$\rightarrow$ obyčejná diferenciální rovnice (1. ř.)

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

$a \quad x_0 \quad b$

$x^2 - y^2 = 3$ není dif.

počáteční (Cauchyho) úloha

$\exists\ y = y(x)$ pro každé $x$ platí rovnice

obecné řešení d. r. $\quad g(x, y, C) = 0$ $\quad$ $C$ libov. konst.

a pro každé $C$ $\ y = y(x)$ splňuje d. r.

$y' = y \qquad y = C \cdot e^x \qquad \boxed{C(e^x)' = Ce^x}$

$y' = y^2$ separovaná d. r.

separovaná d. r.

$y' = f(x) \cdot g(y)$

$y' = \ln(x+y)$ není separovaná rovnice

$y' = 1 \cdot y^2$

$f(x) \quad g(y)$

$$\frac{dy}{dx} = y^2$$

$\frac{dy}{dx} = y', \quad dy = y'\,dx$

$y \neq 0 \qquad \frac{1}{y^2} \cdot dy = dx,$

$y = 0$ – také je řeš.

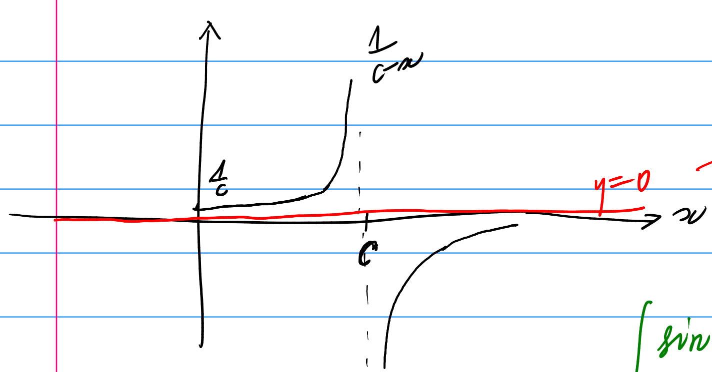
$$\int \frac{1}{y^2}\,dy = \int dx$$

$$C - \frac{1}{y} = x$$

$$\int y^{-2}\,dy = \frac{y^{-2+1}}{-2+1} = \frac{y^{-1}}{-1} = -\frac{1}{y}$$

$C - x = \frac{1}{y}, \qquad y = \frac{1}{C - x}$

$C$ je libov. konst.




v tomto vzorci není – zvláštní řeš.

$y = 0$

$y' = f(x)$

$$\boxed{y = \int f(x)\,dx}$$

$\int \sin(x^2)\,dx$

$\int \frac{dx}{Q(x)} = \int \frac{1}{Q(x)} dx$

$$\boxed{y' = f(x) \cdot g(y)}$$

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\frac{dy}{g(y)} = f(x) dx$$

$$\int \frac{dy}{g(y)} = \int f(x) dx$$

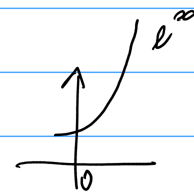
$\Downarrow$

$y = y(x)$ $\qquad$ $\frac{dy = y' dx}{g(y)}$ $\qquad$ $\frac{dy(x)}{g(y(x))} = \frac{y'(x)\, dx}{g(y(x))}$

$(e^x)' = e^x$

$y' = 2y$

$\frac{dy}{dx} = 2y$, $\quad \frac{dy}{y} = 2 dx$, $\quad \int \frac{dy}{y} = 2 \int dx$




$\ln|y| = 2x + C$

$|y| = e^{\ln|y|} = e^{2x+C} = e^{2x} \cdot e^{\tilde{C}}$ $\qquad$ $\tilde{C}$ je libovolné

$y = \pm e^{2x} \cdot e^{\tilde{C}}$ $\qquad$ $e^{\tilde{C}} > 0$

$\pm e^{\tilde{C}} = C$

$\boxed{y = C e^{2x}}$, kde $C \in \mathbb{R}$ (libovolné)

$\downarrow$ obecné řešení

$y' = f(x, y) =$
$= \underline{k(x)} \cdot \underline{y} + b(x)$ – lineární

$y' = ky$ $\qquad$ lineární homogenní rovnice

# Homogenní diferenciální rovnice.

$y' = f(x,y)$

homogenní, jestliže se dá přepsat do tvaru

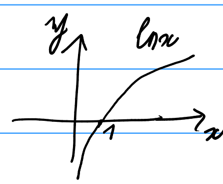
$$\boxed{y' = g\left(\frac{y}{x}\right)}$$

$x \to \lambda x,\ y \to \lambda y \Rightarrow$ rovnice je zachována.

