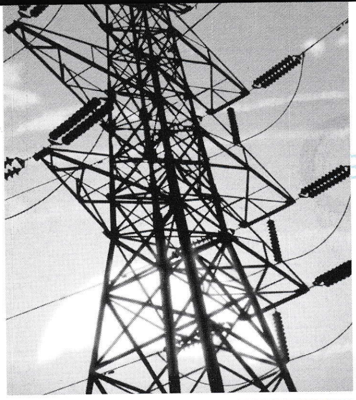


# CHAPTER 13

# Complex Numbers and Functions

**Complex numbers** and their geometric representation in the **complex plane** are discussed in Secs. 13.1 and 13.2. Complex analysis is concerned with complex **analytic functions** as defined in Sec. 13.3. Checking for analyticity is done by the **Cauchy–Riemann equations** (Sec. 13.4). These equations are of basic importance, also because of their relation to **Laplace's equation**.

The remaining sections of the chapter are devoted to elementary complex functions (exponential, trigonometric, hyperbolic, and logarithmic functions). These generalize the familiar real functions of calculus. Their detailed knowledge is an absolute necessity in practical work, just as that of their real counterparts is in calculus.

*Prerequisite:* Elementary calculus.
*References and Answers to Problems:* App. 1 Part D, App. 2.

## 13.1 Complex Numbers. Complex Plane

Equations without *real* solutions, such as $x^2 = -1$ or $x^2 - 10x + 40 = 0$, were observed early in history and led to the introduction of complex numbers.[1] By definition, a **complex number** $z$ is an ordered pair $(x, y)$ of real numbers $x$ and $y$, written

$$z = (x, y).$$

$x$ is called the **real part** and $y$ the **imaginary part** of $z$, written

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

(0, 1) is called the **imaginary unit** and is denoted by $i$,

$$i = (0, 1). \tag{1}$$

[1]First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501–1576), who found the formula for solving cubic equations. The term "complex number" was introduced by CARL FRIEDRICH GAUSS (see the footnote in Sec. 5.4), who also paved the way for a general use of complex numbers.

## Addition, Multiplication. Notation $z = x + iy$

**Addition** of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2). \tag{2}$$

**Multiplication** is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1). \tag{3}$$

In particular, these two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

and

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for real numbers $x_1, x_2$. Hence the complex numbers ***"extend"*** the real numbers. We can thus write

$$(x, 0) = x. \qquad \text{Similarly,} \qquad (0, y) = iy$$

because by (1) and the definition of multiplication we have

$$iy = (0, 1)y = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, \quad 0 \cdot 0 + 1 \cdot y) = (0, y).$$

Together we have by addition $(x, y) = (x, 0) + (0, y) = x + iy$:

***In practice, complex numbers $z = (x, y)$ are written***

$$z = x + iy \tag{4}$$

or $z = x + yi$, e.g., $17 + 4i$ (instead of $i4$).

Electrical engineers often write $j$ instead if $i$ because they need $i$ for the current.

If $x = 0$, then $z = iy$ and is called **pure imaginary.** Also, (1) and (3) give

$$i^2 = -1 \tag{5}$$

because by the definition of multiplication, $i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$.

For **addition** the standard notation (4) gives [see (2)]

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

For **multiplication** the standard notation gives the following very simple recipe. Multiply each term by each other term and use $i^2 = -1$ when it occurs [see (3)]:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).\end{aligned}$$

This agrees with (3). And it shows that $x + iy$ is a more practical notation for complex numbers than $(x, y)$.

If you know vectors, you see that (2) is vector addition, whereas the multiplication (3) has no counterpart in the usual vector algebra.

**EXAMPLE 1 Real Part, Imaginary Part, Sum and Product of Complex Numbers**

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$, Then $\text{Re } z_1 = 8$, $\text{Im } z_1 = 3$, $\text{Re } z_2 = 9$, $\text{Im } z_2 = -2$ and

$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i,$$

$$z_1 z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27) = 78 + 11i.$$ ■

## Subtraction, Division

**Subtraction** and **division** are defined as the inverse operations of addition and multiplication, respectively. Thus the **difference** $z = z_1 - z_2$ is the complex number $z$ for which $z_1 = z + z_2$. Hence by (2),

