$y' = a(x)\cdot y$ lineární o. d. r. 1. ř. (homogenní)

$\frac{dy}{dx} = a(x)\cdot y$ $\qquad y' = \underline{f(x)}\cdot\underline{g(y)}$

$\int \frac{dy}{y} = \int a(x)\,dx$

$$y(x) = C\cdot e^{\int a(x)\,dx}$$

nehomog. $(b \neq 0)$

$[\; y' = a(x)\,y + \underline{b(x)}$

$$y = C(x)\cdot e^{\int a(x)\,dx}$$

obecné řeš. nehomog. r. = obecné řeš. homog. rovn. + partikul. řeš. nehomog.

$$\boxed{y' = -xy} + x^2$$

$y' = -xy$

$y = C e^{\int(-x)\,dx} = \boxed{C e^{-\frac{x^2}{2}}}$ – obecné řeš. hom. r.

variace konst.: $y = C(x)\,e^{-\frac{x^2}{2}}$ $\qquad y' = C'(x)\,e^{-\frac{x^2}{2}} + C(x)\cdot e^{-\frac{x^2}{2}}\cdot(-x) =$

$= C'(x)\,e^{-\frac{x^2}{2}} - x\,C(x)\,e^{-\frac{x^2}{2}}$

$C'(x)\,e^{-\frac{x^2}{2}} \cancel{- x\,C(x)\,e^{-\frac{x^2}{2}}} = \cancel{-x\cdot C(x)\,e^{-\frac{x^2}{2}}} + x^2$

$C'(x)\,e^{-\frac{x^2}{2}} = x^2$, $\quad C'(x) = x^2\cdot e^{\frac{x^2}{2}}$

$$C(x) = \underline{\int x^2 e^{\frac{x^2}{2}}\,dx}$$

$$y_{\text{part. řeš.}} = e^{-\frac{x^2}{2}}\int x^2 e^{\frac{x^2}{2}}\,dx$$

$\int(-1)\,dx = -x$

$y' = -y + x$

hom. rovn. : $y' = -y$ $\quad y = Ce^{-x}$

part. řeš. $y = u\cdot v$; $\quad y' = u'v + uv'$

$u'v + uv' = -uv + x$ $\quad v' = -v$

$\quad\quad\quad\quad\; \| \; 0$

Bernoulli subst.

$uv' = -uv$, $\quad uv' + uv = 0$

$u(v' + v) = 0$

pokud platí $\boxed{v' + v = 0}$,

pak $u'v = x$, $\quad u' = \frac{x}{v}$

$\hookrightarrow u = \int \frac{x}{v(x)}\,dx$

$y' = a(x)\,y$

$y' = a(x) + b(x)$

$y = y_0 + y_1$

řeš. hom. r. $\quad$ part. řeš. nehomog.

$y'' + p(x)\,y' + q(x)\,y = b(x)$ — lineární o. d. r. 2. řádu (nehomog.)

$y'' + py' + qy = b(x)$ $\quad$ rovnice s konst. koef.

$y'' + py' + qy = 0$

$\boxed{y' = ay} + b(x)$

$(e^x)' = e^x$, $\quad (e^x)'' = e^x$

$y'' - y = 0$ $\quad y = e^x$ $\quad (e^{-x})' = -e^{-x}$

$(e^{-x})'' = e^{-x}$

$\sin x$, $\cos x$ $\quad \| \quad y'' + y = 0$

$(\sin x)'' = -\sin x$

$(\cos x)'' = -\cos x$

$\to$ splňují rovnici

$e^{\lambda x}$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$

$\cos x \pm i \sin x = e^{\pm ix}$

$$y'' + py' + qy = 0$$ (homog. rovnice 2. ř. s konst. koef.)

λ: $e^{\lambda x}$ splňovalo rovn. $\lambda = ?$

$y = e^{\lambda x}$

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = 0$$

charakteristická rovnice (l. o. d. r. 2 ř. s konst. k.)

$$\lambda^2 + p\lambda + q = 0$$ – rovnice pro určení λ

pro rovnice 2. ř.: kvadratická

$$\boxed{y'' + y' - 2y = 0}$$

$y = e^{\lambda x}$ (hledáme řešení tohoto tvaru)

$y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

$$\lambda^2 \cancel{e^{\lambda x}} + \lambda \cancel{e^{\lambda x}} - 2\cancel{e^{\lambda x}} = 0$$

$y'' + 1 y' - 2 y = 0$

$$\lambda^2 + \lambda - 2 = 0 \qquad \lambda = \frac{-1 \pm \sqrt{1 + 4\cdot 2}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} \begin{matrix} \nearrow -2 \\ \searrow 1 \end{matrix}$$

Má kořeny -2, 1.