$y' = \frac{y}{x}$ → také separovaná: $y' = \frac{1}{x} \cdot y$ →

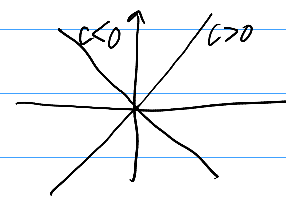
$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln|y| = \ln|x| + \ln C = \ln C|x|$$




$$e^{\ln|y|} = e^{\ln C|x|}$$

$$|y| = C|x|$$




$y' = g(y/x)$

$g(t) = t$

$y = Cx$

$y' = C \qquad \frac{y}{x} = C$

$$y' = \frac{y}{x} - \frac{y^2}{x^2}$$

$y' = g\left(\frac{y}{x}\right)$, kde $g(t) = t - t^2$

$u$ je nová proměnná

pro homogenní rovnice $\frac{y}{x} = u$;

$y = xu, \qquad y' = (xu)' = u + xu'$

$u + xu' = g(u), \qquad xu' = g(u) - u$

$x\frac{du}{dx} = g(u) - u, \qquad \frac{du}{g(u)-u} = x\,dx$

$$y' = \frac{y}{x} - \frac{y^2}{x^2}$$

$u = \frac{y}{x}$, $\quad ux = y$, $\quad u'x + u = y'$

$u'x + \cancel{u} = \cancel{u} - u^2$,

$u'x = -u^2$

$x\frac{du}{dx} = -u^2$, $\quad \frac{du}{u^2} = -\frac{dx}{x}$ $\qquad \int \frac{du}{u^2} = -\frac{1}{u}$

$$\int \frac{du}{u^2} = -\int \frac{dx}{x}$$

$-\frac{1}{u} = -\ln|x| - \ln C$

$\frac{1}{u} = \ln C|x|$, $\quad u = \frac{1}{\ln C|x|}$

$u = \frac{y}{x}$ $\qquad y = \frac{x}{\ln C|x|}$

$\boxed{\text{homogenní rovnica: } u = \frac{y}{x} \to \text{separov. rovn.}}$

poč. podmínka: $y(1) = 2$

$x = 1, \; y = 2$ $\qquad 2 = \frac{1}{\ln C}$, $\quad \ln C = \frac{1}{2}$, $\quad C = \sqrt{e}$

$$y = \frac{x}{\ln \sqrt{e}|x|}$$

$$y' = g(\alpha x + \beta y)$$ $\alpha, \beta$ čísla

$\alpha x + \beta y = u$

$u' = \alpha + \beta y'$, $\quad u' - \alpha = \beta y'$, $\quad y' = \frac{u' - \alpha}{\beta}$

$$\frac{u' - \alpha}{\beta} = g(u), \quad u' - \alpha = \beta g(u)$$

$$y' = \frac{1}{e^{2x-3y}} \quad ?$$

$u = 2x - 3y$, $\quad u' = 2 - 3y'$, $\quad -u' + 2 = 3y'$, $\quad y' = \frac{2 - u'}{3}$

$\frac{2 - u'}{3} = \frac{1}{e^u}$, $\quad 2 - u' = \frac{3}{e^u}$, $\quad u' = 2 - \frac{3}{e^u} = \frac{2e^u - 3}{e^u}$

$$\int \frac{e^u}{2e^u - 3} du = \int dx$$

$d(2e^u - 3) = 2e^u du$

$$\frac{1}{2} \int \frac{d(2e^u - 3)}{2e^u - 3} = x$$

$$\frac{1}{2} \ln|2e^u - 3| = x + C$$

$u = 2x - 3y$ $\quad \boxed{\frac{1}{2} \ln|2e^{2x-3y} - 3| = x + C}$

$y = y(x)$

(1.) integrál d. r.

