Why History of Mathematics?

by Glen Van Brummelen

Among the several new courses added to the BC high school curriculum, the History of Mathematics course may raise the most eyebrows. Although topics in geometry and statistics are often offered in other jurisdictions, the history of mathematics seldom exists as its own course except for teacher training in some American colleges. Even then, the goal is to provide background to help prospective teachers design their mathematics classes, not to prepare to teach the subject on its own. Why does the subject deserve sustained attention in its own course?

Among many possible answers to this question, I find mine in the phrase "humanizing mathematics". By definition, mathematics is an activity performed by humans, intended for a human audience. The fact that the phrase exists at all is as clear a sign as can be that many students are missing the point of mathematics. To them it is a *noun*—a collection of abstractions—rather than a *verb*, a kind of activity. Laudably, curricula are growing in the ability to pass along problem solving strategies to our students. But this is not enough. As the new BC math curriculum emphasizes, we need to raise with our students the bigger questions. Why is mathematics so successful in understanding the physical world? Why did we choose to represent unknown quantities with letters? What is different about how we use mathematics to affect our world, compared to those who went before?

After four years of intense study of undergraduate mathematics, I had no idea why the questions my professors were answering in their lectures were important. In a sense, then, I chose my career in the history of mathematics as an escape from irrelevance. Many of our students, especially those not immediately attracted to STEM, need a big picture to feel part of the conversation. The questions above were not written by me. They are taken from a list of "why" questions I solicited from my humanities-oriented history of mathematics students at the beginning of my course. They shied away from math not just because they don't feel competent, but also because they have not heard answers that are meaningful to the way they think.

The history of mathematics can reach these students in several ways:

Motivation: All mathematical subjects arose due to some need, sometimes within mathematics but often from outside of it. Students who participate in an environment where the need comes first will recognize that what they are doing means something, and will be inspired to pursue solutions. In such a setting, it is natural to portray mathematics properly as inquiry rather than as edifice. Now, the demand for mathematics today is often portrayed to come from science and technology, and certainly such motives are more than appropriate for the classroom. But let's broaden the possible options with an example from well outside of science—local practices of art. From basket weavers in Mozambique to modern musicians using 20th century mathematical objects like fractals in their compositions, the creative artistic process has often provoked mathematical questions.

One such episode is records of meetings between artisans and geometers in late 10th-century Iraq, some written by the great mathematician Abū'l-Wafā'. Decorations of walls in palaces were not to include images of people or animals, for religious reasons. Instead, the local artisans designed elaborate geometric patterns, many of which still grace the walls of historic Muslim buildings. The most famous example is the spectacular Alhambra, a palace in Granada, Spain (see Figure 1). These patterns are born from simple geometric constructions that are repeated to tile the entire plane. Then, each line segment or arc in the diagram is elaborated in some way. The results can be stunning. For instance, Abū'l-Wafā' describes how to embed an equilateral triangle in a square, as follows (Figure 2): extend the base GD by an equal distance to E. Draw a quarter circle with centre G and radius GB; draw a half circle with centre D and radius DE. The two arcs cross at Z. Then draw an arc with centre E and radius EZ downward, to H. If you draw AT = GH and connect B, H, and T, you will have formed the equilateral triangle. (If you want to prove it, connect GZ and ZD. Consider first the angles in triangle GZD; next, consider the angles in triangle BGZ. After that, you're on your own!)

