a pound.

Vocabulary and language

Decimals are actually 'decimal fractions'. This means that they are fractions which have denominators (bottom numbers) that are specifically, 10, 100, 1000 and so on. So, decimals are about tenths, hundredths, thousandths and so on.

The sequence of words, 'tenth, hundredth, thousandth, ten thousandth...' can cause confusion.

There are three ways that this confusion can arise. First, the 'th' at the end of each of those words does not make much sound when you say the words. It can be quite tricky to hear the difference in sound between 'four hundred' and 'four hundredth'. The difference in value is that 'four hundred' is ten thousand times bigger than 'four hundredths'.

The other source of potential confusion is that the sequence, one, ten, hundred, thousand, ten thousand is a sequence that gets ten times *bigger* each step. The decimal sequence, tenth, hundredth, thousandth, ten thousandth is a sequence that gets ten times *smaller* each step.

The third source of potential confusion is that there is no 'oneth' (see below).

When we read decimals, such as 147.9, we say them as 'one hundred and forty-seven *point* nine'. The word 'point' warns the listener that the next digit to be spoken will represent a decimal. Maybe this is one of the contributors to the 'decimal point' (to give it its full name) being a focus of attention.

There has often been a strong focus in maths books on the 'decimal point'. This focus can be misleading conceptually.

Images, symbols and concepts

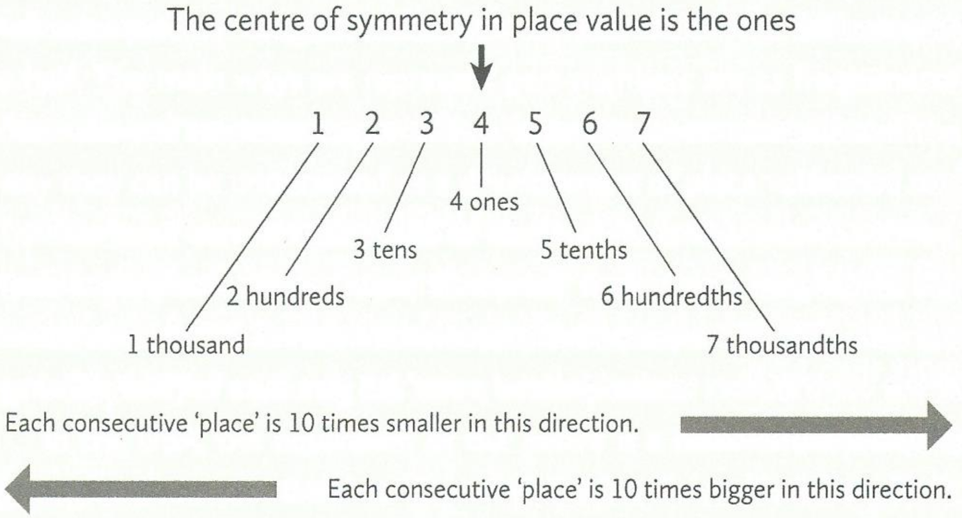
The reason that a focus on the decimal point has the potential for confusion is that the decimal point is not the centre of the symmetry between whole and part numbers. The focus is the ones. To understand the concept of decimals the focus of instruction should be on the ones.

Decimals are an extension of the concept of place value and base 10. The place a digit holds in a number conveys its value, both for whole numbers and for decimal numbers. For example, in 147.9, the 9 is adjacent and to the right of the decimal point and to the right of the ones digit, 7. This is the place where digits have a base 10 value of tenths.

If there was another digit included on the decimal side of the decimal point, for example 147.96, then the 6 is in the one hundredths place. The 6 represents 6/100.

This is the case with money, for example with £16.45, the 4 represent 4 tenths (four 10 pence coins) of £1 and the 5 represents 5 hundredths (five 1 pence coins) of £1.

The place values can be seen in this example:



These rules apply across all the digits and that includes both sides of the decimal point.

One image that might be used to illustrate this place value concept is a metre rule. If one metre represents one, then ten centimetres (a decimetre) represent a tenth (1/10), one centimetre represents a hundredth (1/100) and one millimetre represents one thousandth (1/1000).

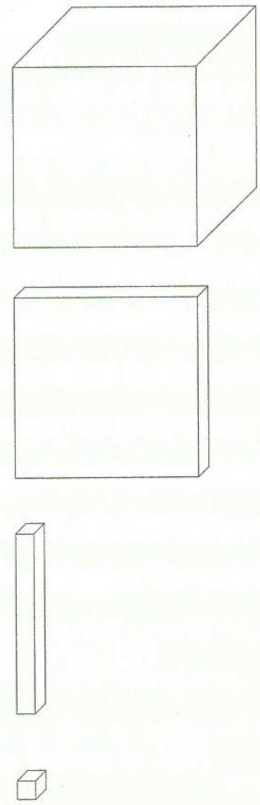
Another image could be created with the base 10 blocks. If the big cube represents 1 (instead of its normal role of representing 1000), then the 'flat square' represents one tenth (because 10 of them make 1), the 'long' block represents one hundredth and the small cube represents one thousandth.