$y_1 = e^{-2x}$, $y_2 = e^{x}$ jsou řešení (lineár. nez. řeš.)

vzhledem k linearitě:

$$\begin{aligned} \cdot c_1 \quad & y_1'' + y_1' - 2y_1 = 0 \\ + \quad & \\ \cdot c_2 \quad & y_2'' + y_2' - 2y_2 = 0 \end{aligned}$$

$y_1 = e^{-2x}$, $y_2 = \frac{3}{2} e^{-2x}$ nejsou lin. nez. ($y_2 = \frac{3}{2} y_1$)

$$c_1 y_1'' + c_2 y_2'' + c_1 y_1' + c_2 y_2' - 2(c_1 y_1 + c_2 y_2) = 0,$$ t. j. funkce

$$\boxed{y = c_1 y_1 + c_2 y_2}$$ je řeš. rovnice

obec. řeš. $$\boxed{y = c_1 e^{-2x} + c_2 e^{x}}$$ c_1, c_2 libovolné konstanty

$$y'' - 4y' = 0$$

$$(y')' - 4y' = 0 \qquad y' = w: \qquad w' - 4w = 0$$

$$w' = 4w$$

$$w = Ce^{4x}$$

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} = 0. \qquad \lambda(\lambda - 4) = 0, \quad \text{kořeny jsou } 0, 4.$$

$$e^{0x} = 1 \qquad e^{4x}$$

$$y = C_1 \cdot 1 + C_2 e^{4x}$$

$$y = C_1 + C_2 e^{4x}$$

$\{1; e^{x}\}$ lin. nez. řeš.

$y'' - 4y = 0$ charakt. rovnice je $\lambda^2 - 4 = 0$

$$\lambda^2 = 4, \quad \lambda = \pm 2$$

$$y = C_1 e^{-2x} + C_2 e^{2x}$$

$$y'' - 2y' + y = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2$$

$$\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$$

$$\lambda = 1, \quad e^{\lambda x} = e^{x} \qquad e^{x} \to x.e^{x}$$

$$(xe^{x})' = e^{x} + xe^{x}$$

$$(xe^{x})'' = e^{x} + e^{x} + xe^{x} =$$

$$= 2e^{x} + xe^{x}$$

$$y = C_1 e^{x} + C_2 x e^{x}$$

$$\cancel{2e^{x}} + \cancel{xe^{x}} - \cancel{2(e^{x} + xe^{x})} + \cancel{xe^{x}} \qquad y = xe^{x} \text{ také řeš.}$$

$$y'' + p(x)y' + q(x)y = 0$$

y_1, y_2 – lineárně nez. řeš. $\Rightarrow$ obecné řeš. je $\boxed{y = C_1 y_1 + C_2 y_2}$

C_1, C_2 konst. (libov.)

$\nexists \alpha$: $y_2 = \alpha y_1$ ($c_1 y_1(x) + c_2 y_2(x) = 0$ pro všechna x $\Rightarrow c_1 = c_2 = 0$)

[$5y'' + 6y' + 5y = 0$

charakt. rovnice má tvar

$5\lambda^2 + 6\lambda + 5 = 0 \qquad D = 6^2 - 4\cdot 25 = 36 - 100 = -64$

$\lambda = \frac{-6 \pm \sqrt{-64}}{2\cdot 5} = \frac{-6 \pm 8i}{10} = \frac{-3 \pm 4i}{5}$

$\lambda = -\frac{3}{5} \pm \frac{4}{5} i$

$e^{\left(-\frac{3}{5} \pm \frac{4}{5} i\right)\cdot x} = e^{-\frac{3}{5}x} \cdot \boxed{e^{\pm \frac{4}{5} i x}} = e^{-\frac{3}{5}x} \cdot \left[\cos \frac{4}{5}x \pm i \sin \frac{4}{5} x\right]$