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{6}$$

The **quotient** $z = z_1/z_2$ $(z_2 \neq 0)$ is the complex number $z$ for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting $z = x + iy$, we obtain $x_1 = x_2 x - y_2 y$, $y_1 = y_2 x + x_2 y$. The solution is

$$z = \frac{z_1}{z_2} = x + iy, \qquad x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}. \tag{7*}$$

The ***practical rule*** used to get this is by multiplying numerator and denominator of $z_1/z_2$ by $x_2 - iy_2$ and simplifiying:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}. \tag{7}$$

**EXAMPLE 2 Difference and Quotient of Complex Numbers**

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$ we get $z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$ and

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}\,i.$$

Check the division by multiplication to get $8 + 3i$. ■

Complex numbers satisfy the same commutative, associative, and distributive laws as real numbers (see the problem set).

## Complex Plane

This was algebra. Now comes geometry: the geometrical representation of complex numbers as points in the plane. This is of great practical importance. The idea is quite simple and natural. We choose two perpendicular coordinate axes, the horizontal $x$-axis, called the **real axis,** and the vertical $y$-axis, called the **imaginary axis.** On both axes we choose the same unit of length (Fig. 315). This is called a **Cartesian coordinate system.**

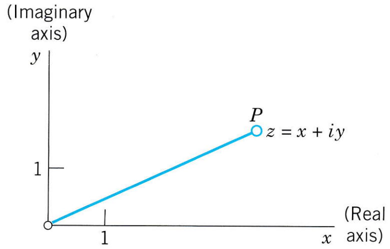



Fig. 315. The complex plane

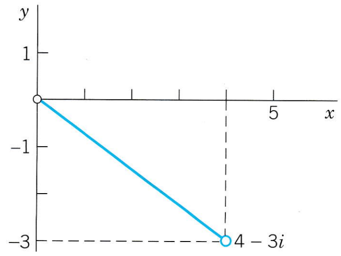



Fig. 316. The number $4 - 3i$ in the complex plane

We now plot a given complex number $z = (x, y) = x + iy$ as the point $P$ with coordinates $x$, $y$. The $xy$-plane in which the complex numbers are represented in this way is called the **complex plane.**[2] Figure 316 shows an example.

Instead of saying "the point represented by $z$ in the complex plane" we say briefly and simply ***"the point z in the complex plane."*** This will cause no misunderstandings.

Addition and subtraction can now be visualized as illustrated in Figs. 317 and 318.

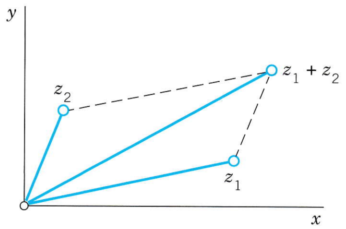



Fig. 317. Addition of complex numbers

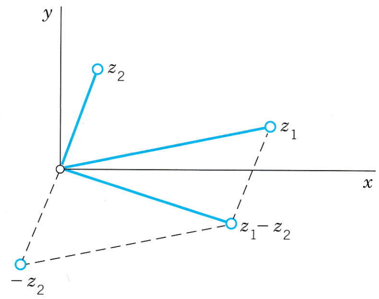



Fig. 318. Subtraction of complex numbers

## Complex Conjugate Numbers

**The complex conjugate** $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point $z$ in the real axis. Figure 319 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

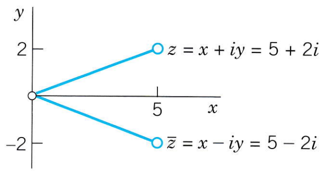



Fig. 319. Complex conjugate numbers

[2]Sometimes called the **Argand diagram,** after the French mathematician JEAN ROBERT ARGAND (1768–1822), born in Geneva and later librarian in Paris. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745–1818), a surveyor of the Danish Academy of Science.

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part $x$ and the imaginary part $y$ (not $iy$!) of $z = x + iy$ the important formulas

$$\text{(8)} \qquad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

If $z$ is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

$$\text{(9)} \qquad \begin{aligned} &\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad &&\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2, \\ &\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad &&\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

**EXAMPLE 3** **Illustration of (8) and (9)**

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$

$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$

■

## PROBLEM SET 13.1

1. **(Powers of $i$)** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **(Rotation)** Multiplication by $i$ is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing $z$ and $iz$ and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.
3. **(Division)** Verify the calculation in (7).
4. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.
5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **(Laws for conjugates)** Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

