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How algebra spoiled recreational problems: A case study in the cross-cultural dissemination of mathematics

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Abstract

This paper deals with a sub-class of recreational problems which are solved by a simple memorized rule resulting from an elementary arithmetical or algebraic solution, called proto-algebraic rules. Their recreational aspect is derived from a surprise or trick solution which is not immediately obvious to the subjects involved. Around 1560 many such problems wane from arithmetic and algebra textbooks to reappear in the eighteenth century. Several hypotheses are investigated why popular Renaissance recreational problems lost their appeal. We arrive at the conclusion that the emergence of algebra as a general problem solving method changed the scope of what is considered recreational in mathematics.

Sommario

Questo saggio tratta di una sottoclasse di problemi ricreativi risolti tramite memorizzazione di una semplice regola risultante da una soluzione algebraica o aritmetica, chiamata regola proto-algebraica. L'aspetto ricreativo di questi problemi deriva da una soluzione a sorpresa o da un trucco non immediatamente ovvi ai soggetti coinvolti. Intorno al 1560 svariati problemi di questo tipo sparirono dai manuali di algebra e aritmetica, per riapparire nel diciottesimo secolo. Diverse ipotesi sono vagliate sul perché problemi ricreativi popolari nel Rinascimento persero attrattività, per giungere alla conclusione che l'emergere dell'algebra come metodo generale di risoluzione di problemi cambiò la portata di ciò che era considerato ricreativo in matematica.

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1. Introduction

Anyone engaging in a study of recreational mathematics soon discovers that many of the problems that are still popular today go back a long time in history. In a recent book on the mathematics behind card tricks by two professional mathematicians (Diaconis and Graham, 2012, 106–114) the authors traced the history of a popular three-object divination problem to Prevost (Prevost, 1584) and Bachet (Bachet, 1612). However, after twenty years of research they were surprised to learn that the problem is a frequently recurring one in abbaco manuscripts of the *quattrocento* and appears in even earlier sources.

Equally surprising is the fact that the same or similar problems appear in very different cultures and geographical regions. One such example is the sliding ladder problem about a ladder of a given length which stands against a wall and which is moved from its original position on the ground. It features in several Old-Babylonian tablets BM 85196 (Høyrup, 2007, 275–6), BM 34568 (Friberg, 1981, 307–8), in Egyptian papyri, Cairo JdE. 89127-30 (Parker, 1972, 13–43), in Chinese classics, *Jiŭ zhāng suàn shù* 九章算術, *Nine chapters of the mathematical art* (Chemla and Shuchun, 2004), in Sanskrit mathematical texts, Bhāskara I. 629. *Commentary on the Āryabhatīya*, II.16 (Keller, 2006, 79–83) and also in the works by Bhāskara II, in Arabic texts, Hasib Tabarī, *Miftāḥ al-mu*ʻ*āmalāt* (Bagheri, 1999), in many abbaco manuscripts, e.g. Gherardi, *Libro di ragioni* (Florence, BNCF, Magl. Cl. XI, 86; Arrighi, 1987b) as well as in Fibonacci's *De Practica Geometrie* (Hughes, 2008, 77) and Pacioli's *Summa* (Pacioli, 1494, part 2, ff. 54^v–55^v), and in seventeenth century works on recreational mathematics (van Etten, 1624, prob. 89; Ozanam, 1725, 320–1, prob. 42). The interest in this specific problem thus not only spans a period of more than three millennia but also five very different cultures and mathematical practices! Though conceptualization, contextual meaning and solution methods may differ amongst these cultures, at least we can induce from this example that problems we now consider as recreational mathematics are omnipresent in mathematical practice and that these travel easily between different cultures.

One of the reasons for the multi-cultured aspect of recreational mathematics is that mathematical knowledge is often embedded in folk stories, riddles, tricks, tangible practices which form the basis for some types of recreational problems. Solution methods, rules and mnemonic devices are in concert with the problems. Many recreational problems have been disseminated as folk stories through merchant connections and trade routes. Embedding mathematics in cultural practices which can be adapted to suit the cultural context allows problems to cross cultural boundaries. That is the reason why the same problems turn up in such diverse cultures.

This paper presents a case study on a sub-class of recreational problems which are based on some elementary arithmetical or algebraic solution. Their recreational aspect stems from a surprise or trick solution which is not immediately obvious to the subjects involved. The legacy problem about a dying father with an unknown number of children (discussed below) is a challenge to solve unless you know the simple rule of thumb which gives you the answer to this and similar problems. The first book to coin the term recreational mathematics in the title (van Etten, 1624) often uses the qualification "wonder to those that are ignorant in the cause".[1] As in divination problems, it is important to conceal the mathematical principles behind the trick or problem to make it surprising and appealing. The title of this paper somewhat polemically states that this recreational aspect of problems disappears when a general solution method – as is algebra – is applied to solve such problems. I will indeed demonstrate that this is the case for some specific problems, while other types of recreational problems are less exposed to algebraic solutions.

I will first discuss the practical context of arithmetical problems. Renaissance recreational problems are often situated in a practical context to give them some flair or alleged utility. In the next part it is shown

[1] I have previously challenged the attribution of this book to the Jesuit Jean Leurechon, and proposed the printer/engraver Jean Appier Hanzelet as the compiler of the problems (Heeffer, 2006b). For further discussion on the authorship see my forthcoming critical edition (Heeffer, forthcoming).

how rules or solution recipes are embedded in such problems. I have termed these 'proto-algebraic' as the rule may have been derived by algebra but solving the problem by such rules is not algebraic. Eventually these problems also turn up in algebra textbooks becoming exercises and cease to be recreational.[2] In the following part I will give two related examples of a problem which was very popular before 1560 and then virtually disappears from mathematics books for two centuries. Several hypotheses will be discussed to explain such remarkable discontinuity. In a conclusion, I will provide arguments for the claim in the title that the scope of recreational mathematics was altered by the advent of algebra.

2. On Renaissance problems and their functions

Records of posing and solving Renaissance problems in the history of mathematics provide us the empirical grounds for the study of the development of problem-solving methods and techniques as well as the socio-cultural context in which they were formulated. These problems have functioned as vehicles for the transmission of new ideas, procedures and concepts in arithmetic, geometry and algebra from Babylonian times to the age of printing. The concrete setting in which the problems were situated facilitated memory, education and tradition and provided a justification for the activities involved. Alcuin's *Propositiones ad Acuendos Juvenes* (*Propositions for Sharpening Youths*) contains 53 problems of which many are repeated over and over again in medieval and Renaissance works.[3] As the title suggests, the problems were intended to be used for their educational value and to be read aloud for students to copy and solve.[4] The long history and tradition of recreational problems lets us connect sixteenth-century problems with much earlier sources. Some type of Renaissance problems occur already in the cuneiform texts of the Babylonians dating back to 1900–1700 BC. Others originated more recently and evolved within decades. Some problems remained unchanged during several centuries while others developed into more complex and challenging forms over time.

Most of these problems are formulated in a practical context which gives them a panache and purpose and make them acceptable within that context. For example, Thomas Digges published a book (Digges, 1579), reprinted in 1590, named *An arithmeticall warlike treatise named Stratioticos compendiously teaching the science of nombers as well in fractions as integers, and so much of the rules and aequations algebraicall, and art of nombers cossicall, as are requisite for the profession of a soldier*. As the title makes clear, the book is intended for army officers and every problem is thus set into the context of an army or warfare. Claiming a practical purpose was an important incentive for selling and producing textbooks on arithmetic from the sixteenth century onwards.

Many text books on arithmetic of the fifteenth to the seventeenth century carried the term 'practical' in the title; e.g. (Ciruelo, 1505; Peurbach, 1536; Ghaligai, 1521; Feliciano, 1526; Finé, 1532; Glarianus, 1539; Frisius, 1540; Des Barres, 1545; Morsianus, 1553; Pérez de Moya, 1562; Petri, 1567; Clavius, 1583; Cortes, 1604; Metius, 1611; Neufville, 1620; Malapert, 1620; Schott, 1662; Coets, 1698). Despite the advertisement as 'practical' by the authors, most of these books were of little practical value for the craftsman and the merchant class of that time. Notable exceptions are: (Feliciano, 1526) with practical geometry for surveying, and some problems of (Ghaligai, 1521) that have practical utility. Vernacular books as by Feliciano and Ghaligai were better suited for practical purposes than the Latin works listed above. They dealt with subjects such as the *Welsche praktijk*, methods for facilitating exchange calculations and commercial transactions (Swetz, 1989, 10–18; Kool, 1999, 157–167). Ad Meskens gives an excellent overview

[2] For a survey of the changing function of problems in algebra textbooks see Heeffer (2012a).

[3] The attribution to Alcuin (c. 800) is not fully certain. Translations are available only recently: a translation into German (Folkerts and Gericke, 1993); Hadley provided an English translation, which was published with notes by Singmaster (Hadley and Singmaster, 1992; Folkerts, 1978) gives a summary of problems from Alcuin in several medieval manuscripts.

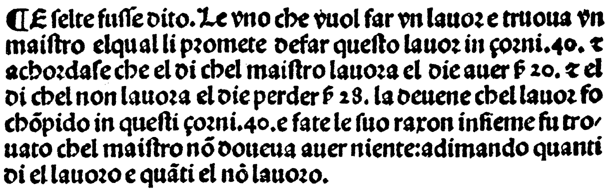
[4] For the educational context in which these problems were used, see (Bayless, 2002).

of the real mathematical needs for the practice of merchants and craftsmen during the Renaissance: money exchange, calculation of interests and annuities, navigation, surveying, gauging, fortification and ballistics (Meskens, 2013). The mathematicians from the Low Countries that contributed most to these practical problems in the sixteenth century were Valentijn Mennher and Michiel Coignet (Mennher, 1556; Coignet, 1597).[5] These authors deal with practical geometrical problems, such as determining the volume of vessel, which get no treatment in the frequently printed and reissued text books of the Latin tradition (Frisius, 1540; Finé, 1532). The determination of the practical utility of many of examples and problems in arithmetic books is problematic. We have to make a distinction between problems and examples that are formulated with objects and situations common to daily life and those that were intended for real practical use. Embedding problems in a concrete cultural context had more an educational purpose than a practical one. In fact, several sixteenth-century authors expressed their doubt about the practical use of some of their problems. William Kempe, who translated Peter Ramus' arithmetic book (Ramus, 1569) into English, writes in the chapter about cistern problems: "Of this sort also are devised divers other questions, which never or seldome happen in practice, and therefore have scarce anie use or fruite at all" (Kempe, 1592, 52).

3. A preliminary case study of a recreational problem and its solution methods

3.1. The role of recipes

In the beginning of the sixteenth century, we find many determinate problems with one or more equations and unknowns that can easily be solved algebraically. However, an algebraic solution is less common as would be expected. Let us look at one specific problem in some detail. The problem has become known as 'the lazy worker problem' and deals with a worker who has to perform a job in a number of days. He is paid a sum for every day he works but has to return a payment for every day he does not work. After the given period he ends up with a certain sum, in most cases zero. One is asked to find how many days he worked and did not work. Our first acquaintance with this problem was from the *Libro de Abacho*, one of the earliest printed book on arithmetic, shown in Figure 1 (Borghi, 1484, f. 111^v, literal translation mine):

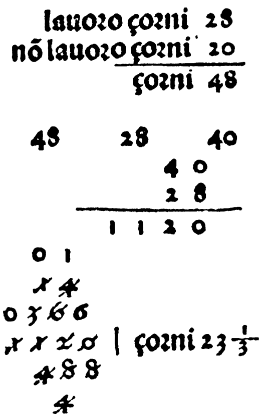

¶ E felte fusse dito. Le vno che vuol far vn lauor e truoua vn maistro elqual li promete defar questo lauor in çorni.40. e achordase che el di chel maistro lauora el die auer ß 20. e el di chel non lauora el die perder ß 28. la deuene chel lauor fo chõpido in questi çorni.40. e fate le suo raxon insieme fu trouato chel maistro nõ doueua auer niente: adimando quanti di el lauoro e quãti el nõ lauoro.

Someone wants to do a job and finds a master whom he promises to conclude the work in 40 days agreeing that he is paid 20 soldi for every day he works and that he has to pay back 28 soldi for every day he does not work. When doing his calculation it was found that the master does not have to pay anything. The question is how many days have been worked and how many did he not work.

Figure 1. Illustration of the lazy worker problem from Borghi (1484).

Let us look at the solution in some detail. Borghi gives a solution which applies "always when you have similar problems". In a loose translation the proposed method is as in Figure 2:

[5] Specific for interest calculation we can add Jean Tenchant, Martinus Wentsel and Simon Stevin. See the dissertation of Waller Zeper for a comprehensive overview (Waller Zeper, 1937).