***Figure 1**: A wall at the Alhambra (Granada, Spain)*

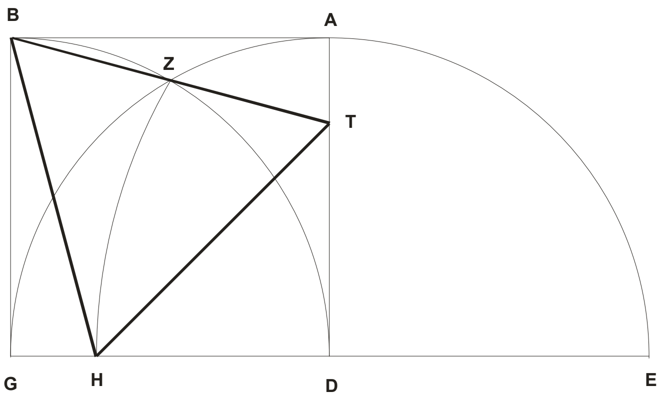

***Figure 2:** Abū'l-Wafā''s construction of an equilateral triangle in a square*

Research: Having posed bigger problems, we are responsible to provide resources for tackling them. History is an obvious tool here as well; after all, Euclid, Newton, Euler and others became who they are because of their ability to solve the big problems. We learn how to approach problems from those who went before us. Often their major insights have involved some sort of leap across the gap between the divide in mathematics between arithmetic and geometry. For instance, the recognition that the function $f(x)=e^x$ is its own derivative allowed Euler and his successors to solve many differential equations — and, with the introduction of $i=\sqrt{-1}$, he was able to combine the inherently geometric discipline of trigonometry with the algebraic analysis of exponential growth into an amazing unity.

Closer to the 11th grade classroom, we turn to ancient Babylonian schoolchildren's solutions to the quadratic equation, which were based on "cut-and-paste" geometry. Consider one tablet where a child solves $x^2+\frac{2}{3}x=\frac{7}{12}$. Although his geometry operates below the surface of the text, it is clear that the reasoning is as follows: x^2 is a square whose side is the unknown *x*; $\frac{2}{3}x$ is a rectangle with sides and *x*. Combine them (Figure 3a), and the resulting rectangle has area $\frac{7}{12}$. Cut off half the original rectangle and move it below the square. The result is a shape that is almost a square, but with a small shaded square left out. Its area is $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$. If we add the shaded square, we know that the larger square of Figure 3b has area $\frac{7}{12}+\frac{1}{9}=\frac{25}{36}$, so its side length is $\frac{5}{6}$. But the side length is also $x+\frac{1}{3}$, so $x=\frac{5}{6}-\frac{1}{3}=\frac{1}{2}$.

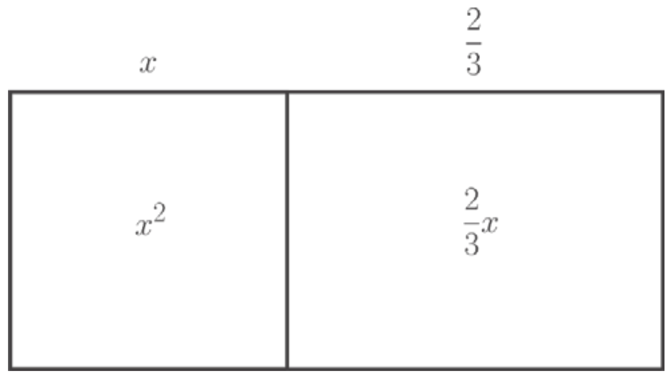

***Figure 3a:** A Babylonian quadratic equation*

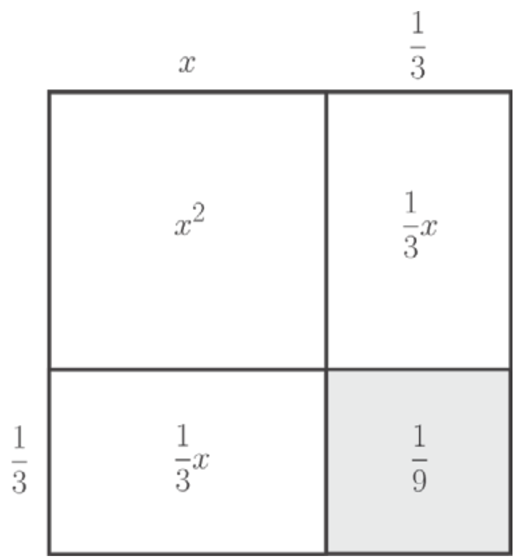

Figure 3b

This process does a lot more than solve a quadratic equation; it *proves the quadratic formula*. Besides gaining the quadratic formula as a tool for future use, students learn several significant lessons about problem solving. Observing the successes of our predecessors can lead to ideas for our own successes as well. Changing the way a problem is represented (in this case, from algebra to geometry) can lead to surprising insights and novel solutions. Most importantly, they see that mathematics moves from the "mess" of the struggle to the discovery of paths to resolution.