**7–15 COMPLEX ARITHMETIC**

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$
8. $\bar{z}_1 \bar{z}_2$
9. $\operatorname{Re}(1/z_1^{\,2})$
10. $\operatorname{Re}(z_2^{\,2})$, $(\operatorname{Re} z_2)^2$
11. $z_2/z_1$
12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
13. $(4z_1 - z_2)^2$
14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$
15. $(z_1 + z_2)/(z_1 - z_2)$

**16–19** Let $z = x + iy$. Find:

16. $\operatorname{Im} z^3$, $(\operatorname{Im} z)^3$
17. $\operatorname{Re}(1/\bar{z})$
18. $\operatorname{Im}[(1 + i)^8 z^2]$
19. $\operatorname{Re}(1/\bar{z}^2)$
20. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \quad \textit{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad \textit{(Associative laws)}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

# 13.2 Polar Form of Complex Numbers. Powers and Roots

The complex plane becomes even more useful and gives further insight into the arithmetic operations for complex numbers if besides the $xy$-coordinates we also employ the usual polar coordinates $r$, $\theta$ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

$r$ is called the **absolute value** or **modulus** of $z$ and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}\,. \tag{3}$$

Geometrically, $|z|$ is the distance of the point $z$ from the origin (Fig. 320). Similarly, $|z_1 - z_2|$ is the distance between $z_1$ and $z_2$ (Fig. 321).

$\theta$ is called the **argument** of $z$ and is denoted by arg $z$. Thus (Fig. 320)

$$\theta = \arg z = \arctan\frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, $\theta$ is the directed angle from the positive $x$-axis to $OP$ in Fig. 320. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense.***

For $z = 0$ this angle $\theta$ is undefined. (Why?) For a given $z \neq 0$ it is determined only up to integer multiples of $2\pi$ since cosine and sine are periodic with period $2\pi$. But one often wants to specify a unique value of arg $z$ of a given $z \neq 0$. For this reason one defines the **principal value** Arg $z$ (with capital A!) of arg $z$ by the double inequality

$$-\pi < \operatorname{Arg} z \leqq \pi. \tag{5}$$

Then we have Arg $z = 0$ for positive real $z = x$, which is practical, and Arg $z = \pi$ (not $-\pi$!) for negative real $z$, e.g., for $z = -4$. The principal value (5) will be important in connection with roots, the complex logarithm (Sec. 13.7), and certain integrals. Obviously, for a given $z \neq 0$ the other values of arg $z$ are arg $z = \operatorname{Arg} z \pm 2n\pi$ $(n = \pm 1, \pm 2, \cdots)$.

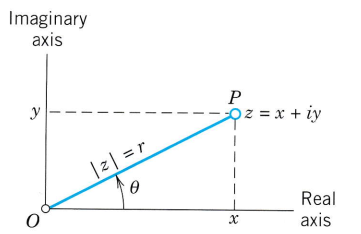



Fig. 320. Complex plane, polar form of a complex number

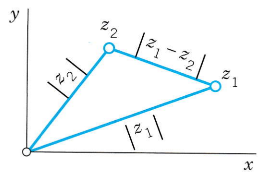



Fig. 321. Distance between two points in the complex plane

**EXAMPLE 1** **Polar Form of Complex Numbers. Principal Value Arg z**

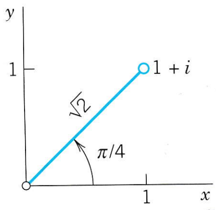



Fig. 322. Example 1

$z = 1 + i$ (Fig. 322) has the polar form $z = \sqrt{2}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$. Hence we obtain

$$|z| = \sqrt{2}, \qquad \arg z = \tfrac{1}{4}\pi \pm 2n\pi \ (n = 0, 1, \cdots), \qquad \text{and} \qquad \operatorname{Arg} z = \tfrac{1}{4}\pi \quad \text{(the principal value).}$$

Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, $|z| = 6$, and $\operatorname{Arg} z = \frac{1}{3}\pi$. ■

**CAUTION!** In using (4), we must pay attention to the quadrant in which $z$ lies, since $\tan\theta$ has period $\pi$, so that the arguments of $z$ and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan\theta_1 = \tan\theta_2 = 1$.