Put first how many soldi he earns for the days that he has worked and then how many soldi for the days he has not worked and then proceed as I will demonstrate. Thus for the days that he earns money we put 20 soldi and for the days that he loses money 28 soldi. Thus we put against the days that he works 28 soldi in such a way against the days that he does not work, 20 soldi. Together 28 soldi and 20 soldi makes 48 and this is the work he has done in 48 days of which 28 days he has worked and 20 days he has not worked, and has received nothing. But because the problem tells us that the work is completed in 40 days, say thus that in 48 days he worked 28 and as he should have worked 40 days, subtract from the days that he should have worked 23 1/3. Thus to know how many days he did not work: subtract 23 1/3 from 40 which leaves 16 2/3 and this is the number of days he did not work. From (Borghi, 1484, f. 111^{v}).

Figure 2. Borghi's solution to the lazy worker problem.

The problem can be solved by two linear equations in two unknowns, x for the days working and y for the days absent:

$$\begin{array}{l} x+y=c \\ ax-by=d \end{array} \quad \text{with solutions} \quad x=\frac{bc+d}{a+b} \quad \text{and} \quad y=\frac{ac-d}{a+b} \tag{0.1}$$

The problem values will be denoted as (a, b, c, d)

However, an algebraic solution with two equations in two unknowns, as represented here, turns up only in the eighteenth century![6] Vogel, in a section on *Der Arbeiter im Weinberg* (Vogel, 1954, 222–3), Tropfke in *Der faule Arbeiter* (Tropfke, 1980, 603) and Singmaster (Singmaster, 2004, section 7.AK) give a history of the problem. Sesiano calls the problem *l'ouvrier paresseux* (Sesiano, 1984, 147). The oldest documents in which the problem can be located is in Arabic texts such as the *Fakhrī* by Al-Karkhī from the early eleventh century. Al-Karkhī gives three instances of the problem set in the same context of a day-worker (Sections 2, 12, 13 and 14), Woepcke (Woepcke, 1853, 83) lists three instances of the problems with values

$$\left(\frac{10}{30}, \frac{6}{30}, 30, 0\right), \qquad \left(\frac{10}{30}, \frac{6}{30}, 30, 4\right) \quad \text{and} \quad \left(\frac{10}{30}, \frac{6}{30}, 30, -2\right).$$

Unfortunately, Woepcke does not include a transcription of the solution text, only the equations as below:

$$\begin{aligned} \frac{1}{3}x &= \frac{1}{5}(30-x) \\ \frac{1}{3}x &= 4+\frac{1}{5}(30-x) \\ \frac{1}{3}x+2 &= \frac{1}{5}(30-x) \end{aligned}$$

The most likely route of transmission to Europe seems to go through Fibonacci who gives two instances of the problem in his *Liber Abbaci* (Boncompagni, 1857, 160–1 and 323; Sigler, 2002, 250–1 and 453–4) and through the abbaco tradition with *La practica di geometria*, by Gherardo de Dino (Florence, Biblioteca

6 Algebraic solutions using one unknown are discussed below.

Riccardiana, codice 2186; Arrighi, 1966, f. 323). The problems are, as with Al-Karkhī, formulated as a payment per month, instead of per day. The first problem *De laboratore laborante in quodam opere* (7/30, 4/30, 30, 1), is solved by what Fibonacci calls "the method of companies". This rather unique procedure is illustrated in the *Liber Abbaci* by a marginal drawing as in Figure 3.[7] In order to explain the method, Figure 3 replaces the quantities of the problems (using the wages per month) by the symbolic expressions from (0.1). In this way we can demonstrate that adding c to a, b and d is part of the solution procedure.

$$\begin{array}{ccc} d+b & & a-d \\ a+c & & c-b \\ & d+c & \end{array}$$

Figure 3. A marginal illustration of Fibonacci's 'method of companies'.

The 'method of companies' is a term used for proportional division. It is derived from the principle that business companions should receive a part of the profit proportional with their contribution to the capital. In this case, proportional division refers to c, the numbers of days, to be divided in proportion to the parts $b+d$ and $a-d$. Thus dividing $c=30$ (not accidently, the number of days in a month) in the proportion of $b+d=5$ to $a-d=6$, he thus arrives at the solution

$$13\frac{7}{11} \quad \text{and} \quad 16\frac{4}{11}.$$

Except for a numerical test he gives no further argumentation why the method is correct.[8] Adding $b+d$ and $a-d$ gives $a+b$, thus using proportional division we get:

$$x = c\frac{b+d}{a+b} \quad \text{and} \quad y = c\frac{a-d}{a+b}.$$

Dividing the monthly wages by c, we should arrive at

$$\frac{a}{c}x - \frac{b}{c}y = d \quad \text{or} \quad a\frac{b+d}{a+b} - b\frac{a-d}{a+b} = d$$

and indeed $\frac{(a+b)d}{a+b} = d$.

The second, *De laboratore, question notabilis* (7/30, 4/30, 30, 30), is solved by the general method of double false position, which he borrows from the Arabs by the name *elchataym*. We find solutions to this problem with double false position over the next centuries: in Munich, Bayerischen Staatsbibliothek, Cod. Lat. Mon. 14908, f. 48^{r} (Curtze, 1895c, 41), in the *Summa* (Pacioli, 1494, f. 99^{r}), the very rare *Compendion de l'Abaco* (Pellos, 1492, f. 71^{v}–f. 72^{r}), the second *Rechenbuch* of Adam Ries (Ries, 1522; 1574, ff. 60^{v}–61^{r}), in the *General Trattato* (Tartaglia, 1551–1560, Bk. I, f. 275^{r}), and Robert Recorde (Recorde, 1552, 312–6). It is important to note that in both cases, Fibonacci does not give an algebraic solution. It might seem plausible that Fibonacci has the problem from Al-Karkhī, because we know that several of his problems from the last chapter are taken from this source.[9]

[7] In the English translation (Sigler, 2002, 624), a footnote is added: "Leonardo uses 37 down to 26 instead of 7 down to −4 in order to avoid negative numbers". However, this interpretation makes no sense.

[8] A numerical test involves checking the correctness of the solution by placing the calculated values back in the problem enunciation.

[9] This has been argued by (Woepcke, 1853).

There could be an alternative path of transmission. A Greek manuscript, Paris, BNF, Cod. suppl. Gr. 387, written c. 1308, has the same problem with values (10, 4, 30, 132) on f. 127^v (Vogel, 1968, 69). The μέϑοδος in the manuscript gives the following recipe:

> Add 10 and 4, which makes 14. Also takes four times 30 days, makes 120. This added to 132 gives 252. From this you take the 14th part which gives 18. That is the number of days he worked, and 12 days he did not work.

The recipe thus corresponds with the procedure

$$x = \frac{bc + d}{a + b}.$$

This rule is applied to problems without any derivation or justification. Although this manuscript is dated after the *Liber abbaci*, the Byzantine tradition preceding Pachymeres and Planudes, could be a plausible source for several of Fibonacci's problems. His use of *bezants* as currency unit, in this and many other problems, gives strong support to a Byzantine influence. Kurt Vogel, who has studied and published several Byzantine arithmetic manuscripts, has always stressed the role of Byzantium as an intermediary in the transmission of Ancient and Arabic mathematics to the West.[10] Fibonacci, as often in the *Liber Abbaci*, is not contended with the application of meaningless recipes. The application of "the method of companies" is most likely an original solution method of his own.

3.2. Identifying the solution type

We have shown three different methods for solving the lazy worker problem in some early sources: the solution from the *Liber Abbaci* using proportional division, the resolution of an expression in one unknown by the rule of double false position and the Byzantine recipe. These solution types can now be used for further identification of influences. Before doing so, we have to add a simplified recipe, specifically for the case in which $d = 0$. This type of problems is found in some early Italian and German manuscripts. An anonymous manuscript of c. 1290, New York, Columbia University, X511, solves an instance of the problem with values (5, 9, 30, 0) as well as (4, 7, 30, 1) (Vogel, 1977, 86–7).[11] In our symbolic representation this leads to $ax = by$, $x + y = c$. The solution then amounts to the proportional division of c into a and b. This can be accomplished as follows:

$$x = c\frac{b}{a + b},$$
$$y = c\frac{a}{a + b}.$$

Representing the procedure in this way, we can see that the recipe can be understood as an application of the rule of three: if someone earns $a + b$ in c days, in how many days does he earn b? The abbaco text indeed formulates the procedure referring to the rule of three.[12]

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10 In fact, the last part of this sentence is the title of one of his articles (Vogel, 1978). See also (Vogel, 1962, 1964, 1968).

11 Vogel dates the manuscript at 1350 but Høyrup has argued that 1288–1290 is a better estimate (Høyrup, 2007, 31, note 70). This makes the treatise the second oldest extant abbaco text.

12 Columbia, X511 (Vogel, 1977, 86–7): "Queste la sua diritta reghola, chomo si dei fare, che tur dei dire 5 e 9 fanno 14 e chosi di, ch'ello die sia 14 ore che le 9 ore lavora e le 5 si posa. Si voli fare in 30 die si die: se 14 fusse 30, che seria 9? Di: vua 30 fanno 270 a partire 14, che nne viene 19 die e 4/14 di die e tanto lavorào e si noi [volemo] savere, quanto [no a] lavorato, si di: se 14

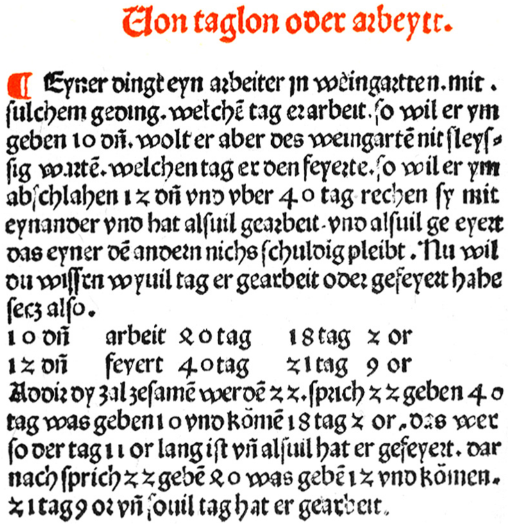
It is difficult to establish if the recipes were derived from the rule of three or that the rule of three was given as a justification for the validity of an earlier formulated recipe. Given that the more general form of the recipe appears in previous Byzantine manuscripts, the second seems more plausible. An additional argument is that the author makes some mistakes in applying the recipe. Namely, in the second example with $d = 1$, he again applies the recipe and obviously reaches the wrong solution.[13]

Tartaglia approaches the case with $d = 0$ with the recipe but solves the two cases with $d \neq 0$ by double false position (problem 39 and 42) (Tartaglia, 1551–1560, Bk. I, f. 275^r). Also manuscripts from German monasteries use this very recipe. A collection of 350 problems is found in the so-called *Algorismus Ratisbonensis* written c. 1450 and preserved in several Munich manuscripts. Number 183 is our problem with values (10, 12, 40, 0) in Munich, Bayerischen Staatsbibliothek, Cod. Lat Mon. 14908, f. 100^v.[14] The recipe also refers to the rule of three:

> Do as such: add 10 and 12, arriving at 22, the divisor. Say 22 gives 40, what shall give 10? Makes 18 days and 2 hours. When a day makes 11 hours, this is how much he took free.

Exactly the same problem and the solution is repeated in the *Bamberger Rechenbuch* of 1483 (see Figure 4).

In some cases the rationale of the proportional division is missing and only the recipe survives. This the case in the *Memoriale* of Francesco Bartoli (c. 1420, Avignon, Archives départementales du Vaucluse, cote 1F 54) who applies it to the case (9, 11, 30, 0) (Sesiano, 1984, 137). This notebook of an Italian business man travelling between Italy and the south of France at the beginning of the fifteenth century gives us an unique insight into the paths of transmission of problems and solution methods throughout Europe.


Von taglon oder arbeytt.
Eyner dingt eyn arbeiter jn weingartten.mit.
sulchem geding.welchẽ tag er arbeit.so wil er ym
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das eyner dẽ andern nichs schuldig pleibt. Nu wil
du wissen wyuil tag er gearbeit oder gefeyert habe
secz also.
10 dñ arbeit 40 tag 18 tag 2 or
12 dñ feyert 40 tag 21 tag 9 or
Addir dy zal zesamẽ werdẽ 22. sprich 22 geben 40
tag was geben 10 vnd komẽ 18 tag 2 or. das wer
so der tag 11 or lang ist vñ alsuil hat er gefeyert. dar
nach sprich 22 gebẽ 40 was gebẽ 12 vnd komen.
21 tag 9 or vñ souil tag hat er gearbeit.

Figure 4. The lazy worker problem from the Bamburger Rechenbuch, 1483, f. 46^v.

fusso 30 che seria 5? Di: 5 via 30 fanno 150, parti in 14 che nne vie[ne] 10 die 10/14 e die e tanto [no] lavora e per questo modo si fa tutte le altre die più o di meno".

