THE SAND-RECKONER.

"THERE are some, king Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognising that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe. Now you are aware that 'universe' is the name given by most astronomers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account (*τὰ γραφόμενα*), as you have heard from astronomers. But Aristarchus of Samos brought out a book consisting of some hypotheses, in which the premisses lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. Now it is easy to see that this is impossible; for, since the centre of the sphere has no magnitude, we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must however take Aristarchus to mean this: since we conceive the earth to be, as it were, the centre of the universe, the ratio which the earth bears to what we describe as the 'universe' is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars. For he adapts the proofs of his results to a hypothesis of this kind, and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the 'universe.'

I say then that, even if a sphere were made up of the sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that, of the numbers named in the *Principles**, some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made.

1. *The perimeter of the earth is about* 3,000,000 *stadia and not greater.*

It is true that some have tried, as you are of course aware, to prove that the said perimeter is about 300,000 stadia. But I go further and, putting the magnitude of the earth at ten times the size that my predecessors thought it, I suppose its perimeter to be about 3,000,000 stadia and not greater.

* Ἀρχαί was apparently the title of the work sent to Zeuxippus. Cf. the note attached to the enumeration of lost works of Archimedes in the Introduction, Chapter II., *ad fin.*

2. *The diameter of the earth is greater than the diameter of the moon, and the diameter of the sun is greater than the diameter of the earth.*

In this assumption I follow most of the earlier astronomers.

3. *The diameter of the sun is about* 30 *times the diameter of the moon and not greater.*

It is true that, of the earlier astronomers, Eudoxus declared it to be about nine times as great, and Pheidias my father* twelve times, while Aristarchus tried to prove that the diameter of the sun is greater than 18 times but less than 20 times the diameter of the moon. But I go even further than Aristarchus, in order that the truth of my proposition may be established beyond dispute, and I suppose the diameter of the sun to be about 30 times that of the moon and not greater.

4. *The diameter of the sun is greater than the side of the chiliagon inscribed in the greatest circle in the* (*sphere of the*) *universe.*

I make this assumption† because Aristarchus discovered that the sun appeared to be about $\frac{1}{720}$th part of the circle of the zodiac, and I myself tried, by a method which I will now describe, to find experimentally (*ὀργανικῶς*) the angle subtended by the sun and having its vertex at the eye (*τὰν γωνίαν, εἰς ἂν ὁ ἅλιος ἐναρμόζει τὰν κορυφὰν ἔχουσαν ποτὶ τᾷ ὄψει*)."

[Up to this point the treatise has been literally translated because of the historical interest attaching to the *ipsissima verba* of Archimedes on such a subject. The rest of the work can now be more freely reproduced, and, before proceeding to the mathematical contents of it, it is only necessary to remark that Archimedes next describes how he arrived at a higher and a lower limit for the angle subtended by the sun. This he did

* *τοῦ ἁμοῦ πατρὸς* is the correction of Blass for *τοῦ Ἀκούπατρος* (*Jahrb. f. Philol.* CXXVII. 1883).

† This is not, strictly speaking, an assumption; it is a proposition proved later (pp. 224—6) by means of the result of an experiment about to be described.

by taking a long rod or ruler (κανών), fastening on the end of it a small cylinder or disc, pointing the rod in the direction of the sun just after its rising (so that it was possible to look directly at it), then putting the cylinder at such a distance that it just concealed, and just failed to conceal, the sun, and lastly measuring the angles subtended by the cylinder. He explains also the correction which he thought it necessary to make because "the eye does not see from one point but from a certain area" (ἐπεὶ αἱ ὄψιες οὐκ ἀφ' ἑνὸς σαμείου βλέποντι, ἀλλὰ ἀπό τινος μεγέθεος).]

The result of the experiment was to show that the angle subtended by the diameter of the sun was less than $\frac{1}{164}$th part, and greater than $\frac{1}{200}$th part, of a right angle.

