**Příklad 4:** Vypočítejte:

(a) $\lim_{n\to\infty} \frac{2n^2+1}{3-n^2} = \lim_{n\to\infty} \frac{\frac{2\cancel{n^2}}{\cancel{n^2}} + \frac{1}{n^2}}{\frac{3}{n^2} - \frac{\cancel{n^2}}{\cancel{n^2}}} = -2$

$\frac{1}{n^2} \to 0$, $\quad \frac{3}{n^2} \to 0$, $\quad \frac{n^2}{n^2} = 1$

(b) $\lim_{n\to\infty} \left(\frac{2n^2}{n+1} - \frac{6n^3}{n^2-3}\right) = [\infty - \infty]$

$= \lim_{n\to\infty} \frac{2n^2\cdot(n^2-3) - 6n^3\cdot(n+1)}{(n+1)\cdot(n^2-3)} = \lim_{n\to\infty} \frac{2n^4 - 6n^2 - 6n^4 - 6n^3}{n^3+n^2-3n-3}$

$= \lim_{n\to\infty} \frac{-4n^4 - 6n^2 - 6n^3}{n^3+n^2-3n-3} = \lim_{n\to\infty} \frac{-4n^4}{1\cdot n^3} = -\infty$

(c) $\lim_{n\to\infty} \frac{(2n+1)^4-(n-2)^4}{(n+1)^4+(n-1)^4} = \lim_{n\to\infty} \frac{(2n)^4 - n^4}{n^4 + n^4}$ $\quad (2n)^4 = 16n^4$

$= \lim_{n\to\infty} \frac{15n^4}{2n^4} = \frac{15}{2}$

(d) $\lim_{n\to\infty} \frac{\sqrt[4]{n^5+2} - \sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2} - \sqrt{n^3+1}} = \lim_{n\to\infty} \frac{n^{\frac{5}{4}}}{-n^{\frac{3}{2}}} = 0$

$\frac{5}{4} < \frac{3}{2}$

(e) $\lim_{n\to\infty} \left(\overset{a}{\sqrt{n+1}} - \overset{b}{\sqrt{n}}\right) = [\infty - \infty]$

$(a-b)\cdot(a+b) = a^2 - b^2$

$$= \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n+1}\right)^2 - \left(\sqrt{n}\right)^2}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{n}+1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \left[\frac{1}{\infty}\right] = 0$$

(j) $\lim_{n\to\infty} \frac{n!}{n^2} = \infty$

$$\lim_{n\to\infty} \frac{n\cdot(n-1)\cdot(n-2)!}{n^2} = \lim_{n\to\infty} \underbrace{\frac{n^2-n}{n^2}}_{\to 1} \cdot \underbrace{(n-2)!}_{\to \infty} =$$

$$= [1\cdot\infty] = \underline{\underline{\infty}}$$

(k) $\lim_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+3)!} = \left[\frac{\infty+\infty}{\infty}\right] = \left[\frac{\infty}{\infty}\right]$

$$= \lim_{n\to\infty} \frac{(n+2)(n+1)! + (n+1)!}{(n+3)(n+2)(n+1)!} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,1}\cdot \cancel{\overbrace{[n+2+1]}^{n+3}}^{\,1}}{\cancel{(n+1)!}_{\,1}\cdot \cancel{(n+3)}_{\,1}\cdot(n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{n+2} = 0$$

(l) $\lim_{n\to\infty} \frac{1}{n^2}\underbrace{(1+2+3+\ldots+n)}_{\text{aritm. posloupnost}}$ $= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n\cdot(1+n)}{2} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$

$$\boxed{S_n = \frac{n\cdot(a_1+a_n)}{2}}$$

(n) $\lim_{n\to\infty} \frac{\overbrace{1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}}^{\text{geom. posl. } q=\frac{1}{2}}}{\underbrace{1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^n}}_{\text{geom. posl. } q=\frac{1}{3}}} = \frac{\frac{1}{1-\frac{1}{2}}}{\frac{1}{1-\frac{1}{3}}} = \frac{\frac{1}{\frac{1}{2}}}{\frac{1}{\frac{2}{3}}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$

Pro součet nekonečně mnoha prvků geometrické posloupnosti s kvocientem $\boxed{0<|q|<1}$ platí tento vzorec: $s = \frac{a_1}{1-q}$

(o) $\lim_{n\to\infty} \left(1-\frac{1}{3n}\right)^n$ (pomůže znalost $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$)

subst.: $-\frac{1}{3n} = \frac{1}{k} \quad /\cdot 3n$

$-1 = \frac{3n}{k} \quad /\cdot k$

$-k = 3n \quad /:3$

$-\frac{k}{3} = n$

$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{-\frac{k}{3}} = \lim_{k\to\infty}\Big[\underbrace{\left(1+\frac{1}{k}\right)^k}_{\to e}\Big]^{-\frac{1}{3}} = e^{-\frac{1}{3}}$

(p) $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{9n-7} =$ (pomůže znalost $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$)

subst. $3n = k \to n = \frac{k}{3}$

$= \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{9\cdot\frac{k}{3}-7} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{3k-7}$

$= \frac{\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{3k}}{\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{7}} = \frac{e^3}{\lim_{k\to\infty}\Big[\underbrace{\left(1+\frac{1}{k}\right)^k}_{\to e}\Big]^{\frac{7}{k}}} = \frac{e^3}{\lim_{k\to\infty} e^{\frac{7}{k}}} = \frac{e^3}{1} = e^3$

$\frac{1}{k} \to 0$, $\quad \frac{7\cdot k}{k} = k\cdot\frac{7}{k}$, $\quad \frac{7}{k} \to 0$

**Příklad 2:** Najděte všechny hromadné body daných posloupností a určete limitu superior a limitu inferior daných posloupností:

(a) $a_n = (-1)^{n+3}$

$a_1 = (-1)^{1+3} = 1$

$a_2 = (-1)^{2+3} = -1$

$a_3 = 1, \; a_4 = -1, \dots$

1. $n = 2k-1, \; k \in \mathbb{N}$:

   $a_n = 1 \longrightarrow$ hromadný bod

2. $n = 2k, \; k \in \mathbb{N}$:

   $a_n = -1 \rightarrow$ hromadný bod

Výsledek: $H(a_n) = \{-1, 1\}$

$\liminf a_n = -1$ ... nejmenší hromadný bod

$\limsup a_n = 1$ ... největší hromadný bod