13 Vogel speculates that the author jumbles up several problems and that this solution corresponds with $7x - 4y = 15$ (Vogel, 1977, 87). Instead, we believe that he did not have a recipe for the case that $d \neq 0$. Apparently he divides the surplus florin into two and adds and subtracts the halves from the florins per hour, which are our coefficients ("vedi che tenne per florino l'ora che in unna ora", f. 32^r).

14 (Vogel, 1954, 86): "Machβ also: addir 10 et 12, erit 22, divisor. Dic 22 dant 40, quod dat 10, facit 18 dies 2 horas, hoc est, quando dies est 11 horarum und alz vil hat er gefeirt".

In addition to arithmetic tools such as the rules of exchange, multiplication tables, the itinerary from Florence to Avignon and price lists, it contains a collection of thirty problems of the recreational sort. We can assume that Bartoli was just one of the many links in the trade routes by which the tradition of arithmetical problem solving was passed from Italy to France and the Low Countries.

In conclusion we can state that a single type of recreational problems as the lazy worker problem can be solved by different means: solution recipes for specific cases or general rules such as the rule of three and the rule of false position. We can identify Borghi's solution as an application of the recipe of proportional division given the case that $d = 0$. Although this simplified recipe, which appears first around 1350, could have been derived from an algebraic solution, the method applied by Borghi is not algebraic. Vogel makes a similar observation in relation to the Byzantine text.[15]

3.3. Algebraic solutions

As the main thesis of this paper is that some types of recreational problems lost their appeal once they became solved by algebra, we will now look at some early algebraic solutions to the lazy worker problem. Given that algebra was introduced in Western Europe since the twelfth century, it is rather surprising that an algebraic approach to such simple problem appears only in the late fifteenth century. The earliest Italian source is Piero della Francesca's *Trattato d'abaco*, written in 1480, Florence, Biblioteca Medicea Laurenziana, Ashburn. 359* (Arrighi, 1970). Piero has several linear problems which he solves by *La regula de la positione*, but he also adds some algebraic solutions. The lazy worker problem is presented as follows, Florence, Biblioteca Medicea Laurenziana, Ashburn. 359*, f. 38^v (Arrighi, 1970, 99):

> Uno dà a fare uno suo lavoro e fa pacto cello maestro de darli 20 soldi el dì che lui lavora, et il dì che non lavora vole che il maestro dia 16 soldi a lui, e sono d'acordo, et che il maestro de' fenire i' lavoro in 30 dì. Et capo de 30 dì, il maestro à finito il lavoro; fano ragione, nisuno à dare l'uno a l'altro. Domando quanti dì lavoro, et quanti non lavorè.

Piero uses the unknown for the days worked and thus arrives at

$$20x - 16(30 - x) = 0 \quad \text{(without actually writing the zero).}$$

This leads to $36x = 480$ or,

$$13\frac{1}{3} \quad \text{for the days worked} \quad \text{and} \quad 16\frac{2}{3} \quad \text{for the days not worked.}$$

He adds a numerical test as a proof of the validity of the solution.

But Piero was not the first. The earliest algebraic solution we could find is by Frederic Amann who wrote an algebraic text in 1461, preserved in Munich, Bayerischen Staatsbibliothek, Cod. Lat. Mon. 14908.[16] After solving some problems by the regula falsi, he adds algebraic solutions to the same problems. Our problem with values (5, 3, 28, 0) is formulates as follows (Curtze, 1895a, 68):

> Quidam conventus est ad laborandum per 28 dies. Die laboris 5 habet, die vacacionis restituet 3; in fini nullum lucrum habuit. Quot dies laboravit?

---

15 (Vogel, 1968, 149): "Und doch benützt der Verfasser in zahlreichen Fällen rezeptmässichen Formeln, die aber nur in vorhergehenden algebraischen Operationen gefunden sein konnten, wobei er sich keinerlei Gedanken über deren Herleitung macht". The question of how these recipes could have been derived from algebraic solution is discussed in (Heeffer, 2006a).

16 (Tropfke, 1980, 603) only mentions the solution with regula falsi, (Singmaster, 2004) has neither.

Using the *coss* x for the days worked, the earning is $5x$. The money returned is the number of days not worked $(28 - x)$ times 3. Both are equal and Amann constructs the equation '84 minus $3x$ equande $5x$', which gives the solution 10 1/2.

The problem with (7, 5, 40, 0) appears with an algebraic solution in the margins of the Dresden codex C80. The text, written in 1481 is attributed to the monk, Aquinas Dacus.[17] The marginal notes appear to be of the hand of Johannes Widmann. His solution is the same as that of Amann on f. 355^r (Wappler, 1899, 543):

> Pone eum laborasse ad $1x$, quare $1x$ per 7 multiplica, et erunt $7x$, et 40 dies ocij $-1x$ per 5 multiplica, et erunt $200 - 5x$, que ex ypothesi equivalent. Restaura ubique $5x$ addendo, et stabunt $12x$ ex una parte 200 Ø, quare iuxta primi capituli preceptum operare, et patet valor x, scilicet laboravit 16 diebus 2/3, vacauit ad 23 dies 1/3 mervit 116 2/3, et tantum opportuit ipsum restituere.

Adam Ries writes in his *Coss* (Dresden, Sächsische Landesbibliothek, Codex C80) that he found the example in the gloss of 'ein alten Buch', identified by Berlet as from Andreas Alexander (Berlet, 1860, 20). This is in fact, Munich, Bayerischen Staatsbibliothek, Cod. Lat. Mon. 14908, discussed above.[18] Ries gives an algebraic solution for the problem with values $(3, 5, 28, 0)$ (problem 37) and another one with $(11, 8, 36, -12)$ in problem 110 (Berlet, 1860, 29). Christoff Rudolff solves the problem with values $(7, 5, 30, 6)$ using a single unknown as $7x - 5(30 - x) = 6$ and arrives at 13 and 17 (Rudolff, 1525, f. Oii^r; Stifel, 1553, ff. 278^r–278^v). He then discusses the cases with $d = -30$ and $d = 0$. The youngest source in which we find the problem is treated algebraically is from the sixteenth century by Caspar Peucer[19]:

> Oeconomus conducit operarium in dies triegenta, ea conditione, ut promittat soluturum se ei in dies singulos 7 detracturum se rursus minetur 5 mumulos in singulos dies, quibus opere intermisso cessarit. Circumactis 30 diebus neuter elteri quidquam debet. Quaeritur ergo quot diebus vacarit operi imposito, quot cessarit?

This is Rudolff's third case with values $(7, 5, 30, 0)$. Peucer's solution is noteworthy as he combines algebra with the rule of three. He uses x for the number of days worked, thus the number of days not working is $30 - x$. As one day work earns 7, x days must earn $7x$. Also, if one day not working pays 5, $30 - x$ days not working results in $150 - 5x$. As both these are equal he constructs the equation $7x = 150 - 5x$ and arrives at the solution 12 1/2 and 17 1/2.

3.4. Proto-algebraic rules

We have looked at one problem from Borghi's book which corresponds with the arithmetic of determining the value of x in

$$x = \frac{bc + d}{a + b}.$$

[17] Little is known about this intriguing figure. Aquinas Dacus was a monk of the Prediger order (Vogel and Stein, 1981, 10). Regiomontanus mentions him in his letter of 4 July 1471 to Christian Roder: "Multa equidem de tua excellentia cum ex aliis plerisque omnibus Erfordia venientibus, tum ex fratre Aquino volupe intellexi" (Curtze, 1902, 325). He was a teacher of Andreas Alexander and Andreas Stöberl (Stiborius) and put his knowledge of algebra available in exchange for money.

[18] Not only problem 37 of Ries, but also his previous one nicely correspond with the gloss from the Clm 14908.

[19] From (Peucer, 1556, ff. Tv^r–Tv^v), not mentioned by (Tropfke, 1980) or (Singmaster, 2004). Peucer's *Logistice* has received some attention only very recently (Meiner and Deschauer, 2005) with a fairly complete description and a German translation of some fragments.

The procedure in itself is not algebraic but depends on the application of a prescribed recipe. Given the right recipe for the right type of problem, the method guarantees a solution. We have also seen that in some cases the application of a recipe to a different problem type results in errors. Such recipes have been called ‘proto-algebraic rules’, because these rules can be and could have been derived algebraically though its application as such does not constitute algebra (Heeffer, 2007). The notebook of Bartoli is a rare historical record that shows how problems and their corresponding proto-algebraic rules disseminate over Europe during the Renaissance. The many instances we have found for this problem and its solution by a proto-algebraic rule show the popularity of the problem type and the widespread use of solution recipes. The many authors we have listed seemed to have found pleasure in presenting and solving this problem in their treatises.

In the later part of this problem’s life cycle we find more and more algebraic solutions in textbooks. After Peucer, the problem disappears from textbooks for almost two centuries. We have even evidence of a case in which Bombelli, some years later, decided to retract this and similar problems from a book published in 1572 (discussed below). In order to understand why such problem disappears, let us see if this was a systematic trend at the second half of the sixteenth century.

4. Some instances of discontinuity

In order to demonstrate that several types of recreational problems virtually disappear from arithmetic and algebra books by the end of the sixteenth century, we will now give two typical examples. It is difficult to prove that there are no instances to be found of these problems between the 1560s and the eighteenth century. Possibly, some versions of the problems here discussed do appear somewhere in books we have not consulted. However, we have traced the most representative ones and our findings can be confirmed from the existing classifications (Tropfke, 1980; Singmaster, 2004). In fact, Tropfke’s lists of these problems ends around the 1560s, an indication that little was found after this date. To show that some other types of problems did continue to be discussed after 1560, we will conclude with one counterexample.

4.1. The monkey and coconut problem

One type of a problem that was popular before the end of the sixteenth century is called *Schachtelaufgaben* (Tropfke, 1980, 592) but better known under its twentieth-century name “monkey and coconut problem” (see Table 1). The name is derived from a short story by Ben Ames Williams, published in a newspaper in 1926.[20] It is about five sailors that are shipwrecked on an island. They gather coconuts all day. During the night each of the sailors takes one fifth of the remaining coconuts and gives one to the monkey. In the morning they each get one fifth, again leaving one coconut for the monkey. The question is how many coconuts they had gathered. This type of problem developed from a simpler version, popular during the Renaissance. For example, a Flemish manuscript by van Varenbrakens of 1532 gives the following formulation[21]:

> There was a hermit who was in serious need for money. This hermit went to the church to stand for Saint Paul and prayed him to double the money in his purse in exchange for six groat. Saint Paul agreed to this. Then he went to Saint Peter and also asked him to double what was left in his purse. Saint Peter agreed, and the hermit gave him six groat. Then he went to Saint Francis and also asked him to double what was left in his purse in exchange for six groat, which he agreed to. In the end he was left with nothing because six groat was all that he had. The question is: how much had the hermit in his purse to begin with.

20 The story about the publication and the many reactions it provoked is described by Martin Gardner (Gardner, 1958). See also (Singmaster, 1997, 2004).

21 (van Varenbrakens, 1532, Ghent, Universiteitsbibliotheek, Hs. 214, f. 158). Transcription by Kool (Kool, 1988). English translation mine.

Table 1
An overview of the Monkey and Coconut problem, showing the discontinuity of its treatment in arithmetic books.