$y_1(x) = \cos \frac{4}{5} x \cdot e^{-\frac{3}{5}x} \qquad y_2 = \sin \frac{4}{5} x \cdot e^{-\frac{3}{5}x}$

$$y'' + p(x)y' + q(x)y = 0$$

$$y'' + py' + qy = 0$$

$$\boxed{y'' + py' + qy = b(x)} \quad \text{nehomogenní}$$

$$y'' + y' - 6y = 36x$$

$$y'' + y' - 6y = 0 \qquad \lambda^2 + \lambda - 6 = 0, \quad \lambda = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} \begin{matrix} -3 \\ 2 \end{matrix}$$

$$\lambda_1 = -3 \quad \lambda_2 = 2$$

$$y = c_1 e^{-3x} + c_2 e^{2x} \quad (\text{řeš. hom. rovn.})$$

$$y_{\substack{\text{obecné} \\ (\text{nehom.})}} = y_{\substack{\text{obecné} \\ (\text{hom.})}} + \underbrace{y_{\text{part.}}}_{\text{nehom.}}$$

$$y'' + py' + qy = b(x)$$

$y = C_1 y_1 + C_2 y_2$ – řeš. hom. rovnice

Pro nehomog. rovnici: hledáme y ve tvaru

$$y = C_1(x) \cdot y_1 + C_2(x) \cdot y_2$$

$$y' = \boxed{C_1'(x) y_1 + C_2'(x) y_2} + C_1(x) \cdot y_1' + C_2(x) y_2'$$

$$y'' = C_1'(x) y_1' + C_2'(x) y_2' + C_1(x) y_1'' + C_2(x) y_2''$$

$$C_1' y_1' + C_2' y_2' + C_1 y_1'' + C_2 y_2'' +$$

$$+ p[C_1 y_1' + C_2 y_2'] + q[C_1 y_1 + C_2 y_2] = b(x)$$

y_1, y_2 jsou řeš. hom. rovnice $(y_i'' + p y_i' + q y_i = 0)$

$$\left\{ \begin{aligned} C_1'(x) y_1 + C_2'(x) y_2 &= 0 \\ C_1'(x) y_1' + C_2'(x) y_2' &= b(x) \end{aligned} \right. \rightarrow \text{lineární algeb. soustava rovnic pro nalezení } (C_1'(x), C_2'(x))$$

$$C_1'(x) = \dots \Rightarrow C_1(x) = \int \dots$$

$$C_2'(x) = \dots, \Rightarrow C_2(x) = \int \dots$$

$$y'' - 5y' + 4y = e^{2x} = f(x)$$

$e^{\alpha x}$

$\cos x$, $\cos \beta x$

$$y'' - 5y' + 4y = \sin 4x$$

$[P(x)\cos(\beta x) + Q(x)\sin \beta(x)]e^{\alpha x}$

$y'' + y = \sin 2x$ 2 ғ. l. нehom.

Hom. r.: $y_1 = \cos x$, $y_2 = \sin x$

$y = C_1(x)\cos x + C_2(x)\sin x$

$$\begin{cases} C_1'(x) y_1 + C_2'(x) y_2 = 0 \\ C_1'(x) y_1' + C_2'(x) y_2' = \boxed{b(x)} \end{cases}$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ b(x) \end{pmatrix}$$

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0 \\ -C_1'(x)\sin x + C_2'(x)\cos x = \sin 2x \end{cases}$$

$$C_1'(x)\cos^2 x + C_2'(x)\sin x\cos x = 0$$

$$+ \quad C_1'(x)\sin^2 x - C_2'(x)\cos x\sin x = -\sin 2x \cdot \sin x$$

$$2C_1'(x) = -\sin 2x \cdot \sin x$$

$$C_1'(x) = -\frac{1}{2}\sin 2x \sin x$$

$$C_1(x) = -\frac{1}{2}\int \underbrace{\sin 2x}_{2\sin x\cos x} \sin x\, dx = -\int \sin^2 x \underbrace{\cos x\, dx}_{d(\sin x)} = -\int \sin^2 x\, d(\sin x) = -\frac{\sin^3 x}{3}$$

$C_2(x) = ?$

$$C_1'(x)\cos x\sin x + C_2'(x)\sin^2 x = 0$$

$$-C_1'(x)\sin x\cos x + C_2'(x)\cos^2 x = \sin 2x\cos x$$

$$2C_2'(x) = \sin 2x\cos x$$

$$C_2(x) = \frac{1}{2}\int \sin 2x\cos x\, dx = \int \sin x\cos^2 x\, dx = -\int \cos^2 x\, d(\cos x)$$

$$= -\frac{\cos^3 x}{3}.$$