### Triangle Inequality

Inequalities such as $x_1 < x_2$ make sense for *real* numbers, but not in complex because *there is no natural way of ordering complex numbers.* However, inequalities between absolute values (which are real!), such as $|z_1| < |z_2|$ (meaning that $z_1$ is closer to the origin than $z_2$) are of great importance. The daily bread of the complex analyst is the **triangle inequality**

$$|z_1 + z_2| \leqq |z_1| + |z_2| \tag{6}$$
(Fig. 323)

which we shall use quite frequently. This inequality follows by noting that the three points 0, $z_1$, and $z_1 + z_2$ are the vertices of a triangle (Fig. 323) with sides $|z_1|$, $|z_2|$, and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 35). (The triangle degenerates if $z_1$ and $z_2$ lie on the same straight line through the origin.)

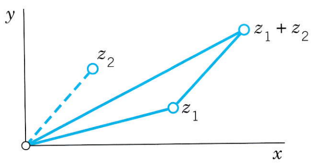



Fig. 323. Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \tag{6*}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

**EXAMPLE 2** **Triangle Inequality**

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$ ■

## Multiplication and Division in Polar Form

This will give us a "geometrical" understanding of multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i \sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i \sin\theta_2).$$

**Multiplication.** By (3) in Sec. 13.1 the product is at first

$$z_1 z_2 = r_1 r_2[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in App. A3.1] now yield

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \tag{7}$$

Taking absolute values on both sides of (7), we see that *the absolute value of a product equals the* ***product*** *of the absolute values of the factors,*

$$|z_1 z_2| = |z_1||z_2|. \tag{8}$$

Taking arguments in (7) shows that *the argument of a product equals the* ***sum*** *of the arguments of the factors,*

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{9}$$

**Division.** We have $z_1 = (z_1/z_2)z_2$. Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{10}$$

Similarly, $\arg z_1 = \arg[(z_1/z_2)z_2] = \arg(z_1/z_2) + \arg z_2$ and by subtraction of $\arg z_2$

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{11}$$

Combining (10) and (11) we also have the analog of (7),

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \tag{12}$$

To comprehend this formula, note that it is the polar form of a complex number of absolute value $r_1/r_2$ and argument $\theta_1 - \theta_2$. But these are the absolute value and argument of $z_1/z_2$, as we can see from (10), (11), and the polar forms of $z_1$ and $z_2$.

**EXAMPLE 3** **Illustration of Formulas (8)–(11)**

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1 z_2 = -6 - 6i$, $z_1/z_2 = 2/3 + (2/3)i$. Hence (make a sketch)

$$|z_1 z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\text{Arg}\, z_1 = 3\pi/4$, $\text{Arg}\, z_2 = \pi/2$,

$$\text{Arg}(z_1 z_2) = -\frac{3\pi}{4} = \text{Arg}\, z_1 + \text{Arg}\, z_2 - 2\pi, \qquad \text{Arg}(z_1/z_2) = \frac{\pi}{4} = \text{Arg}\, z_1 - \text{Arg}\, z_2. \quad \blacksquare$$

**EXAMPLE 4 Integer Powers of z. De Moivre's Formula**

From (8) and (9) with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

$$z^n = r^n (\cos n\theta + i \sin n\theta). \tag{13}$$

Similarly, (12) with $z_1 = 1$ and $z_2 = z^n$ gives (13) for $n = -1, -2, \cdots$. For $|z| = r = 1$, formula (13) becomes **De Moivre's formula**[3]

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{13*}$$

We can use this to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos \theta$ and $\sin \theta$. For instance, for $n = 2$ we have on the left $\cos^2 \theta + 2i \cos \theta \sin \theta - \sin^2 \theta$. Taking the real and imaginary parts on both sides of (13*) with $n = 2$ gives the familiar formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \qquad \sin 2\theta = 2 \cos \theta \sin \theta.$$

This shows that *complex* methods often simplify the derivation of *real* formulas. Try $n = 3$. ■

## Roots

If $z = w^n$ $(n = 1, 2, \cdots)$, then to each value of $w$ there corresponds *one* value of $z$. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely $n$ distinct values of $w$. Each of these values is called an ***n*th root** of $z$, and we write

$$w = \sqrt[n]{z}. \tag{14}$$

Hence this symbol is ***multivalued,*** namely, *n-valued.* The $n$ values of $\sqrt[n]{z}$ can be obtained as follows. We write $z$ and $w$ in polar form

$$z = r(\cos \theta + i \sin \theta) \qquad \text{and} \qquad w = R(\cos \phi + i \sin \phi).$$