Author	Date	Book	Problem	Page
Gherardi	1328	*Libro di Ragioni*	(unnumbered)	47–48, 100
Anonymous	1330	*Libro d'Abaco* (Lucca, Biblioteca Statale, Ms. 1754)	(unnumbered)	26r–27r, 59v
Paolo dell'abbaco	1339	*Trattato di Tutta l'Arte dell'Abacho*	(unnumbered)	25r–25v
Biagio	1340	*Trattato di praticha d'arismetricha*		
Gilio	1384	*Arithmetica e geometrica*	40, 41	145r–146r
Anonymous	13xx	*D'astucie algorismi*	6	8r–11r
		BL, Royal 12 C XII		
Anonymous	13xx	*Quaestiones arithmeticae*	3	17v–21r
		BL, Cotton Cleop. B IX		
Anonymous	1417	Turin, N.III.53	(unnumbered)	11v
Munich Clm 14684	14xx	*Incipiunt subtilitates enigmatum*	5, 6, 36	30r, 32v
Bartoli	1420	*Memoriale*	9	75v
Paolo Dagomeri	1440	*Trattato d'arithmetica*	47, 71	44, 65–7
Fridericus Gerhart	1450	*Algorismus Ratisbonensis*	185, 187	
Pierro della Francesca	1480	*Trattato d'abaco*	73, 74, 97, 98, 104	23r, 37v, 41v
Chuquet	1484	*Triparty*	30–33	
Johannes Widmann	1489	*Behende und hubsche Rechenung*	3 problems	97v–98v
Calandri	1491	*Arimethrica*	(unnumbered)	66v
Luca Pacioli	1494	*Summa di Arithmetica et Geometria*	22	150v
Luca Pacioli	1500	*De viribus quantitates*	67	120v
Johannes Köbel	1514	*Rechenbuechlein auf den linien mit Rechenpfeningen*	(unnumbered)	89r
Ghaligai	1521	*Practica d'Arithmetica*	29	66v
Tunstall	1522	*De Arte Supputandi*	43, 44	173–4
Adam Riese	1524	*Die Coss*	53, 58, 65, 66, 92, 94, 109, 110	
Christoff Rudolff	1525	*Coss*	5	
van Varenbrakens	1532	*Die Edel Conste Arithmetic*		
vanden Hoecke	1537	*Een sonderlinghe boeck*		
Nicolo Tartaglia	1556	*General Trattato*	(Bk, par) 12, 34; 16, 47; 17, 9; 17, 20	199v, 113–6, 246r, 253v–254r
Jean Trenchant	1558	*L'arithmétique*, Book 3	6	(1618) 321
Johannes Buteo	1559	*Logistica*	6, 13, 19, 20, 21	334–350
Humfrey Baker	1562	*Well spring of sciences*	7, 8	193r–193v
Jacques Ozanam	1725	*Recreations Mathematiques*	28	211–2
Thomas Simpson	1745	*A treatise of algebra*	XV, XXVII	86–89
Leonhard Euler	1770	*Vollständige Anleitung zur Algebra*	added in (Euler, 1822)	204

This formulation as an oblation by a poor man or hermit originated in the fifteenth century and was popular until the early sixteenth century but the problem is still older. Other versions include a merchant who gains and spends money at different places and a girl who collects apples in a garden but has to give the guards a share of the apples. Singmaster (2004) gives a comprehensive history of this type of problems, which also includes indeterminate versions of the forms

$$x_{i+1} = \left(\frac{n-1}{n}\right)x_i \quad \text{and} \quad x_{i+1} = \left(\frac{n-1}{n}\right)(x_i - 1).$$

Its history goes back to Babylonian, Chinese and Indian sources, but several formulations of the problem appear in medieval Europe until halfway the sixteenth century. A Byzantine source lists two versions (Vogel, 1968, 109 & 135, problem 89 and 119). The *Libro d'Abaco* of c. 1330 has three versions, Lucca, Biblioteca Statale, Ms. 1754 (Arrighi, 1973, 64–5 & 135). The *Trattato d'Aritmetica* of the early fourteenth century gives the apple garden version twice (see Figure 5).[22]

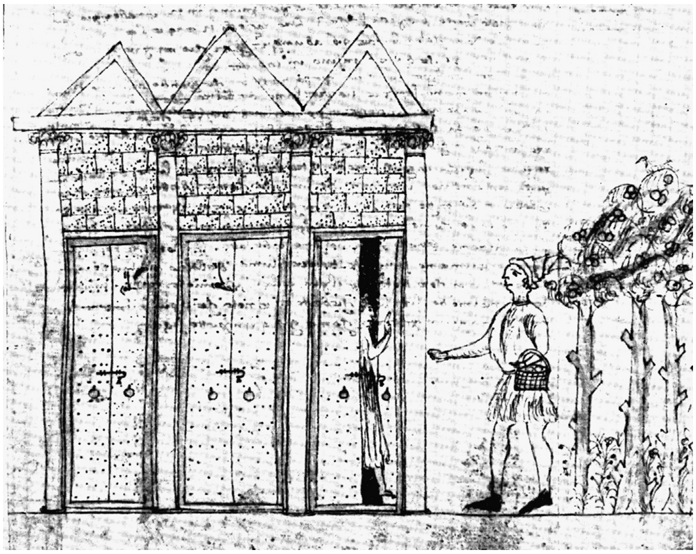


Figure 5. Three porters in an apple garden (from the *Trattato d'Arithmetica*, c. 1340).

Folkerts found 17 instances of this type of problems in fourteenth and fifteenth-century manuscripts (Folkerts, 1968). The *Algorismus Ratisbonensis* (c. 1450), has 2 examples of the problem (Vogel, 1964, prob. 185 and 187). All important Renaissance treatises deal with the problem: Bartoli's *Memoriale* (Sesiano, 1984, 138 & 148), Munich, Bayerischen Staatsbibliothek, Cod. Lat. Mon. 14684 (Curtze, 1895a, prob. 5), (Chuquet, Paris, BNF, Français 1346, prob. 30–33; Calandri, 1491, f. 66v & 74r; Pacioli, 1494, prob. 22; Köbel, 1514; Ghaligai, 1521; Tunstall, 1522, question 43 and 44; Riese, Dresden, Sächsische Landesbibliothek, Codex C80, *aufgabe* 53; Tartaglia, 1551–1560, Book 17, art 9 and 20; Trenchant, 1558, prob. 6; Buteo, 1559, prob. 21). The problem, as given here, can easily be solved by reversing the order and calculating backwards. In Indian mathematics the method was called *viparītakarma* or *Rule of inversion*. Āryabhata I prescribes the *viparītakarma* for the four arithmetical operations, but this was extended to include squaring and roots by Brahmagupta and Bhāskara. The Bakhshālī manuscript has examples formulated as business trips, c. 700 (Hayashi, 1995, Sūtra C1).

[22] BNCF, Magl. XI, 86 (Arrighi, 1964, 44 & 65–7), problem 47 and 71. Arrighi attributes the manuscript to Paolo dell'Abaco but this is disputed (Van Egmond, 1977; 1980, 114–5).

Fibonacci gives over twenty variations of the problem under the heading *De Viagiorum propositionibus* (business trips). He solves the problem by calculating the fractional parts (Sigler, 2002, 372–383) and gives an alternative solution using the rule of double false position (Sigler, 2002, 460). In the *Algorismus Ratisbonensis* the problem is also solved by the double regula falsi (Vogel, 1954, Prob. 185).

The inversion method is also found in later European texts. Widman gives a problem of business trips by three times halving and adding four, leaving 3 1/2 (f. 97^{v}), and another one adding two, leaving one (Widman, 1489, f. 100^{r}). He coins the name *Regula transversa* for the rule of calculating backwards.[23]

Pacioli discusses the possible solutions in his unpublished manuscript *De viribus Quantitates* (see Figure 6).[24] According Pacioli, the problem can be solved, (1) by a standard rule, (2) in the ordinary way, (3) by the rule of double false position and (4) by algebra. As he has "no intention to fatigue" his audience, he proposes the "general rule which can be applied without any mental exertion". His general rule is calculating backwards. The ordinary way is presumably the calculation of the fractions. Some years before, he discussed two versions of the problem in the *Summa* (Pacioli, 1494, f. 105^{v}, problem 21 and 22). For the first he calculates the fractions and the second he solves by calculating backwards. Four more complex versions are given in the distinction (i.e. chapter) dealing with business trips (Pacioli, 1494, ff. 187^{r}–188^{r}). These are solved by algebra. For example, problem 10 concerns an unknown number of business trips in which each trip contributes 20% to the starting capital. At the end, it is found that he the capital has become the square of the starting capital.

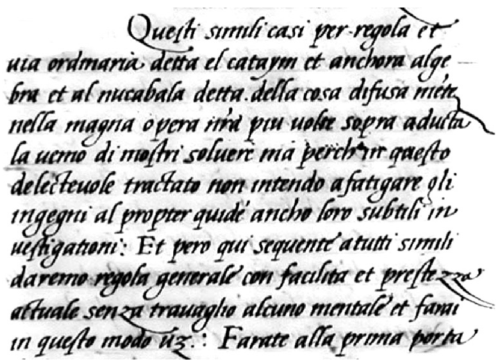

Questi simili casi per regola et
uia ordinaria detta el catayn et anchora alge-
bra et almucabala detta della cosa difusa mente
nella magna opera nrã piu uolte sopra adutta
la uemo di nostri soluere ma perche in questo
delecteuole tractato non intendo afatigare gli
ingegni al propter quidē ancho loro subtili in
uestigationi: Et pero qui sequente atutti simili
daremo regola generale con facilita et prestezza
attuale senza trauaglio alcuno mentale et fanno
in questo modo uz: Farate alla prima porta

Figure 6. Pacioli discusses the possible solutions to the monkey and coconut problem.

Pacioli was not the first to apply algebra to the problem with business trips. Such problems and their algebraic solution appear regularly in the *Trattato di praticha d'arismetricha* of Biagio The Old, c. 1340, Siena, Biblioteca Comunale, L.IV.21, and the *Arithmetica e geometrica* of Maestro Gilio, 1384, Siena, Biblioteca Comunale, L.IX.28.[25]

Algebraically, our problem from van Varenbrakens results in a simple determinate equation with one unknown:

$$2\big[2(2x-6)-6\big]-6=0 \quad \text{with solution} \quad x=5\ 1/4.$$

[23] An alternative name for the *Regula transversa* is the (first) *Regula pulchra*. For further discussion on these rules, see below.

[24] Bologna, Biblioteca universitaria, no. 250, f. 120^{v}. A comprehensive study of Pacioli's *De Viribus* can be found in (Amedeo Agostini, 1924) and a transcript is published by (Peirani, 1997). Dario Uri has pictures of the complete manuscript on-line and is in the process of doing the transcription. See http://www.uriland.it/matematica/. A recent publication includes a facsimile and commentaries (Honsell and Bagni, 2009).

[25] See Heeffer (2006a) for further references. The problems are BTP058, BTP065, BTP066, BTP092, BTP094, BTP109, BTP110, GQA030-33, GQA040 and GQA041.

However, at the beginning of the sixteenth century a straightforward translation into such an equation is still uncommon. A manuscript of the early fifteenth century gives a rare algebraic solution to a simpler version of a man making two business trips, c. 1417, Turin, Bibliotheca Nazionale, N. III; 53, f. 37^r (Rivoli, 1983, 53–4). At the first, he doubles his money and spend 8 denari, at the second he makes 3 denari from 2, and spend 10. He is left with nothing. The anonymous author uses a *cosa* for the starting capital and arrives at $2x - 8$ after the first trip. After the second trip he is left with $3x - 12 - 10$ or $3x - 22 = 0$ ("Avemo che 3 cose meno d. 22 sono equale a niente, perchè tu voi che non li resta nula"), with the solution $x = 7\ 1/3$.

Ries has nine instances of the problem in his *Coss*. One is alike van Varenbrakens', but doubles and spends 12. Ries proceeds as follows[26]:

> Suppose they had x and thus at first they had $2x$ from which they gave away 12 d. thus they kept $2x - 12$ Ø. Now, in the second this amount to $4x - 24$ Ø. Gives away 12 d. leaves $4x - 36$ Ø. In the third they arrive at $8x - 72$ Ø, give away 12 and keep therefore $8x - 84$ Ø. This is the amount they ended up with, thus 0. This gives two parts that are equal, one $8x$ and the other 84 Ø. Divide, and so is 10 1/2 the amount they started with.

Although the solution applies algebra, it is different from what would become common from the eighteenth century. Instead of translating the problem in symbolic form, the unknown is used to build an argumentation which ultimately leads to an equation. The same approach is taken in (Rudolff, 1525; Stifel, 1553, f. 183^r, problem 5).

Formulated as a symbolic expression, it becomes understandable that an algebraic formulation of the problem spoils the recreational aspect.[27] In contrast with some other problems, this type did indeed disappear by the end of the sixteenth century. Baker was the last one to be "troubled" by this problem (Baker, 1562).[28] The development of algebra changed the scope of recreational mathematics. Problems that can be easily solved algebraically loose their appeal in the seventeenth century. Only from the early eighteenth century onwards, the problem reappears as an easy exercise in works on algebra (Euler, 1770, 204; Wingate and Kersey, 1650, 397; Kersey, 1673, 67).

4.2. *Similar problem but all getting the same*

In the early thirteenth century a problem very similar to the monkey and coconut problem appears in which the shares are of equal value. Curiously, the condition that the shares are all equal – which is essential for solving the problem – is often concealed or implicit. The problem cannot be solved by calculating backwards.[29] Here is an example from a Byzantine arithmetic book of c. 1308[30]:

[26] Dresden C80. My translation from the transcription (Berlet, 1860, 22). A possible source for Ries is the problem from the Vienna codex 7255, f. 14^r.

[27] I presented this problem as an informal test to twelve test subjects. All of them solved the problem algebraically and not a single person used the rule of inversion which is actually the easiest method in this case. This is an illustration of how symbolic algebra strongly influences and narrows our scope of possible problem solving methods.

[28] (Baker. 1562, ff. 193^r–193^v): "A merchant did ride into three severall fayres: at the first hee doubled his money and spent 10 crowns, at the second he did also double his money and spent 10 crowns. And likewise at the third fayre, he did double his money and spent 10 crowns, and in the ende, hee founde that he had remaining but 2 crowns".