Critical Thinking: Every mathematical community makes shared decisions about the validity and power of various competing approaches. For instance, medieval Indian mathematical astronomers were comfortable using *iterative* solutions to equations—a series of guesses or approximations, each one improved over the previous one using some procedure. However, ancient Greek and medieval Islamic astronomers preferred direct arguments and calculations. This may be due to a commitment to pure mathematics as the only path to true knowledge, propounded by the great astronomer Claudius Ptolemy. It is difficult for modern students to understand that such commitments are also present today. For instance, we represent geometric magnitudes like lengths and areas using numbers—a choice that practitioners of the Euclidean tradition rejected. As a result, our mathematics leans heavily toward arithmetic and algebra, exploiting the power that these algorithmic methods provide. In order to think creatively, one needs to make informed judgments about such alternate avenues of attack; one must know what the community's rules are before one decides to bend or break them.

Consider, for example, the curious case of 12^{th}-century Iranian scholar Ibn Yahyā al-Samaw'al al-Maghribī, who late in his life composed *Exposure of the Errors of the Astronomers* in which he pointed out dozens of what he perceived to be mistakes in the works of his colleagues and ancestors. One of these episodes involved finding a value for $\sin 1^\circ$ from a given value of $\sin 3^\circ$. To find a precise solution turns out to require solving a cubic equation, and this cannot be done with geometric methods—it is equivalent to the problem of trisecting the angle with ruler and compass, now known to be impossible. Violating his own commitment to the perfection of mathematics, Claudius Ptolemy had been forced into approximating (an equivalent of) $\sin 1^\circ$. Al-Samaw'al's creatively restored the role of pure mathematics in philosophy by redefining the number of degrees in a circle from 360 to 480. The sine of 3° becomes the sine of four units, and he can apply the sine half-angle formula twice to find the sine of one unit. Sometimes, changing the rules of one game allows one to conform better to the rules of another.

Implications: Students motivated by humanistic questions admit readily that mathematics has had a major effect on our world, but usually they can speak only vaguely about its uses in modern science and technology. This lack of clarity contributes significantly to the widespread misperception that mathematics is only a toolbox of algorithms allowing manufacturers and tech firms to build new devices. Witnessing the various breakthroughs that have changed how people live their lives and perceive their world can bring these students to a deeper appreciation of mathematics as much more than a calculating machine, but rather a living discipline and driver of social change.

The biggest driver of social change these days in fact a calculating machine, the computer. In both the 19^{th} and 20^{th} centuries, mathematicians studied algorithms themselves, recognizing both their power and their limitations. The person usually considered to be the first to compose an algorithm implementable on a computer was Ada Lovelace (1815-1852), daughter of poet Lord Byron. She maintained a working relationship with Charles Babbage, whose Analytical Engine would have been easily the first real computer, if it had ever been built. Lovelace gave a prescription for computing Bernoulli numbers (fundamental in number theory), and was one of the few at this time to consider the possibility that the Analytical Engine or a machine like it could do much more than just calculation—although she rejected the possibility of true artificial intelligence. This remarkably early analysis of algorithms reached an extraordinary conclusion with the work of Julia Robinson (1919-1985), an important figure in the resolution of Hilbert's tenth problem. This problem asked whether a computer algorithm can be written to determine whether a Diophantine equation has any integer solutions. The answer, "no", implies that as powerful as computers are, there are some things—meaningful things—that they simply cannot do.