$$y' = \frac{y}{x} + tg\,\frac{y}{x}$$ homogenní

$$y' = g\left(\frac{y}{x}\right) \qquad g(t) = t + tg\,t$$

$$\frac{y}{x} = u \qquad ux = y, \qquad u'x + u = y' \qquad cotg\,x = \frac{1}{tg\,x}$$

$$u'x + \cancel{u} = \cancel{u} + tg\,u$$

$$u'x = tg\,u, \qquad x\frac{du}{dx} = tg\,u, \qquad \int \frac{du}{tg\,u} = \int \frac{dx}{x}$$

$$\int cotg\,u\,du = \int \frac{dx}{x}$$

$$\int \frac{\cos u\,du}{\sin u} = \int \frac{d(\sin u)}{\sin u} = \ln|\sin u|$$

$$d(\sin u) = (\sin u)'du = \cos u\,du$$

$$\ln|\sin u| = \ln|x| + \ln C = \ln C|x|$$

$$e^{\ln|\sin u|} = e^{\ln C|x|}$$

$$\sin u = Cx$$

$$u = \arcsin Cx$$

$$y' = ay \qquad \text{lineární d. r. 1. ř. s konst. koef.}$$

$$y = Ce^{ax} \qquad a \in \mathbb{R} \text{ je konst.}$$

$$\boxed{y' = a(x)\cdot y} \qquad a = a(x) \text{ je funkce} - \text{lineární rovnice (s prom. koef.)}$$

↓ (je separovaná)

$$\frac{dy}{y} = a(x)\,dx, \qquad \int \frac{dy}{y} = \int a(x)\,dx$$

$$\ln|y| = \int a(x)\,dx + \tilde{C}$$

$$|y| = e^{\int a(x)dx + \tilde{C}} = \underbrace{e^{\tilde{C}}}_{>0} \cdot e^{\int a(x)dx}$$

$$\boxed{y = Ce^{\int a(x)dx}} \qquad \text{obecné řešení lineární (hom.)}$$

$$\left[\; y' = Ce^{\int a(x)dx} \cdot \left(\int a(x)dx\right)' = Ce^{\int a(x)dx} \cdot a(x) = C\,y(x)\cdot a(x) \;\right]$$

$y' = a(x)\cdot y + \boxed{b(x)}$ — lineární d. r. (nehomogenní)

funkce

není separovaná (pro $b\neq 0$)

**Metoda variace konstanty:**

$y' = a(x)\cdot y$

$\Downarrow$

ob. řeš. je $y = C e^{\int a(x)dx}$

hledáme řešení ve tvaru

$$\boxed{y = C(x)\cdot e^{\int a(x)}}$$

funkce

nehomog. rovnice:

$y' = a(x)y + b(x)$

$$y' = C'(x)\, e^{\int a(x)dx} + C(x)\, e^{\int a(x)dx}\cdot a(x)$$

$$C'(x)e^{\int a(x)dx} + \boxed{\cancel{C(x)e^{\int a(x)dx}\, a(x)}} = \boxed{\cancel{a(x)\cdot C(x) e^{\int a(x)dx}}} + b(x)$$

$$C'(x)\, e^{\int a(x)dx} = b(x), \quad \boxed{C'(x) = e^{-\int a(x)dx}\cdot b(x)}$$

$$C(x) = \int e^{-\int a(x)dx}\, \boxed{b(x)}\, dx$$

$Ax = 0$, $Ax = b$

$x = \boxed{(\text{obecné řeš. } Ax=0)} + (\text{partikul. řeš. } Ax=b)$

$y' = a(x)\, y$

$y = C e^{\int a(x)dx}$

$y' = a(x)y + b$

$y = \underline{C e^{\int a(x)dx}} + (\text{partikulární řešení } \underbrace{y' = a(x)y + b(x)}_{v})$

$= a(x)(y_0 + v) + b.$

$y = y_0 + v$

$y_0' = a(x)y_0$

$v' = a(x)v + b$

$y' = y_0' + v' = a(x)y_0 + a(x)v + b =$