To prove that (on this assumption) the diameter of the sun is greater than the side of a chiliagon, or figure with 1000 *equal sides, inscribed in a great circle of the 'universe.'*

Suppose the plane of the paper to be the plane passing through the centre of the sun, the centre of the earth and the eye, at the time when the sun has just risen above the horizon. Let the plane cut the earth in the circle EHL and the sun in the circle FKG, the centres of the earth and sun being C, O respectively, and E being the position of the eye.

Further, let the plane cut the sphere of the 'universe' (i.e. the sphere whose centre is C and radius CO) in the great circle AOB.

Draw from E two tangents to the circle FKG touching it at P, Q, and from C draw two other tangents to the same circle touching it in F, G respectively.

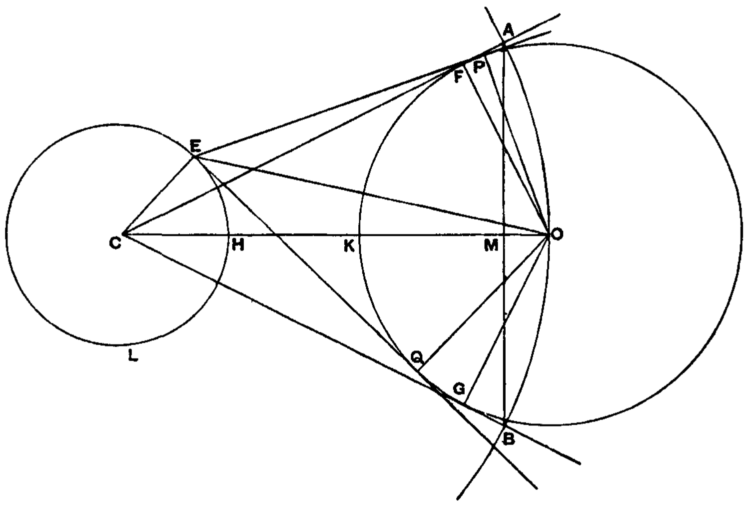
Let CO meet the sections of the earth and sun in H, K respectively; and let CF, CG produced meet the great circle AOB in A, B.

Join EO, OF, OG, OP, OQ, AB, and let AB meet CO in M.

Now $CO > EO$, since the sun is just above the horizon.

Therefore $$\angle PEQ > \angle FCG.$$

And $\angle PEQ > \frac{1}{200}R$ \
but $\quad\quad\quad < \frac{1}{164}R$ } where R represents a right angle.



Thus $\quad\quad \angle FCG < \frac{1}{164}R$, *a fortiori*,

and the chord AB subtends an arc of the great circle which is less than $\frac{1}{656}$th of the circumference of that circle, i.e.

$AB <$ (side of 656-sided polygon inscribed in the circle).

Now the perimeter of any polygon inscribed in the great circle is less than $\frac{44}{7}CO$. [Cf. *Measurement of a circle*, Prop. 3.]

Therefore $\quad\quad AB : CO < 11 : 1148,$

and, *a fortiori*, $\quad\quad AB < \frac{1}{100}CO$ ....(α).

Again, since $CA = CO$, and AM is perpendicular to CO, while OF is perpendicular to CA,

$$AM = OF.$$

Therefore $\quad AB = 2AM =$ (diameter of sun).

Thus $\quad\quad$ (diameter of sun) $< \frac{1}{100}CO$, by (α),

and, *a fortiori*,

(diameter of earth) $< \frac{1}{100}CO$. $\quad\quad$ [Assumption 2]

Hence $CH + OK < \frac{1}{100}CO$,

so that $HK > \frac{99}{100}CO$,

or $CO : HK < 100 : 99$.

And $CO > CF$,

while $HK < EQ$.

Therefore $CF : EQ < 100 : 99$..................(β).

Now in the right-angled triangles CFO, EQO, of the sides about the right angles,

$OF = OQ$, but $EQ < CF$ (since $EO < CO$).