Then the equation $w^n = z$ becomes, by De Moivre's formula (with $\phi$ instead of $\theta$)

$$w^n = R^n(\cos n\phi + i \sin n\phi) = z = r(\cos \theta + i \sin \theta).$$

The absolute values on both sides must be equal; thus, $R^n = r$, so that $R = \sqrt[n]{r}$, where $\sqrt[n]{r}$ is positive real (an absolute value must be nonnegative!) and thus uniquely determined. Equating the arguments $n\phi$ and $\theta$ and recalling that $\theta$ is determined only up to integer multiples of $2\pi$, we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where $k$ is an integer. For $k = 0, 1, \cdots, n - 1$ we get $n$ *distinct* values of $w$. Further integers of $k$ would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$,

[3] ABRAHAM DE MOIVRE (1667–1754), French mathematician, who pioneered the use of complex numbers in trigonometry and also contributed to probability theory (see Sec. 24.8).

hence the $w$ corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the $n$ distinct values

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \tag{15}$$

where $k = 0, 1, \cdots, n - 1$. These $n$ values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of $n$ sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg $z$ and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n - 1. \tag{16}$$

These $n$ values are called the ***n*th roots of unity.** They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 324–326 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

If $\omega$ denotes the value corresponding to $k = 1$ in (16), then the $n$ values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if $w_1$ is any $n$th root of an arbitrary complex number $z\ (\neq 0)$, then the $n$ values of $\sqrt[n]{z}$ in (15) are

$$w_1, \qquad w_1\omega, \qquad w_1\omega^2, \qquad \cdots, \qquad w_1\omega^{n-1} \tag{17}$$

because multiplying $w_1$ by $\omega^k$ corresponds to increasing the argument of $w_1$ by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

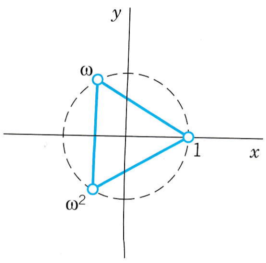



Fig. 324. $\sqrt[3]{1}$

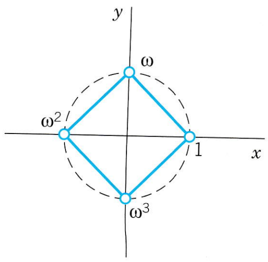



Fig. 325. $\sqrt[4]{1}$

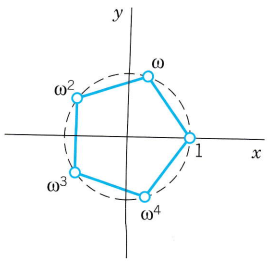



Fig. 326. $\sqrt[5]{1}$

## PROBLEM SET 13.2

### 1–8 POLAR FORM

Do these problems very carefully since polar forms will be needed frequently. Represent in polar form and graph in the complex plane as in Fig. 322 on p. 608. (Show the details of your work.)

1. $3 - 3i$
2. $2i, -2i$
3. $-5$
4. $\frac{1}{2} + \frac{1}{4}\pi i$
5. $\dfrac{1 + i}{1 - i}$
6. $\dfrac{3\sqrt{2} + 2i}{-\sqrt{2} - (2/3)i}$

7. $\dfrac{-6 + 5i}{3i}$

8. $\dfrac{2 + 3i}{5 + 4i}$

### 9–15 PRINCIPAL ARGUMENT

Determine the principal value of the argument.

9. $-1 - i$
10. $-20 + i, \ -20 - i$
11. $4 \pm 3i$
12. $-\pi^2$
13. $7 \pm 7i$
14. $(1 + i)^{12}$
15. $(9 + 9i)^3$

### 16–20 CONVERSION TO $x + iy$

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos \frac{1}{2}\pi + i \sin (\pm\frac{1}{2}\pi)$
17. $3(\cos 0.2 + i \sin 0.2)$
18. $4(\cos \frac{1}{3}\pi \pm i \sin \frac{1}{3}\pi)$
19. $\cos (-1) + i \sin (-1)$
20. $12(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$

### 21–25 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$
22. $\sqrt[8]{1}$
23. $\sqrt[4]{-1}$
24. $\sqrt[3]{3 + 4i}$
25. $\sqrt[5]{-1}$

26. **TEAM PROJECT. Square Root. (a)** Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}\right],$$

$$(18) \quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1.$$

**(b)** Obtain from (18) the often more practical formula

$$(19) \quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\operatorname{sign} y)\, i\sqrt{\tfrac{1}{2}(|z| - x)}\right]$$

where $\operatorname{sign} y = 1$ if $y \geqq 0$, $\operatorname{sign} y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