One 1

One tenth $\frac{1}{10}$ 0.1

One hundredth $\frac{1}{100}$ 0.01

One thousandth $\frac{1}{1000}$ 0.001



The relevance of the topic/concept to developing maths skills

Decimals are primarily about base 10 and about place value for quantities less than 1. They can be used to express any value less than 1. They can be linked to both fractions and percentages. They are involved in many aspects of everyday life, from money to measuring.

Decimal fractions are much easier to compare (for value) than fractions.

THE TASK

The task here is to be able to use decimal numbers when calculating with any of the four operations.

Addition and subtraction

The procedure and concept are the same as with whole numbers. The numbers to be added or the numbers to be subtracted must be lined up vertically according to place value. For example, to add 1.02 + 13.4 + 7.983 + 0.004 rewrite the numbers with the corresponding place values above each other:

$$\begin{array}{r} 1.02 \\ 13.4 \\ 7.983 \\ +\ 0.004 \\ \hline \end{array}$$

Note that the decimal points also line up (as they must if you line up place values).

The trading or renaming, if required, follows the same principle as with whole numbers, for example, 1 can be exchanged for 10 tenths, 1 tenth for 10 hundredths (and vice-versa).

Multiplication and division by decimal numbers

This requires the link between a decimal when written as 0.4 and as a fraction 4/10 to remind the student that the 4 in 0.4 is 4 ÷ 10. The decimal disguises the ÷ sign.

So, for multiplication by a decimal number that is less than 1, for example 52 × 0.4 is really two operations:

$52 \times 4 = 208$ and $208 \div 10 = 20.8$

And 52 × 0.04 is also two operations, one of them a division because 0.04 = 4/100 = 4 ÷ 100:

52 × 4 = 208 and 208 ÷ 100 = 2.08

Dividing by a decimal number that is less than 1 also requires you to think about the concept of decimals.

For example, dividing by 5, 0.5 and 0.05, is dividing by 5, 5/10 and 5/100. The dividing number is 10 times smaller from 5 to 0.5 and 100 times smaller from 5 to 0.05.

When a number is divided by numbers that are progressively smaller, the answer gets progressively bigger. So, if the dividing number is 10 times smaller, the answer will be 10 times bigger, for example:

10 ÷ 5 = 2 (How many 5s in 10?)
0.5 is 10× smaller than 5, so
10 ÷ 0.5 = 20 (How many halves in 10?)
0.05 is 10× smaller than 0.5, so
10 ÷ 0.05 =200 (How many five hundredths in 10?)

The answers to such divisions will be understood if the concepts of decimals and of division are understood.

Meta-cognition (thinking about how and what you are thinking)

To understand decimals, it is necessary to see them as an extension of the base 10 place value system.

It is easier to compare quantities expressed as decimals than when expressed as fractions. If the fractions that are difficult to compare are converted to decimals, then the comparison becomes much more straightforward. So, when dealing with quantities less than 1 it is useful to be able to draw on an interlinking knowledge of fractions, decimals and percentages and an understanding of that interlinking.

One way to check out a basic understanding of decimals is to count up in tenths...0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9...what comes next? It's *not* 0.10. That is not the required change in place value for the 1.

It sometimes helps to think of decimals as fractions, as tenths, hundredths, thousandths...1 tenth, 2 tenths, 3 tenths, 4 tenths, 5 tenths, 6 tenths, 7 tenths, 8 tenths, 9 tenths, 10 tenths. 10 tenths are 1. So, the tenths sequence ends with...0.8, 0.9, **1.0**. As with all counting in the base 10 system, when you reach 10 of whatever value you are counting, that takes you to the next place value.

Decimals can also be an experience that changes thinking about multiplication and division. If a number (whether bigger or smaller than 1) is *multiplied* by a decimal number that is less than 1, the answer will be smaller than the original number.

If some number (whether bigger or smaller than 1) is *divided* by a decimal number that is less than 1, the answer will be bigger than the original number.

The same rules apply when using fractions.

Things to do and practise

Practise counting on and back in decimals and do this across place values.

Round up decimals when shopping or looking at prices, for example, £1.95 to £2 and £399.99 to £400.

Add up the rounded prices on a shopping trip. Estimate the adjustment to make the addition accurate, for example, £2.95 + £7.99 is £3 + £8 = £11 when rounded up. The adjustment is 5p + 1p, so the accurate price is slightly lower than £11.00. It is £11.00 – 6p = £10.94.

Chapter 18

Percentages

Percentages are probably the most user-friendly way of presenting numbers that are less than 1. They are also used for values greater than 1, as in the classic soccer coach quote, 'The lads gave it 110% today.' Percentages up to, but below, 100% are less than 1. A typical bank interest rate (2017) on savings is around 2%. A typical rate on a credit loan from a shop is 34%. (You don't need a degree in maths to see who gets the advantage there!)