Therefore $\angle OEQ : \angle OCF > CO : EO$,

but $< CF : EQ$*.

Doubling the angles,

$$\angle PEQ : \angle ACB < CF : EQ$$

$$< 100 : 99, \text{ by } (\beta) \text{ above.}$$

But $\angle PEQ > \frac{1}{200}R$, by hypothesis.

Therefore $\angle ACB > \frac{99}{20000}R$

$> \frac{1}{203}R$.

It follows that the arc AB is greater than $\frac{1}{812}$th of the circumference of the great circle AOB.

Hence, *a fortiori*,

$AB >$ (side of chiliagon inscribed in great circle),

and AB is equal to the diameter of the sun, as proved above.

The following results can now be proved:

(diameter of 'universe') $< 10{,}000$ *(diameter of earth)*,

and (diameter of 'universe') $< 10{,}000{,}000{,}000$ *stadia.*

* The proposition here assumed is of course equivalent to the trigonometrical formula which states that, if α, β are the circular measures of two angles, each less than a right angle, of which α is the greater, then

$$\frac{\tan\alpha}{\tan\beta} > \frac{\alpha}{\beta} > \frac{\sin\alpha}{\sin\beta}.$$

(1) Suppose, for brevity, that d_u represents the diameter of the 'universe,' d_s that of the sun, d_e that of the earth, and d_m that of the moon.

By hypothesis, $d_s \not> 30d_m$, [Assumption 3]
and $d_e > d_m$; [Assumption 2]
therefore $d_s < 30d_e$.

Now, by the last proposition,

$$d_s > \text{(side of chiliagon inscribed in great circle)},$$

so that

$$\text{(perimeter of chiliagon)} < 1000d_s < 30{,}000d_e.$$

But the perimeter of any regular polygon with more sides than 6 inscribed in a circle is greater than that of the inscribed regular hexagon, and therefore greater than three times the diameter. Hence

$$\text{(perimeter of chiliagon)} > 3d_u.$$

It follows that $d_u < 10{,}000d_e$.

(2) (Perimeter of earth) $\not>$ 3,000,000 stadia. [Assumption 1]

and (perimeter of earth) $> 3d_e$.

Therefore $d_e <$ 1,000,000 stadia,
whence $d_u <$ 10,000,000,000 stadia.

Assumption 5.

Suppose a quantity of sand taken not greater than a poppy-seed, and suppose that it contains not more than 10,000 grains.

Next suppose the diameter of the poppy-seed to be not less than $\frac{1}{40}$th of a finger-breadth.

Orders and periods of numbers.

I. We have traditional names for numbers up to a myriad (10,000); we can therefore express numbers up to a myriad myriads (100,000,000). Let these numbers be called numbers of the *first order*.

Suppose the 100,000,000 to be the unit of the *second order*, and let the *second order* consist of the numbers from that unit up to $(100{,}000{,}000)^2$.

Let this again be the unit of the *third order* of numbers ending with $(100{,}000{,}000)^3$; and so on, until we reach the 100,000,000*th order* of numbers ending with $(100{,}000{,}000)^{100{,}000{,}000}$, which we will call P.

II. Suppose the numbers from 1 to P just described to form the *first period.*

Let P be the unit of the *first order of the second period,* and let this consist of the numbers from P up to $100{,}000{,}000P$.

Let the last number be the unit of the *second order of the second period,* and let this end with $(100{,}000{,}000)^2 P$.

We can go on in this way till we reach the 100,000,000*th order of the second period* ending with $(100{,}000{,}000)^{100{,}000{,}000} P$, or P^2.

III. Taking P^2 as the unit of the *first order of the third period,* we proceed in the same way till we reach the 100,000,000*th order of the third period* ending with P^3.