[29] For an extensive discussion of this problem see (Høyrup, 2008). Høyrup notes that the problem can be solved directly, without algebra, from the equality of the last two shares, but this is on the condition that one believes that there is a solution. However, at the time that the problem was formulated this was not understood in this way.

[30] Cod. Par. Suppl. Gr. 387, f. 135^r, My translation from (Vogel, 1968, 103–104).

> At breakfast apples are served. The first person gets 1 apple and the 7th part of the rest of the apples, the second gets two apples and the 7th part of the rest, the third, three apples and the 7th part of the rest, the fourth, four apples and the 7th part of the rest, and the others also according to this order. One will answer how many there were at breakfast and how many apples have been eaten.

The problem could have evolved from an older version, possibly of Arabic origin, where the number of parts is given. In *Liber augmenti et diminutionis* we find a chapter on division problems *Capitulum de pomis*, with a division of apples leaving one.[31] Høyrup however dismisses an Arabic origin and believes that the evidence of advanced arithmetical and algebraic experimentation in the *Liber mahamaleth* and the *Liber abbaci* makes it more likely to have been created in the Maghreb or al-Andalus of the 12th century (Høyrup, 2008).

The same problem is often formulated as a legacy problem in which a dying man distributes gold pieces to his children, all receiving the same amount. He dies before finishing his sentence. With i children, each child gets ai plus $1/n$th of the rest (with a being an arbitrary factor). The question is how many gold pieces he possessed and how many children he had. The problem is known in German writing by the name "Der unbekannte Erbschaft" (Figure 7) (Vogel, 1968, 173; Tropfke, 1980, 586). Fibonacci is our earliest source to this version of the problem. Although we find the problem treated in chapter 12 of the *Liber Abbaci* on *elchataym*, it is not solved by double false position but by the following rule: if the rest is divided by seven, subtract one from seven and square the result, which is 36. This is the original amount (Boncompagni, 1857, 279; Sigler, 2002, 399). The solution thus depends on the following rule of thumb, without any explanation provided:

$$x = (n-1)^2.$$

Fibonacci adds two more problems after the unknown heritage problem which Sigler places under the heading "On the Separation of a Number into Parts" (Sigler, 2002, 399). These are discussed by Høyrup and provide a solution to the same problem using the *regula recta*, a form of linear algebra using 'the thing' (Høyrup, 2008). Interestingly, the problems that are presented as abstract partitioning problems are solved by algebra, while the 'recreational' version of the heritage problem is solved by the recipe. This seems to indicate that while all three problems are mathematically equivalent, Fibonacci already makes a distinction between a recreational version of the problem to be solved by a recipe and an abstract version to be solved by algebra.

The same rule of thumb for the identical problem is repeated in Planudes' Ψηφηφορια κατ' *Ινδους* η Λεγομενη Μεγαλη of c. 1300 (Allard, 1981, 192–194; 233–234). Also in Paolo Dagomeri (Arrighi, 1964, 140) with 1000 and 1/10th and another one with 1/6th, *Libro d'Abaco* of c. 1330 (Arrighi, 1973, 199). In the *Trattato d'Abacho* of Maestro Benedetto, c. 1470, Florence, BNCF, Magl. Cl. XI, 76, f. 53r (Arrighi, 1987a) the problem appears with 1000 and 1/10th.

In the *Algorismus Ratisbonesis* (c. 1450) three instances of the problem are solved with the same rule of thumb (Vogel, 1954, problem 114, 115 and 352). The rule continues to be prescribed well into the sixteenth century and it takes some time before algebra is applied to the problem. Ghaligai (Ghaligai, 1521, Chap. 9, problem 25, 26) and Tartaglia (Tartaglia, 1551–1560, book 9, quest. 2, 98^r–98^v) do not solve the problem

[31] Paris, BNF, Lat. 7377A (Libri, 1838, 336–337): "Quod si quis dixerit quidam vir intravit viridarium, et collegit in eo poma; viridarium vero habebat tres portas, quarum quamque hostiarius custodiebat. Vir ergo ille partitus est poma cum primo, et insuper donavit ei duo, et partitus est cum secundo et donavit ei duo, et partitus est cum tertio et donavit ei duo, et egressus est habens unum: quantus ergo fuit numerus pomorum que collegit?" The resulting equation is: $x - x/2 - 2 - (x - x/2 - 2)/2 - (x - x/2 - 2 - (x - x/2 - 2)/2)/2 = 1$. The original author of this treatise according to (Hughes, 1994, 2001) is Abraham ben Meir ibn Ezra. This attribution is disputed by Jens Høyrup. The problem is solved by double false position.

algebraically. Buteo, who solves 67 recreational problems by the use of algebra in book V, lists it in the fourth book on arithmetical problems instead of the algebra section (Buteo, 1559, quaestio 78, 286–288).

The earliest algebraic solution I could find (apart from the partioning problem of Fibonacci) is from Rudolff for the problem with $n = 10$ (Rudolff, 1525; 1553, 252^r, RBH1110). Using the unknown for the legacy, he expresses the share of the first person as $\frac{x-1}{10} + 1$ and calculates the share of the second as $\frac{9x-29}{100} + 2$. As all shares are equal, the two can be equated and he arrives at a solution of 81. Cardano gives the solution by rule of thumb but adds "potest etiam fieri per algebra" (it can also be done by algebra). He adds an algebraic solution to the problem with first giving 1/7th of the remainder and then 100 (Cardano, 1539, Chap. 66, problem 65).


¶ Ein Testament

Jtm̃ es lygt ein vatter am todtbet vnd stirbt auch vñ er lest kinder vñ sagt nicht wieuil/ vñ lest gelt vñ sagt auch nit wie vil/ vnnd bestelt seinē letstē willē also/das man einē kind so vil sol gebē als dem andern. Vnnd dē ersten gibt man 1 fl vnd $\frac{1}{10}$ des übrigē gelts Vñ dem andern 2 fl vnd auch $\frac{1}{10}$ des übrigē gelts. Vnd also fürt alweg einem 1 fl mer dan dē andern

Figure 7. The unknown legacy problem from (Widmann, 1489, f. 97v).

He equates the share of the first son with that of the second and arrives at the equation:

$$\frac{1}{7}x + 100 = \frac{1}{7}\left(\frac{6}{7}x - 100\right) + 200$$

which is easily solved to $x = 4200$.

We find the algebraic solution reproduced by Valentin Mennher in his revised *Practicque pour brievement apprendre à ciffrer* (Mennher, 1556, 21b).[32] After Mennher the problem virtually vanishes from arithmetic books. One notable exception is Bachet's *Problèmes plaisants*, can be considered as the first book fully devoted to recreational mathematics (Bachet, 1612, 149–154). The problem gets a full algebraic treatment in the later editions of Bachet's works (Lasbosne, 1874, 1879). With the further exception (Ozanam, 1725, 67–68) the problem reappears only with Euler (Euler, 1770, 227). Euler treats many problems, which we now consider as recreational, as exercises on algebra. Euler considers the legacy problem to be of particular interest.[33]

32 (Chuquet, Paris, BNF, Français 1346) gives a general solution based on a progression. He does not formulate the problem as an equation which is solved algebraically. See (Flegg et al., 1985, 225).

33 (Euler, 1770): "Diese Frage ist von einer gantz besondern Art und verdienet deswegen bemercket zu werden" (Weber, 1911, 227).

It appears frequently in nineteenth-century textbooks, not always with the most elegant solution. A French classic is Bourdon who gives a general solution with legacy x and the nth child getting na plus one nth of the rest (Bourdon, 1840, 62–3). The first child gets

$$a + \frac{x-a}{n} \quad \text{or} \quad \frac{an + x - a}{n}$$

the second part then becomes

$$x - 2a - \frac{(an + x - a)}{n} \quad \text{divided by } n, \quad \text{or} \quad \frac{nx - 3an - x + a}{n^2}$$

thus, the second child gets

$$2a + \frac{nx - 3an - x + a}{n^2} \quad \text{or} \quad \frac{2an^2 + nx - 3an - x + a}{n^2}$$

equating the share of the first and that of the second gives

$$\frac{an + x - a}{n} = \frac{2an^2 + nx - 3an - x + a}{n^2} \quad \text{which leads to} \quad x = an^2 - 2an + a$$

substituting this in the expression for the part of the first child gives

$$\frac{an + an^2 - 2an + a - a}{n} \quad \text{which simplifies to} \quad a(n-1).$$

As all the shares are the same, dividing the total by the share of the first child gives the number of heirs:

$$\frac{an^2 - 2an + a}{an - a} \quad \text{which results in} \quad n - 1.$$

Hackley, possibly misunderstanding the original problem, goes as far as expanding the expression for the third child, subtracts the three parts from x and then equates to zero. He thus arrives at the value of the legacy as (Hackley, 1846, 164–6):

$$\frac{(6n^2 - 4n + 1)a}{n^2 - 2n + 1}.$$

In contrast, Euler's solution is refreshingly simple and elegant. He uses the example in which the ith child gets $100.i + 1/10$th of the rest. He notices that the differences between subsequent parts are the same and can be expressed as:

$$100 - \frac{x + 100}{10}.$$

But as all children get an equal part, these differences must be zero, therefore

$$1000 - x - 100 = 0 \quad \text{or} \quad x = 900.$$

The problem of the unknown legacy is a suitable example to illustrate our case: It has been very popular for several centuries. It was commonly solved by some trick or rule particular to the problem. A general solution became available in books on algebra around 1560. It looses its appeal after an algebraic solution emerged, to return later as an exercise in elementary algebra.

4.3. A counter example

We have discussed three types of problems for which we observe a discontinuity in arithmetic and algebra books during the second half of the sixteenth century. All the three types have a history which goes back at least to the fourteenth century, but often even earlier sources can be given. These problems have been very popular and we have shown how they appear widespread in medieval and Renaissance texts. Then, around 1560 they virtually disappear from the books only to turn up again about two centuries later. Given all this evidence one might suspect that this is a general phenomenon, that all such recreational problems lost their appeal or function from the late sixteenth century. We will now demonstrate that this is not the case and that the observed discontinuity only occurs for some particular types of problems. Other problems continue to get treated both arithmetically and algebraically in the many textbooks of the late sixteenth and seventeenth centuries.

Cistern problems are a type of division and sharing problems which is as old as mathematics itself. The name of the problem refers to a fountain or cistern which filled by several pipes with a different flow rate. Given the rates of each pipe and the volume, the question is how long it takes to fill the cistern. The older versions deal with a task to be performed by several labourers with different working rates.

Table 2
Meeting and overtaking problems between 1560 and 1700.

Author	Date	Book	Problem	Page
Humfrey Baker	1562	*Well spring of sciences*	Prob. 3 and 4	36^v–37^v
Vander Gucht	1569	*Cijfer bouck*	2 problems	99^v
Dionigi Gori	1571	*Libro di arimetricha*	3 problems	81^v–82^r
Christian Wurstisen	1579	*Elementa Aritmeticae*	2 problems	
Nicolaus Petri	1583	*Practicque om te leeren rekenen*	2 problems	133^v
Giambattista Benedetti	1585	*Diversum Speculationum Mathematicarum*	Theorema CVI–CXIIII	68–77
Christoph Clavius	1583	*Epitome arithmeticae practicae*	Quest. 20	(1607) 256
Thomas Hood	1596	*The elements of arithmeticke*	2 problems	110, 119
Christoph Clavius	1608	*Algebra*	Cap. 31, 2, 3	306–307
Denis Henrion	1634	*Cursus Mathematicus, II Algebra*	Quest. 19	172
John Kersey	1650	*Arithmetic made easy*	Quest. 14	361
Jonas Moore	1660	*Moores arithmetick*, vol. 2 *Algebra*	Quest. 1	55
William Leybourn	1667	*Arithmetical recreations*	16, 17, 18	41–44
Edmund Wingate	1668	*Wingate's Arithmetick*	14–18, 23	450–453
Juan Caramuel	1670	*Mathesis Biceps*	81, 82	139–140
Edward Cocker	1678	*Arithmetic*	32	181
William Leybourn	1694	*Pleasure with profit*	8	38
Edward Wells	1698	*Elementa arithmeticae numerosae*	109, 116	207–208
Peter Lauremberger	1698	*Institutiones Arithmeticae*	12.XII	196
Isaac Newton	1707	*Arithmetica Universalis*	Prob. 5	81–85
Thomas Simpson	1745	*A treatise of algebra*	XIX, L, LXIII, LI	89, 113–116
Leonhard Euler	1770	*Vollständige Anleitung zur Algebra*	16, 25	205–6

Several Babylonian tablets before 1500 BC describe such problems: YBC 7673, 7164 and 4669, BM 85196 (Neugebauer and Sachs, 1945). We find them also in another form in the earliest Chinese writings of the second century BC, e.g. the *Suàn shù shū* 筭數書, *Writings on Reckoning* (Cullen, 2004, 54–6) and they are discussed by the main authors of Hindu algebra. Such problems have always existed and have been solved by different means. Their popularity does not seem to be affected during the seventeenth century.[34]

Even more pronounced is the continuous attention for meeting and overtaking problems. Several versions of the problem exist. Two travellers travel towards each other with different speed. Given the distance, the question is when do they meet? Variations to the problem use the same direction with different starting times and add travelling rates in arithmetic or geometric progression. Our first sources of such problems are Chinese, the *Suàn shù shū* and the *Jiŭ zhāng suàn shù* 九章算術, *Nine chapters of the mathematical art* (Vogel, 1968; Chemla and Shuchun, 2004). They appear very frequently in Indian mathematics and are omnipresent in the West after Fibonacci. As a prime example of a recreational problem which retained its popularity, while many other disappeared, we have collected a list of references from books published between 1560 and 1770 (see Table 2).