Many of the changes brought about by mathematics were not in the realm of science and technology. Consider the birth of non-Euclidean geometry. In the early 19^{th} century, after dozens of failed attempts to prove statements equivalent to the assertion that the angles of a triangle sum to 180°, three separate mathematicians considered the unthinkable: what if it's false? Carl Friedrich Gauss, Nicolai Lobachevsky, and János Bolyai discovered that a new and consistent geometry may be born from denying this assumption. Through this bold step, they created "worlds out of nothing": elliptical and hyperbolic geometry. These non-Euclidean geometries remained creations of the mind for decades, but starting in the early 20^{th} century they have become candidate models for the geometry of the universe in which we live. This episode reveals that seemingly obvious assumptions that have stood for millennia are not necessarily on solid ground.

Students aware of cosmic shifts like these participate in a rich educational experience. They are able to orient themselves within the intellectual landscape, and are can act in their profession with reflectiveness and purpose.

Communication: The rich settings exemplified here provide ample opportunity for our students to expand their options for writing and otherwise presenting ideas in a mathematics class. History encompasses entire narratives from initial conception to final product and societal impact, providing rich opportunities for interpretive short answers and essays that can go far beyond learning how to write up an effective solution.

Diversity: Attempts to improve intercultural understanding, including with indigenous societies, have been spreading throughout the school curriculum. Mathematics poses a unique quandary for these efforts, since most popular accounts send the message that mathematics is separate from culture. "Isn't for everyone?" Curricular projects have been helping to bring to the classroom certain examples of indigenous mathematics realized in daily life, extending the project of *ethnomathematics* that has been working on such matters for a few decades. However, history also can help to deepen considerably our questioning of how culture affects mathematics. Why are western cultures the only ones to use axiomatic-deductive systems to verify mathematical knowledge? Are other ways of knowing mathematics possible or desirable? Can one think differently than we do now?

A simple example of a positive answer to this question comes from pre-modern China. Figure 4a is a standard diagram representing two similar triangles; we all recognize that $\frac{a}{b}=\frac{c}{d}$. Chinese geometers and surveyors rarely used this ubiquitous tool. Rather, they relied on the *in-out complementarity principle*. In Figure 4b, the large rectangle is cut by a diagonal line. At any point on the diagonal, draw vertical and horizontal lines. The reader might pause here to consider why the two shaded rectangles have the same area … Now that you have paused, we see that the two rectangles' areas are $ad=bc$, which of course is equivalent to $\frac{a}{b}=\frac{c}{d}$. The two methods are able to accomplish the same geometric goals, but using the in-out complementarity principle can lead to diagrams and thought processes that look very different from those produced by the use of similar triangles.