IV. Taking P^3 as the unit of the *first order of the fourth period,* we continue the same process until we arrive at the 100,000,000*th order of the* 100,000,000*th period* ending with $P^{100{,}000{,}000}$. This last number is expressed by Archimedes as "a myriad-myriad units of the myriad-myriad-th order of the myriad-myriad-th period (*αἱ μυριακισμυριοστᾶς περιόδου μυριακισμυριοστῶν ἀριθμῶν μυρίαι μυριάδες*)," which is easily seen to be 100,000,000 times the product of $(100{,}000{,}000)^{99{,}999{,}999}$ and $P^{99{,}999{,}999}$, i.e. $P^{100{,}000{,}000}$.

[The scheme of numbers thus described can be exhibited more clearly by means of *indices* as follows.

FIRST PERIOD.

First order.	Numbers	from	1 to 10^8.
Second order.	„	„	10^8 to 10^{16}.
Third order.	„	„	10^{16} to 10^{24}.
⋮			
(10^8)*th order.*	„	„	$10^{8\cdot(10^8-1)}$ to $10^{8\cdot 10^8}$ (P, say).

SECOND PERIOD.

First order.	Numbers from	$P.1$ to $P.10^8$.
Second order.	" "	$P.10^8$ to $P.10^{16}$.
⋮		
(10^8)*th order.*	" "	$P.10^{8.(10^8-1)}$ to $P.10^{8.10^8}$ (or P^2).
⋮		

(10^8)TH PERIOD.

First order.	" "	$P^{10^8-1}.1$ to $P^{10^8-1}.10^8$.
Second order.	" "	$P^{10^8-1}.10^8$ to $P^{10^8-1}.10^{16}$.
⋮		
(10^8)*th order.*	" "	$P^{10^8-1}.10^{8.(10^8-1)}$ to $P^{10^8-1}.10^{8.10^8}$ (i.e. P^{10^8}).

The prodigious extent of this scheme will be appreciated when it is considered that the last number in the *first period* would be represented now by 1 followed by 800,000,000 ciphers, while the last number of the (10^8)*th period* would require 100,000,000 times as many ciphers, i.e. 80,000 million millions of ciphers.]

Octads.

Consider the series of terms in continued proportion of which the first is 1 and the second 10 [i.e. the geometrical progression 1, 10^1, 10^2, 10^3, ...]. The *first octad* of these terms [*i.e.* 1, 10^1, 10^2, ... 10^7] fall accordingly under the *first order of the first period* above described, the *second octad* [i.e. 10^8, 10^9, ... 10^{15}] under the *second order of the first period*, the first term of the octad being the unit of the corresponding order in each case. Similarly for the *third octad*, and so on. We can, in the same way, place any number of octads.

Theorem.

If there be any number of terms of a series in continued proportion, say $A_1, A_2, A_3, \dots A_m, \dots A_n, \dots A_{m+n-1}, \dots$ *of which* $A_1 = 1$, $A_2 = 10$ [so that the series forms the geometrical progression 1, 10^1, $10^2, \dots 10^{m-1}, \dots 10^{n-1}, \dots 10^{m+n-2}, \dots$], *and if any two terms as* A_m, A_n *be taken and multiplied, the product*

$A_m . A_n$ will be a term in the same series and will be as many terms distant from A_n as A_m is distant from A_1; also it will be distant from A_1 by a number of terms less by one than the sum of the numbers of terms by which A_m and A_n respectively are distant from A_1.

Take the term which is distant from A_n by the same number of terms as A_m is distant from A_1. This number of terms is m (the first and last being both counted). Thus the term to be taken is m terms distant from A_n, and is therefore the term A_{m+n-1}.

We have therefore to prove that

$$A_m . A_n = A_{m+n-1}.$$

Now terms equally distant from other terms in the continued proportion are proportional.

Thus $$\frac{A_m}{A_1} = \frac{A_{m+n-1}}{A_n}.$$

But $$A_m = A_m . A_1, \text{ since } A_1 = 1.$$

Therefore $$A_{m+n-1} = A_m . A_n \qquad\qquad (1).$$

The second result is now obvious, since A_m is m terms distant from A_1, A_n is n terms distant from A_1, and A_{m+n-1} is $(m + n - 1)$ terms distant from A_1.