5. Possible explanations of discontinuities

Several authors have traced the history and evolution of recreational problems, either as a whole (Smith, 1925, II, 532–594; Tropfke, 1980; Singmaster, 2004) or for some particular types of problems (Curtze, 1895a, 1895b; Vogel, 1954, 1968; Folkerts, 1978). Only Vera Sanford, a student of David Eugene Smith, paid specific attention to the phenomenon of discontinuity.[35] She formulates four possible explanation for the disappearance of certain problems (Sanford, 1927, 79–94):

> Problems have been eliminated from elementary textbooks when the business and economic conditions upon which they are based have become obsolete; when their subject matter has lost interest; when the mathematical ideas they illustrate have been forgotten; and when the invention of better mathematical tools makes them trivial or causes their transfer to more advanced work.

Let us look at this fourfold vindication more closely.

5.1. Conditions that have become obsolete

The conditions that have become obsolete deal with problems such as “the assize of bread”, in German *Brotordnung*. This problem asks for the calculation of the size or weight of a bread according to the cost of wheat, for a given price, or the calculation of the price of a bread for a given weight. In order to facilitate such calculations, books where published with tables working our the different combinations of weight, cost of wheat and price of bread (Riese, Dresden, Sächsische Landesbibliothek, Codex C80; Gebhardt, 2004). Sanford shows how such problems materialize in arithmetic books to disappear by the end of the sixteenth century when prices stabilized and the problem became a less pressing one. Other types of problems discussed by Sanford, which met the same fate as the *Brotordnung*, include a specific contract type for shepherds and a type of pasturage problem (Sanford, 1927, 81–86). She also mentions alligation, but this is disputable as the rules for solving alligation problems apply to other contexts as well.

We can add tollet reckoning as a forgotten practice. The word ‘tollet’ is most likely derived from the Italian *tavola*, in Venetian dialect *toleta* (Cantor, 1880–1908, II, 222). An example is shown in Figure 8.

[34] (Singmaster, 2004, 7.H) lists seven references between 1570 and 1700, but several more arithmetic books of the seventeenth century have versions of the problem.

[35] This book is based on extensive research in Plimpton’s vast collection of rare books and manuscripts, now part of Columbia University. Smith himself added more than thirteen thousand volumes on the history of mathematics and astronomy, including some unique manuscripts such as Omar Khayyam’s treatise on algebra and trigonometry. Sanford’s book is often quoted as it contains many problem texts from the original sources, which are difficult to find.

Tollet reckoning assisted in the complex calculations of exchange between the currencies of different cities and countries which used coins of different gold and silver contents. The letters M, C, X, lb, mr, x, lot and quint were used for denominations of marks, the unit of weight.

In any case, the type of recreational problems we have discussed above, is not dependent on such socio-economical conditions and therefore this explanation provided by Sanford does not help us any further.

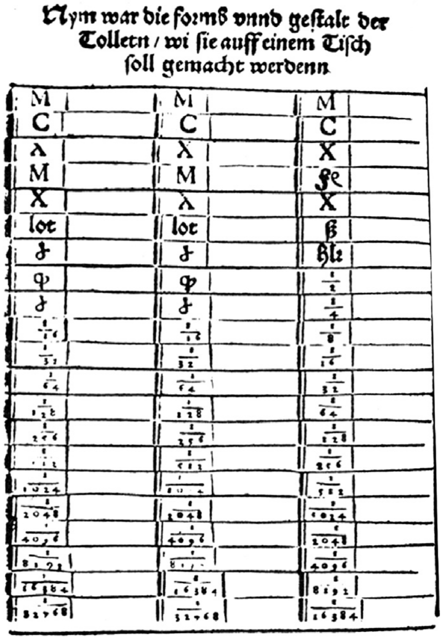


Figure 8. A layout of a table for tollet reckoning, from (Apianus, 1527, f. Aaiiij^{v}).

5.2. *Subject matter that has lost its interest*

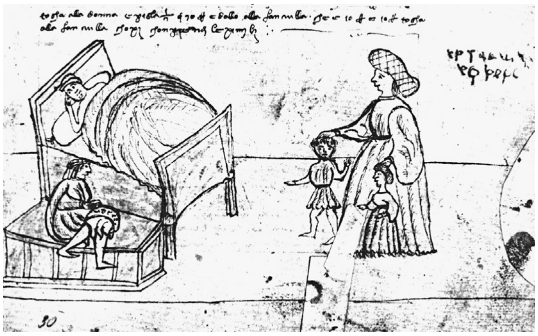
For the problems whose subject matter has lost its interest, Sanford refers to Biblical themes such as the computation of the number of angels in heaven. This medieval theme appeared regularly within writings of the magic tradition, such as Albertus Magnus who conjectured their number at 6.666^2 and cabbalist writings where angels were determined to number exactly 301.655.722. These calculations became part of arithmetical exercises in textbooks such as the *General Trattato* (Tartaglia, 1551–1560, I. f. 261v). We could add Stifel's preoccupation with Biblical word reckoning. Using the Roman numerals in Latin expressions, numbers can be created. For example using I (= J), V (= U), X, L, C, D and M in "Jesus Nazarenus Rex Judeorum" from his *Endchrist*, allows to reconstruct the publication year as MDXVVVVII (Stifel, 1532; Reich, 2004, 334). As with the obsolete conditions, the recreational problems we have discussed do not touch on subject matters that have lost their interest as these biblical themes.

5.3. *Mathematical concepts now forgotten*

The third explanation provided by Sanford refers somewhat awkwardly to "problems based on mathematical concepts now forgotten". As an example, she gives the kind of testament problems known from Arabic algebra.[36] Arabic law prescribes certain shares of a legacy for sons, daughters and the mother, when

[36] Such problems appear in the third part of al-Kwārizmī's Algebra, in English translation by (Rosen, 1831). For a devastating critique of Rosen's interpretation, see (Gandz, 1937) who interprets the problems within the context of Arab inheritance law.

the father dies. This gives rise to rather complex calculations of proportional division. Such problems became mixed with the rather hypothetical problem of posthumous twins known from Roman law (Cantor, 1880–1908, I, 522–25). Such problems appear commonly between the thirteenth and sixteenth centuries. Sanford cites Frisius, Ortega, Borghi, Tonstall and Buteo. Let us look at one typical instance from the pseudo Paolo dell'abacco (c. 1440) (see Figure 9)[37]:



One makes a testament, and if the [pregnant] wife gives birth to a male child, she will have one third of what he leaves, and his son the other two parts; and if the wife gives birth to a female child, the wife will receive two thirds and the girl one third. It now happens that the woman bears twins, one male and one female. The question is how much comes to the woman, the male and female child when the legacy values 70 lire.

Figure 9. Illustration of the posthumous twins problem (from Magliab. XI, 86).

One wonders what "mathematical concepts now forgotten" Sanford has in mind. The problem is usually solved by establishing the relative portions of the legacy. As the son will get twice the part of his mother and the mother twice the part of her daughter the estate is to be divided in the proportion son : wife : daughter as 4 : 2 : 1. With a legacy of 70 lire, this nicely divides as 40, 20 and 10 lire.[38] Later treatments of the problem such as (Buteo, 1559, 341–2) are algebraical. Therefore the problem does not depend on specific mathematical concepts. We even question that this type of problem lost its appeal. It appears continuously in arithmetic books after the sixteenth century (Vander Schuere, 1600, f. 98; Schott, 1674, 559–60; Leybourn, 1694, 39–40; Ozanam, 1725, 179) and further on during the eighteenth century.

5.4. The invention of new mathematical tools

The final reason for the disappearance of certain problems is ascribed by Sanford to the invention of new mathematical tools. Throughout the book she stresses the role of the new symbolism in the change of the way problems are approached. The best formulation is from her discussion of recreational problems (Sanford, 1927, 54):

> These problems lose their mystery when the algebraic relations they represent are written out. It is significant that the problems shifted from the main body of the work into the group of recreations when algebraic symbolism was sufficiently improved to show this clearly.

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[37] From Florence, BNCF, Magliab. XI, 86 (Arrighi, 1964, 85): "Uno fa testamento e llascia che sse ila donna fae fanciullo maschio, che ella abbi 1/3 di ciò ch'egl'à e il maschio l'altre due partj; e anchora lascia che xxe la donna xua fae fanciulla femjna, che la donna abbj 2/3 di ciò ch'egl'à e ila fanciulla abbj l'altro terzo. Ora adiviene che lla donna sua partoriscie e fae uno fanciullo maschio e una femjna; adomando che ttoccherà alla donna e che ttoccherà al fanciullo e che ttoccherà alla fanciulla, valendo il suo 70 lire", translation mine.

[38] (Arrighi, 1964, 85): "Fa' choxj. Lo intendimento del padre si fue che ila donna avexxi 2 chotantj che ila fanciulla e che il masc[h]io avexxi 2 chotantj che lla donna; adunque puoj dire choxì: avendo la fanciulla i arebbe la donna 2 e avendo la donna 2 arebbe il maschio 4".

This is the explanation we will pursue further. But before going into the role of symbolic algebra, let us explore some alternative explanations which are not discussed by Sanford.

5.5. *The discovery of Diophantus*

The disappearance of some standard arithmetical problems after 1560 coincides with the discovery and dissemination of the *Arithmetica* of Diophantus. Antonio-Mario Pazzi, who thought mathematics at the University of Bologna, contacted Rafael Bombelli in the late 1560s about the Greek manuscript of the *Arithmetica* at the Vatican library. Together they set to translating the six extant books.[39] Unfortunately their translation is lost. However, Bombelli included 143 problems from the *Arithmetica* in his *Algebra* published in 1572. The history of Bombelli's *Algebra* has been traced by Bortolotti after discovering the manuscript in a Bologna library in 1923.[40] The original edition consisted of five books, most likely completed before 1560.[41] Books I and II are included in the printed edition. Book IV and V on geometry have been omitted. The missing books were published in 1929 (Bortolotti, 1929) and later a complete edition was published (Bombelli, 1966). Book III has only later been analysed by Jayawardene (1973). It contains 156 practical and recreational problems solved by algebra, including "men finding a purse" (problems 40 and 41) and "monkey and coconuts" problems, formulated as business trips (problems 101, 112, 113, 133, 135, 147). Jayawardene argues that although the printed book show little signs of the influence of practical arithmetic on the printed *Algebra*, the manuscript certainly does. So after discovering Diophantus, Bombelli decided to replace the practical and recreational problems by 143 more abstract problems from the *Arithmetica* as a better illustration of algebraic practice. This highly unusual decision to leave practical problems "to the dignity of arithmetic" (Bombelli, 1572, 317) is significant. It was common practice in algebraic works before 1560 to include many practical and recreational problems, partitioned by type of equation, as exercises in mastering the "new art". Bombelli clearly distanced himself from this tradition in an attempt to restore the analytic approach of the Greeks.

The disaffection with practical arithmetic was further pronounced by Stevin. In his *L'Arithmétique* (Stevin, 1585), he chose to deal with the theoretical issues first and treated the arithmetic (*La practique d'Arithmétique*) with its mercantile rules and practical problems last. The part on arithmetic was even preceded by his own free translation of Diophantus' *Arithmetica* from Xylander's latin version (Xylander, 1575). Stevin had principle reasons do so. He believed that is was necessary to fully master the theoretical foundation of arithmetic before applying them. Explaining the rule of algebra using practical examples would be illicit from didactic point of view.[42] Although he does not apply algebra to practical problems their influence is clearly present in his choice of 27 problems illustrating the rule of algebra. Let us look at Stevin's 24th problem: "Find four numbers such that the sum of the first, second and third be 10, the second, third and fourth equals 14, the third, fourth and first 13 and the fourth, first and second 11, leading to the equations[43]:

[39] Bombelli mentions seven books, but this is considered a mistake. Ver Eecke identifies Bombelli source based on the missing proposition 28, as Codex Vaticana Gr. 200, which contains the known six books (Ver Eecke, 1926, lxiv).