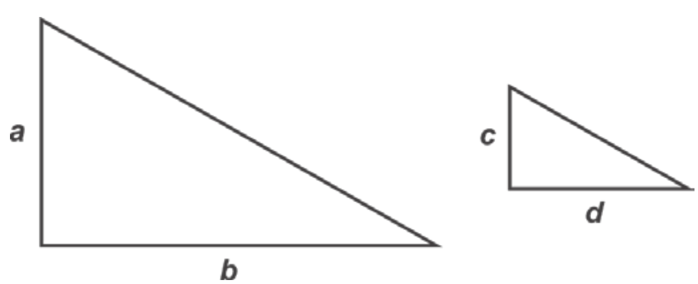

Figure 4a: *Similar triangles*

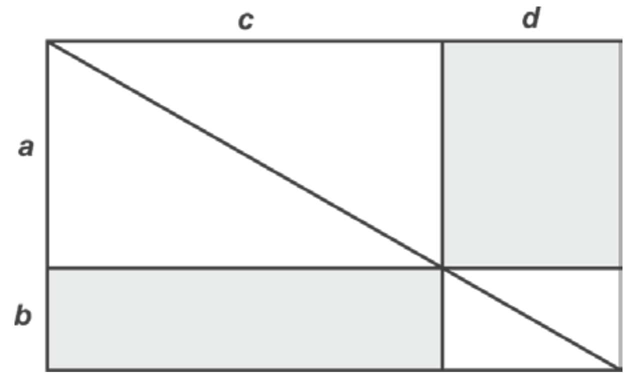

Figure 4b: *The in-out complementarity principle*

Of course, *who* has the opportunity to do the thinking is just as important as how the thinking is done. In this respect, the human race does not have a very good track record in allowing equal participation for everyone in all societal endeavours, including mathematics. It is an uncomfortable but unavoidable fact, for instance, that that very few women have been in a position to be able to contribute to pushing forward the boundaries of mathematical knowledge until quite recently (and even now, equality has not yet close). A history of mathematics course can engage in these issues readily. One approach that historians are starting to take to help bring female stories to the forefront is to illuminate their experiences with mathematics in other ways, especially in their education. This broadens what we mean by "the history of mathematics", making it more inclusive of all aspects of human experience. This approach is in its early stages, but see pp. 59-68 of Jacqueline Stedall's *The History of Mathematics: A Very Short Introduction* for a good start.

My colleague Clemency Montelle (University of Canterbury, NZ) and I are working on a book to bring episodes like these to the classroom and to the general public. In the meantime, we conclude with a list of resources for teachers to help design interesting and effective lesson plans.

References

Parts of this article were modified from "Why use history in a mathematics classroom?", *Canadian Mathematical Society Notes* 47 (1) (2015), 16-17, and a companion article in *Models and Optimisation and Mathematics Journal* ***3*** (1) (2015), 36-41.

Resources:

This list is not intended to be comprehensive. There are a number of general textbooks in the history of mathematics; some are more appropriate than others for a high school course.

Berlinghoff, William; and Gouvea, Fernando. *Math Through the Ages: A Gentle History for Teachers and Others*, 2nd edition, Farmington, ME: Oxton House, 2014.

» A compact and accessible, yet surprisingly deep text. Contains a nutshell history of the subject and 30 short episodes appropriate for the high school classroom. An expanded edition published by the Mathematical Association of America contains 60 extra pages of questions and projects.

Katz, Victor. *A History of Mathematics*, 3rd edition, Boston: Addison Wesley, 2009.

» At a higher level, but as authoritative as history of mathematics textbooks get.

Barrow-Green, June; Gray, Jeremy; and Wilson, Robin. *The History of Mathematics: A Source-Based Approach*, 2 vols., Washington, DC: MAA Press, 2019 (due to appear in May).

» A comprehensive text that emphasizes working with original historical materials.

Shell-Gellasch, Amy; and Thoo, John B. *Algebra in Context: Introductory Algebra from Origins to Applications*. Baltimore: Johns Hopkins University Press, 2015.

» Contains a wealth of historical episodes from number systems to number theory.

Stedall, Jacqueline. *The History of Mathematics: A Very Short Introduction.* Oxford, UK: Oxford University Press, 2012.

» Not a survey of historical developments in mathematics, but rather a short statement of recent approaches to the subject that are opening up the history of mathematics to new ways of thinking.

Joseph, George Gheverghese. *The Crest of the Peacock: Non-European Roots of Mathematics*, 3rd edition, Princeton, NJ: Princeton University Press, 2011.

» The most comprehensive general source for the mathematics of non-western civilizations.

Shell-Gellasch, Amy; and Jardine, Dick, eds. *The Courses of History: Ideas for Developing a History of Mathematics Course*, Washington, DC: MAA Press, 2017.

» A varied collection of approaches to designing a history of mathematics course.

MAA Convergence, https://www.maa.org/press/periodicals/convergence

» A high quality free online journal designed "to help you teach mathematics using its history". Geared to 8th grade through undergraduate.

The Story of Maths, BBC/Open University, 2008.

» This comprehensive four-part documentary series takes a multi-cultural approach. Its last episode, uniquely, takes on the implications of mathematical developments from the 20th century to almost the present day.

Transformational Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), https://blogs.ursinus.edu/triumphs/

» A five-year National Science Foundation-funded project to develop teaching modules for the undergraduate mathematics curriculum based on original sources. Some of the units deal with high school mathematics as well. They are looking for site testers at the moment.