Application to the number of the sand.

By Assumption 5 [p. 227],

(diam. of poppy-seed) $\nless \frac{1}{40}$ (finger-breadth);

and, since spheres are to one another in the triplicate ratio of their diameters, it follows that

(sphere of diam. 1 finger-breadth) $\ngtr$ 64,000 poppy-seeds
$\ngtr 64{,}000 \times 10{,}000$
$\ngtr 640{,}000{,}000$
$\ngtr$ 6 units of *second order* + 40,000,000 units of *first order*
(*a fortiori*) $<$ 10 units of *second order* of numbers.
} grains of sand.

We now gradually increase the diameter of the supposed sphere, multiplying it by 100 each time. Thus, remembering that the sphere is thereby multiplied by 100^3 or 1,000,000, the number of grains of sand which would be contained in a sphere with each successive diameter may be arrived at as follows.

Diameter of sphere.	*Corresponding number of grains of sand.*
(1) 100 finger-breadths	< 1,000,000 × 10 units of *second order*
	< (7th term of series) × (10th term of series)
	< 16th term of series [i.e. 10^{15}]
	< [10^7 or] 10,000,000 units of the *second order*.
(2) 10,000 finger-breadths	< 1,000,000 × (last number)
	< (7th term of series) × (16th term)
	< 22nd term of series [i.e. 10^{21}]
	< [10^5 or] 100,000 units of *third order*.
(3) 1 stadium (< 10,000 finger-breadths)	< 100,000 units of *third order*.
(4) 100 stadia	< 1,000,000 × (last number)
	< (7th term of series) × (22nd term)
	< 28th term of series [10^{27}]
	< [10^3 or] 1,000 units of *fourth order*.
(5) 10,000 stadia	< 1,000,000 × (last number)
	< (7th term of series) × (28th term)
	< 34th term of series [10^{33}]
	< 10 units of *fifth order*.
(6) 1,000,000 stadia	< (7th term of series) × (34th term)
	< 40th term [10^{39}]
	< [10^7 or] 10,000,000 units of *fifth order*.
(7) 100,000,000 stadia	< (7th term of series) × (40th term)
	< 46th term [10^{45}]
	< [10^5 or] 100,000 units of *sixth order*.
(8) 10,000,000,000 stadia	< (7th term of series) × (46th term)
	< 52nd term of series [10^{51}]
	< [10^3 or] 1,000 units of *seventh order*.

But, by the proposition above [p. 227],

(diameter of 'universe') < 10,000,000,000 stadia.

Hence *the number of grains of sand which could be contained in a sphere of the size of our 'universe' is less than* 1,000 *units of the seventh order of numbers* [or 10^{51}].

From this we can prove further that *a sphere of the size attributed by Aristarchus to the sphere of the fixed stars would contain a number of grains of sand less than* 10,000,000 *units of the eighth order of numbers* [or $10^{56+7} = 10^{63}$].

For, by hypothesis,

(earth) : ('universe') = ('universe') : (sphere of fixed stars).

And [p. 227]

(diameter of 'universe') < 10,000 (diam. of earth);

whence

(diam. of sphere of fixed stars) < 10,000 (diam. of 'universe').

Therefore

(sphere of fixed stars) < $(10{,}000)^3$. ('universe').

It follows that the number of grains of sand which would be contained in a sphere equal to the sphere of the fixed stars

< $(10{,}000)^3 \times 1{,}000$ units of *seventh order*

< (13th term of series) × (52nd term of series)

< 64th term of series [i.e. 10^{63}]

< [10^7 or] 10,000,000 units of *eighth order* of numbers.

Conclusion.

"I conceive that these things, king Gelon, will appear incredible to the great majority of people who have not studied mathematics, but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth the sun and moon and the whole universe the proof will carry conviction. And it was for this reason that I thought the subject would be not inappropriate for your consideration."