[40] Bologna, Bibliotheca Comunale dell'Archiginnasio, Codex B. 1569. Another copy of this manuscript is in the University library of Bologna MS 595 (0)J2.

[41] According to Bortolotti before 1556, but Jayawardene writes "between 1557 and 1560" (Jayawardene, 1973, p. 521).

[42] As common at that time, Stevin considered algebra as a rule rather than a general problem solving method. He literally used the term "la reigle d'Algebre".

[43] (Stevin, 1585), question XXIV, 424–425; (Struik, vol. IIB, 689; 707–708). For a earlier history of this type of problem see Heeffer (2010).

$$\begin{aligned} x+y+z&=10\\ y+z+u&=14\\ x+z+u&=13\\ x+y+u&=11 \end{aligned} \quad \text{with the solution:} \quad x=2,\ y=3,\ z=5,\ u=6.$$

This type of problem was at that time commonly set in a concrete situation and appears for example in the correspondence between Regiomontanus and Giovanni Bianchini,[44] and in Peletier's *Algebre*.[45] A more complex problem with 5 men is found in 15th century Italian manuscripts.[46] However, Stevin was not the first to strip the practical context of problems and present them in a purer form. Estienne de la Roche, whose *Larismetique* of 1520 depended heavily on the manuscript of Chuquet (Marre, 1881), treats classic problems such as "If I had one from you..." or *Geben und Nehmen* in Tropfke's terminology, as purely arithmetical problems.[47]

Practical or recreational problems are completely absent in the works of Viète. In fact, Viète is more interested in the structure of an equation and the relations between equations than in the solution to algebraic problems. In Viète we observe a complete shift in interest in objects of analysis. Despite the postscript "*Nullum non problema solvere*" (to leave no problem unsolved), Viète in *In artem analyticam isagoge*, does not solve a single one of the many problems that he must have found in the many books on arithmetic of his time (Viète, 1591). In the *Zetetics* (Viete, 1593) he uses 24 problems from Diophantus as an illustration of how to translate a given problem in algebraic terms, while the application of this new art to practical problems would demonstrate its power and usefulness more convincingly.

The explanation that some recreational problems disappeared with the discovery of Diophantus apparently has some value. At least Bombelli and Stevin have consciously chosen to treat abstract problems from the *Arithmetica* at the expense of the more common practical and recreational problems of their era. But there are some problems with this line of reasoning. It overestimates the importance of Diophantus on the vernacular arithmetic books from the abbaco tradition, the reckoning schools and the *rechenmeister* in Germany and the low countries. Diophantus might have had a profound influence on Bombelli and Viète but the *Arithmetica* did not touch the practioners of arithmetic who were more interested in recipes and algorithms for solving problems of daily concern.

Another problem with the explanation of the discovery of Diophantus is that it was actually discovered much earlier. Copies of the manuscript of the *Arithmetica* were, as all editions of Greek mathematics, eagerly collected by the Italian humanists of the Quattrocento. One of the most industrious was Cardinal Bessarion living in Venice. He succeeded in collecting over 500 Greek manuscripts before his death in 1472.[48] The 1468 catalogue includes a copy of the *Arithmetica*, now known as MS Venice, Biblioteca Marciana, Graeci 308. Regiomontanus had developed a friendly relationship with Bessarion and had the opportunity to study the Greek text around 1463. He reported his find of the six books of the *Arithmetica*

44 (Curtze, 1902, 291): "Tres socii sunt, et quilibet per se habet denarios in marsupio. Duo primi absque tertio habent ducatos 30; duo etiam secundi absque primo habent ducatos 42; sed duo alii absque secundo habent 54: queritur, quot quilibet per se habeat".

45 (Peletier, 1554, 103–105): "Quatre hommes ont chacun certeine somme d'Ecuz. Le premier, second e tiers, ont ensamble 149. Le second, tiers e quart ont 110. Le tiers, quart e premier ont ansamble 125. Le quart premier e second ont ansamble 138. Quele ét la somme particuliere de tous?". In modern notation: $x+y+z=149$, $y+z+u=110$, $x+z+u=125$, $x+y+u=138$ with solution $x=64$, $y=49$, $z=36$, $u=25$.

46 In particular, manuscripts by Giovanni di Bartolo and Maestro Benedetto, see (Franci and Rigatelli, 1985, 54–55).

47 For example, problem 70 from Chuquet (Marre, 1881, 432; Flegg et al., 210–211) is stripped from its concrete context: "Trouves troys nombres tels que leves 12 du second et du tiers et les adiouster au premier, le p*re*mier sera le double des aultres +6. Et leves 13 du tiers et du p*re*mier et les adiouster au seco*n*d. Le seco*n*d sera le quadruple des aultres deux +2 et leves 11 du premier et seco*n*d. Et les adiouster au tiers. Le tiers sera le triple des aultres deux +2". In (de la Roche, 1520, f. 43v) the problem is reproduced literally, but uses a different solution method. For more on the relation of de la Roche and Chuquet see (Heeffer, 2012b).

48 See (Rose, 1975, 44–46 and 90–109), for the role of Bessarion in the transmission of Greek mathematics to Renaissance Italy.

in a letter to Giovanni Bianchini (Curtze, 1902, 256–7). He was then already well-acquainted with the Arab algebra and owned a copy of the manuscript on algebra by Al-Kwārizmī, possibly from his own pen (New York, Columbia University, Plimpton 188).[49] Very receptive to the processes of transmission of mathematical ideas, he found in Diophantus a source of inspiration for Arab algebra. In his *Oratio*, a series of lectures at the University of Padua in 1464, he was the first to propose the view that Arabic algebra descended from the *Arithmetica* of Diophantus (Regiomontanus, 1537). Having discovered Diophantus, Regiomontanus' interest in practical and recreational problems does not seem to be affected as is shown in his correspondence with Giovanni Bianchini, Jacob von Speier and Christian Roder.

A final argument against the role of Diophantus on the disappearance of recreational problems is that only some types vanished while others continued to flourish and be used in arithmetic books. An explanation for the lack of interest in some types of problems should take the particular structure of those problems into account.

5.6. *The replacement of the oral tradition by printing*

The specific structure and function of recreational problems could lead us to another possible explanation why many types disappeared in the sixteenth century. Rhyme and cadence functioned as mnemonic aids and facilitated the oral tradition of asking riddles and telling stories. Many of the older problems are put in verse. Some best known examples are "Going to St-Yves" using the geometric progression $7 + 7^2 + 7^3 + 7^4$ (Tropfke, 1980, 629). We know also many problems in rhyme from Greek epigrams[50] such as Archimedes cattle problem (Hillion and Lenstra, 1999), the ass and mule problem from Euclid (Singmaster, 1999) and age problems (Tropfke, 1980, 575–576). During the middle ages complete algorisms were written this way, taking over 500 verses (Waters, 1929). Even without rhyme, problems were cast into a specific cadence to make it easier to learn by heart. The 53 problems of Alcuin early 9th century clearly show a character of declamation, specific for the medieval system of learning by rote.[51] Medieval students were required to calculate the solution to problems mentally and to memorize rules and examples. Mnemonic aids were not only used in formulating problems, but also in remembering solutions. Memorizing a trick solution, a mathematical short cut, is also typical for many recreational problems. The solution to the problem of Christians and Turks, also called Josephus or survival problem, was commonly learned by heart by use of a vowel mnemonic.[52] To avoid to be thrown overboard, the men have to be arranged into a specific order. This is remembered as a sequence of men of the same type, represented by vowels a, e, i, o, u. *Rex angli cum veste bona dat signa serena* represents the sequence 2, 1, 3, 5, 2, 2, 4, 1, 1, 3, 1, 2, 2, 1, saving the 15 Christians from 15 Jews counted by 10, *Algorismus Ratisbonensis* (Vogel, 1954, 52).

And then, with the emergence of printed books, our collective memory was completely transformed. Collection of problems and recipes for solutions became available in printed form for everyone to read and study. And they were popular indeed! The Latin book of Gemma Frisius on practical arithmetic

[49] Menso Folkerts has since long announced a transcription which will be published in the series *Algorismus* by Erwin Rauner Verlag in Augsburg as *Algebraischen Studien des Regiomontanus. Edition eines Textes aus der Handschrift Plimpton 188*.

[50] The most comprehensive collection of Greek epigrams is in The Greek Anthology (Paton, 1979). A representative selection of these problems appear in the *Récréation Mathématique* of 1624.

[51] As an example, Proposition 5, propositio de emptore denariorum: "Dixit quidam emptor: Volo de denariis C porcos emere; sic tamen, ut verres X denariis emantur; scrofa autem V denariis; duo vero porcelli denario uno. Dicat, qui intelligit, quot verres, quot scorfae, quotve porcelli esse debeant, ut in neutris numerus nec superabundet, nec minuatur? (Folkerts, 1978).

[52] For a comprehensive history of the problem and list vowel mnemonics for the Arab, Latin, English, French and German language see (Singmaster, 2004). One not in his list is: "Rex Darius gentes profanas in mare legat", for 15 Turks and 15 Christians counted by 10, from (van Varenbrakens, Ghent, Universiteitsbibliotheek, Hs. 214, f. 148^{v}).

went through 64 editions before 1600.[53] Vernacular works such as the second arithmetic book of Adam Ries (Ries, 1522) saw no less than 99 editions in the sixteenth century (Gebhardt and Rochhaus, 1996). Mnemonic aids became less relevant for the continued existence of recreational problems and the problems itself were not the only means to convey arithmetical knowledge. A possible hypothesis is therefore that the replacement of oral traditions by printed works caused the disappearance of recreational problems. However, there are some problems with this line of explanation. Again, not all recreational problems disappeared at the end of the sixteenth century, only certain types of problems. We would expect printing technology to affect all types of problems equally. Secondly, this explanation somewhat forces the timing. I consider the period between 1540 and 1560 to be pivotal for the disappearance of problems after 1560. It is true that the growing popularity of arithmetic books lagged behind that of printed bibles and calendars. However, its effect should have manifested some decades earlier. Thirdly, and most importantly, the oral tradition was not at all replaced by printed works. Many recreational problems, such as the river crossing problem are part of our cultural heritage. They have been told over and over again since Alcuin and probably much earlier.[54] I learned them as a child at the camp fire as I continued the tradition by posing the riddles to my children.

5.7. The replacement of standard recipes by algebra

We now come to our explanation for the disappearance of certain types of problems during the second half of the sixteenth century. Sanford has already set the line of explanation by stating that recreational problems lose their mystery when expressed by algebraic equations (Sanford, 1927). We have discussed the example of monkey and coconut problem, reduced to a simple equation as $2(2(2x-6)-6)-6=0$. Indeed, the availability of a general method for solving linear problems by 1560 made some types of standard Renaissance problems less appealing for use in arithmetic and algebra textbooks. Problems for which the solution was based on some standard recipe, or rule of thumb, became trivial with the advent of symbolic algebra. The new art of algebra provided a general solution method which rendered the knowledge of standard recipes or specific rules superfluous.

Arithmetic books before the sixteenth century contained a great many recipes for a wide variety of problems. General methods for solving arithmetical problems were limited to the *regula falsi*, or rule of double false position. This rule applied to simple linear problems of the type $ax+b=c$. Problems involving more unknown quantities or specific linear problems such as $ax+b=cx+d$ were solved through a recipe that applied only to one type of problem. Such recipes appear scattered through arithmetic books and ought to be understood from examples. No formal explanation or justification is given, except for a numerical test of the solution, at least in Medieval and Renaissance texts. By the end of the fifteenth century, some authors felt the need for a more systematic exposition of the vast repository of the rules of mercantile arithmetic. Two noteworthy examples are (Widmann, 1489) and (Pacioli, 1494). They both give a treatment of many rules which have become known under certain Latin names during the fourteenth and fifteenth centuries. Both did not hesitate to coin new names using Latin expressions which they considered appropriate for the procedure described by the recipe. Table 3 gives an overview of the names of rules used by Widmann in his arithmetic of 1489.[55]

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53 Not counting French and Italian translations. Van Ortroy (1920) compiled a list with 62 Latin editions appearing before 1600. Two more editions not included in his listing are Georg Hantzsch, Leipzig, 1558 and Johannes Rhamba, Leipzig, 1566.

54 Alcuin (c. 800) is our first written source. He has four versions of the problem (propositio 17–20), with the wolf, goat and cabbage as the best known formulation today (Folkerts, 1978).