**(c)** Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}i$ by both (18) and (19) and comment on the work involved.

**(d)** Do some further examples of your own and apply a method of checking your results.

### 27–30 EQUATIONS

Solve and graph all solutions, showing the details:

27. $z^2 - (8 - 5i)z + 40 - 20i = 0$ (Use (19).)
28. $z^4 + (5 - 14i)z^2 - (24 + 10i) = 0$
29. $8z^2 - (36 - 6i)z + 42 - 11i = 0$
30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

### 32–35 INEQUALITIES AND AN EQUATION

Verify or prove as indicated.

32. **(Re and Im)** Prove $|\text{Re } z| \leqq |z|$, $|\text{Im } z| \leqq |z|$.

33. **(Parallelogram equality)** Prove

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Explain the name.

34. **(Triangle inequality)** Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.

35. **(Triangle inequality)** Prove (6).

# 13.3 Derivative. Analytic Function

Our study of complex functions will involve point sets in the complex plane. Most important will be the following ones.

## Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 327) has already occurred in Sec. 13.2. Figure 328 shows a general circle of radius $\rho$ and center $a$. Its equation is

$$|z - a| = \rho$$

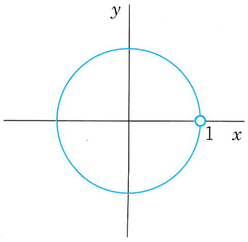



**Fig. 327.** Unit circle

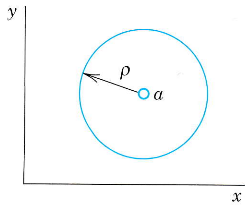



**Fig. 328.** Circle in the complex plane

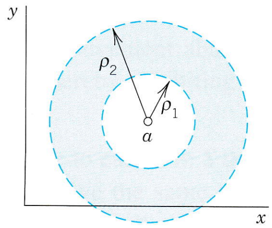



**Fig. 329.** Annulus in the complex plane

because it is the set of all $z$ whose distance $|z - a|$ from the center $a$ equals $\rho$. Accordingly, its interior **("open circular disk")** is given by $|z - a| < \rho$, its interior plus the circle itself **("closed circular disk")** by $|z - a| \leqq \rho$, and its exterior by $|z - a| > \rho$. As an example, sketch this for $a = 1 + i$ and $\rho = 2$, to make sure that you understand these inequalities.

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of $a$ or, more precisely, a *$\rho$-neighborhood* of $a$. And $a$ has infinitely many of them, one for each value of $\rho$ $(> 0)$, and $a$ is a point of each of them, by definition!

In modern literature *any set* containing a $\rho$-neighborhood of $a$ is also called a *neighborhood* of $a$.

Figure 329 shows an **open annulus** (circular ring) $\rho_1 < |z - a| < \rho_2$, which we shall need later. This is the set of all $z$ whose distance $|z - a|$ from $a$ is greater than $\rho_1$ but less than $\rho_2$. Similarly, the **closed annulus** $\rho_1 \leqq |z - a| \leqq \rho_2$ includes the two circles.

**Half-Planes.** By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.

## For Reference: Concepts on Sets in the Complex Plane

To our discussion of special sets let us add some general concepts related to sets that we shall need throughout Chaps. 13–18; keep in mind that you can find them here.

By a **point set** in the complex plane we mean any sort of collection of finitely many or infinitely many points. Examples are the solutions of a quadratic equation, the points of a line, the points in the interior of a circle as well as the sets discussed just before.

A set $S$ is called **open** if every point of $S$ has a neighborhood consisting entirely of points that belong to $S$. For example, the points in the interior of a circle or a square form an open set, and so do the points of the right half-plane $\operatorname{Re} z = x > 0$.

A set $S$ is called **connected** if any two of its points can be joined by a broken line of finitely many straight-line segments all of whose points belong to $S$. An open and connected set is called a **domain.** Thus an open disk and an open annulus are domains. An open square with a diagonal removed is not a domain since this set is not connected. (Why?)

The **complement** of a set $S$ in the complex plane is the set of all points of the complex plane that ***do not belong*** to $S$. A set $S$ is called **closed** if its complement is open. For example, the points on and inside the unit circle form a closed set ("closed unit disk") since its complement $|z| > 1$ is open.