I think of the symbol for percentages, % as having two zero symbols and thus it relates to 100, which also has two zero symbols.

The concept of percentages is, not surprisingly, a variation on the concepts of fractions and decimals.

1 is 100%.

For a half:

The fraction is ½ or 1÷ 2.

The decimal is 0.5 (which you get when you divide 1 by 2).

The percentage is 50%, which you get when you calculate: $\frac{1}{2} \times 100\%$.

A decimal can be converted to a percentage by multiplying by 100, for example, 0.27 × 100 = 27%.

A fraction can be converted to a decimal by dividing the top number by the bottom number, for example, for a half, this will be 1 ÷ 2 = 0.5.

A percentage can be converted to a decimal by dividing by 100, for example 47% ÷ 100 = 0.47.

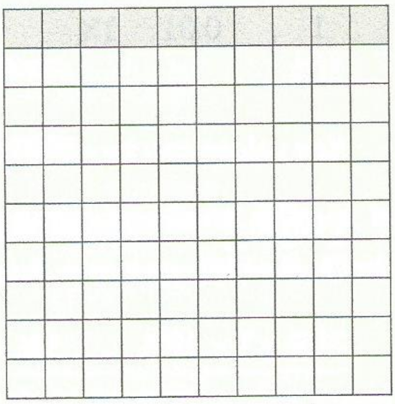
An example of this relationship/sequence in action is a test mark of 17/20.

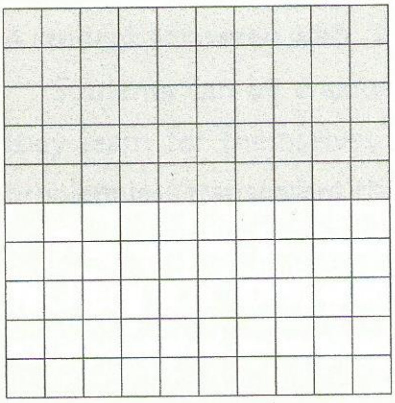
Fraction: $\frac{17}{20}$ Decimal: $17 \div 20 = 0.85$ Percentage: $0.85 \times 100 = 85\%$

A visual image of a 100 square can show the relative values of percentages up to 100. Of course, the image is also that used for a 1 to 100 square hence its familiarity and ease of use.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

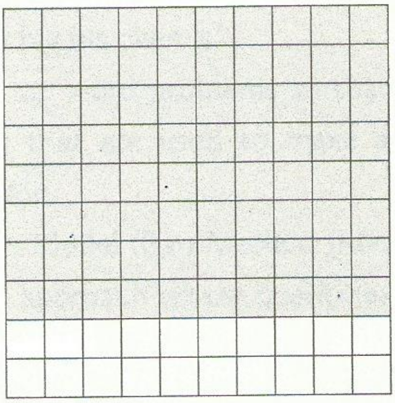
10%





50%

80%



Key values for fractions, decimals and percentages

$\frac{1}{1}$	1.0	100%
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{1}{10}$	0.1	10%
$\frac{1}{100}$	0.01	1%

Key values can be used to access other values, for example:

$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ 75% = 50% + 25% 0.75 = 0.5 + 0.25

20% = 2 × 10% 5% = ½ × 10%

'Another superb book from Steve Chinn – it will be essential reading for all teachers. The educational sector owes much to Steve Chinn and this book is testimony to that.'
– Dr Gavin Reid, Practitioner, Psychologist and author

'In this marvellous book Steve Chinn presents concisely and clearly knowledge and wisdom gained over many years of experience. I urge all teachers of mathematics to read it.'
– Robin Moseley MA MSc OBE

'This is a fantastic book written by one of the most recognised and respected names in the field of maths learning difficulties and dyscalculia. Highly informative and comfortingly practical.'
– Judy Hornigold, Independent Educational Consultant

'This book answers the question, "If they can't learn maths the way you teach, can you teach the way they learn?" Providing clear, practical activities, based on huge experience and expertise, this is teaching genius. Essential reading for all teachers.'
– Dr Kate Saunders, Dyslexia/SpLD Specialist Teacher, previously CEO British Dyslexia Association

Written by a world authority on maths difficulties in children, this accessible guide presents tried-and-tested visual strategies and tailored techniques to help teachers and parents support children with SpLDs who need help with maths.

Drawing on research into areas such as cognition and meta-cognition, along with the authors' decades of teaching experience, the book offers insight into how maths learning difficulties, including dyslexia, dyscalculia and maths anxiety, make maths difficult. Each chapter looks at foundational areas of maths learning that children may struggle with, from early number experiences to basic addition and subtraction, times tables, measurement and more.

Steve Chinn is an independent consultant, researcher and internationally renowned author and regularly presents papers, contributes to conferences worldwide (over 30 countries) and delivers training courses for psychologists, teachers, parents and support assistants.

Jessica Kingsley *Publishers*
ISBN 978-1-78592-579-5
9 781785 925795
www.jkp.com/dyslexia

British Dyslex!a Association

Cover design: www.ianrossdesigner.com