55 (Widmann, 1489), reprinted in Pforzheim, 1508. Our table is compiled from (Treutlein, 1879, 63), Leipzig, Codex 1470 (Kaunzner, 1968), (Tropfke, 1980) and the original text. There are some additional rules formulated by Widmann in some manuscripts: the Leipzig Codex 1470 contains *Regula usuri* (f. 448v), *Regula leporiss* (f. 448v), *Regula camby* (f. 534v) and *Regula de* (*tali*) *tela* (f. 502v), the last one also appears in the Dresden C80, (f. 287v) see (Kaunzner, 1968).

Some of the names have been coined previously. For example the *regula equalitatis* also appears in the *Vorlesung* at Erfurt by Gottfried Wolack (1467–8) and the *Algorithmus* from Peurbach (published 1510) (Tropfke, 1980, 600). But as others do not appear in previous manuscripts, Widmann must have devised many of the names by himself. Typical medieval problems using standard solution recipes, such a woman handing out figs to children, have risen in esteem through the use of Latin names, such as *Regula augmenti et decrementi*, here given by Widmann. But the large number of rules made life not easier for the typical student of vernacular arithmetic of the sixteenth century.

For several of these rules the principle behind it remains a mystery, and in several cases it is hard to understand what the rules are about.[56] What to think of the problem "when 3 times 3 equals 10, how much will 4 time 4 be?", which is "solved" by the *Regula suppositionis* (Widmann, 1489, f. 117r).[57] Even Widmann himself loses track, as he uses one name for several different rules such as the *Regula pulchra* and also uses different names for the same rule, such as *Regula augmenti*.[58] We cannot withhold the suspicion that the fabrication of new rules or devising fancy names for existing recipes was motivated by the commercial aims of selling books. We do not know much about Widmann's life but at least some *rechenmeister* after him, such as Adam Ries made a fortune from selling arithmetic books. Presenting problem solutions by a great number of rules with Latin names legitimized the method as well as the author. The use of many names for different methods of solving arithmetical problems reinforced the efficacy of reckoning school and the value of textbooks.

And then algebra became popular as the general method for solving all arithmetical problems. The recognition of the universal applicability of algebra did not happen in a single day. Within the abbaco tradition in Italy and starting from the second half of the fifteenth century in Germany and France, problems which used to be solved by standard recipes, entered the reach of the new art of algebra. Some problems became an easier prey than others.

Age problems and problems solved by the *regula augmentati* are instances of problems that were subject to an early approach by algebra.[59] For others, such as the legacy problem discussed before, it was not before the sixteenth century that we find the first algebraic solutions. With the wide spread of printing in the sixteenth century, the algebraic method for solving classic problems became known all over Europe. A handful of books laid the foundations for algebraic problem solving, initially, Pacioli's *Summa*, although the major part of his work also deals with typical mercantile recipes (Pacioli, 1494). This was followed by Ghaligai who included a chapter on algebra with almost fifty problems frequently appearing within the abbaco tradition (Ghaligai, 1521). In Germany, the first book covering algebra was (Grammateus, 1518) which demonstrated the general applicability of algebra as an alternative to the *regula falsi* for 15 examples of linear problems. But Grammateus was overshadowed in scope and method by Christoff Rudolff who compiled a repository of four hundred problems from fifteenth-century manuscripts and solved all of them by algebra (Rudolff, 1525). Imagine the impact this must have had on its target public in the early sixteenth century.

Having learned and practiced mercantile recipes for years, there comes a book in which all these different type of problems, on profit and loss, alligation, alloys, exchange, barter, company and company with time, are solved using a single rule (the resolution of linear equations).The power and applicability of the

[56] Witness the comments from Kaunzner's treatment of Widmann, as on the *Regula positionis*: "Wenn kein Beispiele dabeistünde könnten wir mit dieser Erläuterung nichts anfangen" (Kaunzner, 1968, 77).

[57] (Høyrup, 2007, 67) coined the term "counterfactual calculations" for these problems in arithmetic.

[58] See also (Eneström, 1908, 197–8) for a critique on Cantor's treatment of Widmann's rules (Cantor, 1880–1908, II, 234). Widmann's use of the rules is also discussed by Kaunzner (Kaunzner, 1968, 76).

[59] For the early age problems see (Tropfke, 1980, 575) and the sixteenth century versions are covered by (Singmaster, 2004, 7X). Their most common form, as in the Greek Anthology, is $a_1x + \frac{x}{a_2} + \frac{x}{a_3} \cdots + \frac{x}{a_n} = b$. Algebraic solutions to problems that fall under the *regula augmenti* or *augmentationis* will are discussed extensively in Heeffer (2007).

Table 3
An overview of the rules from Widmann's mercantile arithmetic.

Rule	Purpose	1489	1508
Regula residui	$x - \frac{x}{a} = b$	52r	
Regula reciprocationis Regula excessus	$(x - \frac{x}{a_1} - \cdots - \frac{x}{a_n})^2 = x$	53v, 55v	
Regula divisionis	$x + y = a, \frac{x}{y} = b$	55r	
Regula inventionis Regula resolutionis	Variation on Regula detri	76r	57r
Regula fusti	Accounting for tainted food	96r	65r
Regula pulchra I	Calculating backwards	97v	66r
Regula transversa		100r	
Regula detri conversa	Variation on Regula detri	98v	66v
Regula ligar	Alloys and mixtures	102r	68v
Regula positionis	$ax = by = cz, dx + ey + fz = g$	103v	70r
Regula pulchra II	$ax + b(x + e) + c(x + f) = d$	105v	71r
Regula equalitatis	$ax + bx + cx = d$	107r	72r
Regula legis	$ax + bx + cx = d, x + y + z = e$	108v	73r
Regula augmenti	$ax + b = cx + d$	110v	74r
Regula augmenti + Decrementi	$ax + b = cx - d$	112r	75v
Regula plurima	$ax + b = d - cx$	114r	76r
Regula pulchra III	$ax + b = cx - d$	114v	77r
Regula sententiarum	$\frac{1}{a}(\frac{1}{b}x) = c$	115v	77v
Regula suppositionis	Variation on Regula detri	117r	78v
Regula residui II	Profit and loss reckoning	117v	79r
Regula excessus II	$x + y = a$ $xy = b$	118v	79v
Regula collectionis	$x + \frac{x}{a_1} + \frac{x}{a_2} \cdots + \frac{x}{a_n} = b$	119v	80r
Regula pulchra IV	$x + a = b(y - a), y + b = c(x - b)$	120v	81r
Regula quadrata	Pythagorean theorem	122v	82r
Regula cubica	Cubic root approximation	124r	82v
Regula bona	Double other's money problem	125v	84r
Regula lucri	Calculation of interest	127v	85r
(no name)	$x + c = ay, y + c = bx$	146v	98v
Regula pagamenti	Money exchange	76r	107r
Regula alligationis	Also known as Regula virginum	159r	107v

algebraic method is so evident in Rudolff's book that it could not come without a drastic effect on existing mercantile rules and methods. A few years before, a book by de la Roche (1520) was published in Lyon, France's center for commerce and trade in the early sixteenth century. Not only the *maître*, but also the banker, and the merchant in the wool trade or maritime trade, benefited from the way many practical problems were solved by algebra in this *Larismethique* (de la Roche, 1520).[60]

These few books were sufficient to initiate an important change in the selection of problems in the books published during the following years. It is not so that from now on all problems were solved by algebra. Arithmetic books in which problems were abundantly approached by standard recipes and purely arithmetic means continued to be published for centuries. But the simpler type of problems, whose answer depends on knowledge of some rule of thumb, became unattractive knowing that they had a trivial solution in books such as (Rudolff, 1525).

60 The influence of Chuquet and de la Roche on German cossic algebra may have been more important than previously considered, as argued in (Heeffer, 2012b).

During the next years the plethora of mercantile rules for arithmetic problems became an embarrassment. Their use became a subject of ridicule and their recommendation was criticized by authors such as Cardano. Both in his *Arithmetica practicae* and in his *Ars magna*, Cardano hits on Pacioli, who presented many of these rules in their works. In response to the way these authors churned out mercantile rules, Cardano himself devised the *Regula de modo*. This has been misinterpreted as just another rule for solving linear problems with two unknowns.[61] Instead, Cardano presents the method as "the rule, from which all rules of the previous chapters can be extracted" ("regula per quam extrahuntur omnes haec regulae ex superioribus capitulis vocatur regula de modo") (Cardano, 1663, IV, 79). These previous chapters of the *Arithmetica practicae* treat mercantile rules as well as the rule of double false position. "One can fill a book full of these rules in one month" Cardano sneers. He exposes the method by which Pacioli can fabricate new rules. "Just solve the problems by algebra, then omit the unknown and keep the operations on the term together and there you have a general rule".[62] In the *Ars magna* he spends a full chapter on the *Regula de modo* and writes[63]:

> This rule of method (so called because it shows the method for fabricating as many mercantile rules as you wish) is very useful to teachers of arithmetic in teaching that art, certain easier [methods] having been discovered [by means of it]. With its help, we constructed the greater part of the sixth book. This, then, is the rule: Solve any given problem by any means you can, either [by using] an unknown or with the help of the sixth book. Then leave out the unknown and the other rules and use those operations which you can best use, looking always for brevity, and you will have the rule for the method of [solving] any similar problem.

This is a clear statement on the redundancy of mercantile rules. Algebra, as a general problem solving method, subsumes a great number of individual rules. It even allows to generate as many mercantile rules as you wish. Here, Cardano not only reveals the secret of the reckoning masters of the early sixteenth century, he also uncovers the probable origin of some of these mercantile rules. Most likely they are relics from previous algebraic solutions. We tend to use the term *fossils* as a characterization of the rules, as fossilized imprints of previous algebraic practice. Cardano depicts the process as starting from an algebraic solution, leaving out the unknown, formulating the operations of the derivation as a rule and you can apply it to similar problems. Many of the mercantile recipes curiously correspond with rules described by Āryabhata, Brāhamagupta and Bhāskara.[64]

6. Conclusion

As we have shown by several examples, the development of symbolic algebra between (Cardano, 1539) and (Buteo, 1559) is precursory to the disappearance of certain types of problems which became part of recreational mathematics. Curiously, several works after this period refer to the art of solving problems without algebra. A German book by Zacharias Lochner makes explicit reference in the title, that it contains

61 (Tropfke, 1980, 402): "Cardano nennt die lösung eines Systems von zwei linearen Gleichungen mit zwei Unbekannten *Regula de Modo*". This is apparently an interpretation from the editors Vogel, Reich and Gericke, because Tropfke in his original edition (Tropfke, 1930–1940, 45, note 228) points out the special status of this rule, and cites and translates the relevant fragment.

62 (Cardano, 1663, IV, 79–80): "Est etiam regula de Modo a me appellata, quoniam ex ipsa habent regulae infinitae in rebus maxime mercantilibus, et potes replere librum ex ipsis in uno mense, diversarum operationum, quae omnes regulae diversae videbuntur: et ita Frater Lucas, Borgias, Fortunatus, fecerunt libros per Neotericis instruendis, et ita tu lector poteris quotidie novas regular et inusitatis fabricare. Modus est talis solve quaestionem quamvis per algebra deinde detrahe la co. et serva operationes easdem in terminis suis, et erit regula generalis".

63 (Cardano, 1663, IV, 272–3). This translation is slighly adapted from (Witmer, 1968, 180).

64 For a systematic study of a remarkable parallel of three typical problems in Renaissance arithmetic books and their Indian counterparts see (Heeffer, 2007).

problems "with the advantage that they can be solved without the rule of algebra" (Lochner, 1583). Hans Sybrandt advertises his book with solutions to hundred geometrical questions, that they can all be solved "some with lines and some by numbers (though all without cossic numbers)" (Sybrandt, 1612, foreword). John Speidell writes on logarithms recommending that "they require not at all any skill in algebra, or cossike numbers, but may be used by every one that can onely adde and subtract, in whole numbers, according to the common or vulgar arithmeticke, without any consideration or respect of + and –" (Speidell, 1624). Webster (1722) advertises his book by asserting that all demonstrations can be performed by the principles of arithmetick, without the use of algebra. Thomas Rudd gives a selection of "hundred geometricall questions, with their solutions and demonstrations, some of them being performed arithmetically, and others geometrically, yet all without the help of algebra" (Rudd, 1650). Advertising books on arithmetic which do not use algebra may be prompted by the fear for the difficulties of learning algebra. However, there is something more going on when William Leybourn (1667) for his *Arithmetical recreations*, uses the subtitle: "Enchiridion of arithmetical questions both delightful and profitable, all of them performed *without algebra*".[65]

Clearly, symbolic algebra, as a general problem solving method for arithmetical problems, spoils their recreational aspect. Mathematical problems lose their recreational nature if they are easily solved by a general rule or method. We can also turn this reasoning backwards: recreational mathematics functions best in solving problems for which no general solution method is readily available.

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[65] (Leybourn, 1667), emphasis mine.

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