(c) $\lim_{n\to\infty} \frac{(2n+1)^4-(n-2)^4}{(n+1)^4+(n-1)^4} = \lim_{n\to\infty} \frac{(2n)^4 - n^4}{n^4 + n^4}$

$$= \lim_{n\to\infty} \frac{15n^4}{2n^4} = \frac{15}{2}$$

(d) $\lim_{n\to\infty} \frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{n^{\frac{5}{4}}}{-n^{\frac{3}{2}}} = 0$

(e) $\lim_{n\to\infty} (\underbrace{\sqrt{n+1}}_{a} - \underbrace{\sqrt{n}}_{b}) = [\infty - \infty]$

$$= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\overset{a}{\sqrt{n+1}} + \overset{b}{\sqrt{n}}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n}+1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

Použitý vzorec: $(a-b)\cdot(a+b) = a^2 - b^2$

(j) $\lim_{n\to\infty} \frac{n!}{n^2} = \infty$

$$\lim_{n\to\infty} \frac{n\cdot(n-1)\cdot(n-2)!}{n^2} = \lim_{n\to\infty} \underbrace{\frac{n^2-n}{n^2}}_{1} \cdot (n-2)!$$

$$= \lim_{n\to\infty} (n-2)! = \infty$$

(k) $\lim_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+3)!} = \left[\frac{\infty}{\infty}\right]$

$$= \lim_{n\to\infty} \frac{(n+2)\cdot(n+1)! + (n+1)!}{(n+3)\cdot(n+2)(n+1)!} =$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}\cdot[n+2+1]}{\cancel{(n+1)!}\cdot\underbrace{[n^2+5n+6]}_{(n+3)\cdot(n+2)}} = \lim_{n\to\infty} \frac{n+3}{n^2+5n+6} = 0$$

(l) $\lim_{n\to\infty} \frac{1}{n^2} \underbrace{(1+2+3+\ldots+n)}_{\text{aritmetická posl.}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n}{2} \cdot (1+n)$

$= \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$

(n) $\lim_{n\to\infty} \frac{\overbrace{1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}}^{\text{geom. posl. } q=\frac{1}{2}}}{\underbrace{1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^n}}_{\text{geom. posl. } q=\frac{1}{3}}} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{2}}}{\frac{1}{1-\frac{1}{3}}} = \frac{\frac{1}{\frac{1}{2}}}{\frac{1}{\frac{2}{3}}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$

V **čitateli** i jmenovateli je součet **členů** geometrických posloupností s kvocientem $0 < |q| < 1$, **můžeme** tedy použít vzorec. $s = \frac{a_1}{1-q}$

(o) $\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^n$ (pomůže znalost $\boxed{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e}$)

subst. $-\frac{1}{3n} = \frac{1}{k} \quad /\cdot k$

$-\frac{k}{3n} = 1 \quad /\cdot 3n$

$-k = 3n$

$k = -3n \quad /:(-3)$

$-\frac{k}{3} = n$

$\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^n =$

$= \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^{-\frac{k}{3}}$

$= \lim_{k\to\infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{-\frac{1}{3}}$

$= e^{-\frac{1}{3}}$

(p) $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{9n-7}$ (pomůže znalost $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$)

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{\cancel{9}^{3}\cdot\frac{k}{\cancel{3}}-7}=\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{3k-7}=\frac{\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{3k}}{\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{7}}=\frac{e^3}{1}=\underline{\underline{e^3}}$$

Subst. $\frac{1}{3n}=\frac{1}{k}$ $/\cdot k\cdot 3n$

$k=3n$ $/:3$

$\frac{k}{3}=n$

binomickou větou

$$1^7+\binom{7}{1}\cdot\left(\frac{1}{k}\right)^1+\binom{7}{2}\cdot\left(\frac{1}{k}\right)^2+\dots+\binom{7}{7}\cdot\left(\frac{1}{k}\right)^7$$

$\to 0$ $\to 0$ $\to 0$

Příklad 2: Najděte všechny hromadné body daných posloupností a určete limitu superior a limitu inferior daných posloupností:

(a) $a_n=(-1)^{n+3}$

$a_1=(-1)^{1+3}=1$

$a_2=(-1)^{2+3}=-1$

$a_3=(-1)^{3+3}=1$

$a_4=-1$

$\vdots$

1. $n=2k,\ k\in\mathbb{N}$ … sudé prvky $\{a_n\}$

$\lim_{n\to\infty} a_n=-1$

2. $n=2k+1,\ k\in\mathbb{N}\cup\{0\}$ … liché prvky $\{a_n\}$

$\lim_{n\to\infty} a_n=1$

$H(a_n)=\{-1,1\}$

$\liminf a_n=-1$ … nejmenší hromadný bod

$\limsup a_n=1$ … největší hrom. bod

(b) $a_n = (-2)^n = (-1)^n \cdot 2^n$

1. $n = 2k,\ k \in \mathbb{N}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n = \infty$$

2. $n = 2k+1,\ k \in \mathbb{N} \cup \{0\}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -2^n = -\infty$$

$$H(a_n) = \{-\infty, \infty\}$$

$\liminf a_n = -\infty$

$\limsup a_n = \infty$

(b) $a_n = (-2)^n = (-1)^n \cdot 2^n$

1. $n = 2k,\ k \in \mathbb{N}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n = \infty$$

2. $n = 2k+1,\ k \in \mathbb{N} \cup \{0\}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -2^n = -\infty$$

$$H(a_n) = \{-\infty, \infty\}$$

$\liminf a_n = -\infty$

$\limsup a_n = \infty$

(d) $a_n = (-1)^n \cdot \frac{2n}{n+1}$

$$a_1 = -\frac{2}{2} = -1$$
$$a_2 = \frac{2 \cdot 2}{2+1} = \frac{4}{3}$$
$$a_3 = -\frac{6}{3+1} = -\frac{3}{2}$$
$$a_4 = \frac{8}{4+1} = \frac{8}{5}$$

1. $n = 2k,\ k \in \mathbb{N}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+1} = 2$$

2. $n = 2k-1,\ k \in \mathbb{N}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{2n}{n+1} = -2$$

$$\boxed{\begin{aligned} H(a_n) &= \{-2, 2\} \\ \liminf a_n &= -2 \\ \limsup a_n &= 2 \end{aligned}}$$