A **boundary point** of a set $S$ is a point every neighborhood of which contains both points that belong to $S$ and points that do not belong to $S$. For example, the boundary points of an annulus are the points on the two bounding circles. Clearly, if a set $S$ is open, then no boundary point belongs to $S$; if $S$ is closed, then every boundary point belongs to $S$. The set of all boundary points of a set $S$ is called the **boundary** of $S$.

A **region** is a set consisting of a domain plus, perhaps, some or all of its boundary points. **WARNING!** "Domain" is the *modern* term for an open connected set. Nevertheless, some authors still call a domain a "region" and others make no distinction between the two terms.

## Complex Function

Complex analysis is concerned with complex functions that are differentiable in some domain. Hence we should first say what we mean by a complex function and then define the concepts of limit and derivative in complex. This discussion will be similar to that in calculus. Nevertheless it needs great attention because it will show interesting basic differences between real and complex calculus.

Recall from calculus that a *real* function $f$ defined on a set $S$ of real numbers (usually an interval) is a rule that assigns to every $x$ in $S$ a real number $f(x)$, called the *value* of $f$ at $x$. Now in complex, $S$ is a set of *complex* numbers. And a **function** $f$ defined on $S$ is a rule that assigns to every $z$ in $S$ a complex number $w$, called the *value* of $f$ at $z$. We write

$$w = f(z).$$

Here $z$ varies in $S$ and is called a **complex variable.** The set $S$ is called the *domain of definition* of $f$ or, briefly, the **domain** of $f$. (In most cases $S$ will be open and connected, thus a domain as defined just before.)

*Example:* $w = f(z) = z^2 + 3z$ is a complex function defined for all $z$; that is, its domain $S$ is the whole complex plane.

The set of all values of a function $f$ is called the **range** *of* $f$.

$w$ is complex, and we write $w = u + iv$, where $u$ and $v$ are the real and imaginary parts, respectively. Now $w$ depends on $z = x + iy$. Hence $u$ becomes a real function of $x$ and $y$, and so does $v$. We may thus write

$$w = f(z) = u(x, y) + iv(x, y).$$

This shows that a *complex* function $f(z)$ is equivalent to a *pair* of *real* functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables $x$ and $y$.

**EXAMPLE 1** **Function of a Complex Variable**

Let $w = f(z) = z^2 + 3z$. Find $u$ and $v$ and calculate the value of $f$ at $z = 1 + 3i$.

***Solution.*** $u = \text{Re } f(z) = x^2 - y^2 + 3x$ and $v = 2xy + 3y$. Also,

$$f(1 + 3i) = (1 + 3i)^2 + 3(1 + 3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i.$$

This shows that $u(1, 3) = -5$ and $v(1, 3) = 15$. Check this by using the expressions for $u$ and $v$. ■

**EXAMPLE 2** **Function of a Complex Variable**

Let $w = f(z) = 2iz + 6\bar{z}$. Find $u$ and $v$ and the value of $f$ at $z = \frac{1}{2} + 4i$.

***Solution.*** $f(z) = 2i(x + iy) + 6(x - iy)$ gives $u(x, y) = 6x - 2y$ and $v(x, y) = 2x - 6y$. Also,

$$f(\tfrac{1}{2} + 4i) = 2i(\tfrac{1}{2} + 4i) + 6(\tfrac{1}{2} - 4i) = i - 8 + 3 - 24i = -5 - 23i.$$

Check this as in Example 1. ■

### Remarks on Notation and Terminology

**1.** Strictly speaking, $f(z)$ denotes the value of $f$ at $z$, but it is a convenient abuse of language to talk about *the function* $f(z)$ (instead of *the function* $f$), thereby exhibiting the notation for the independent variable.

**2.** We assume all functions to be ***single-valued relations,*** as usual: to each $z$ in $S$ there corresponds but *one* value $w = f(z)$ (but, of course, several $z$ may give the same value $w = f(z)$, just as in calculus). Accordingly, we shall *not use* the term "multivalued function" (used in some books on complex analysis) for a multivalued relation, in which to a $z$ there corresponds more than one $w$.

## Limit, Continuity

A function $f(z)$ is said to have the **limit** $l$ as $z$ approaches a point $z_0$, written

$$\lim_{z \to z_0} f(z) = l, \tag{1}$$

if $f$ is defined in a neighborhood of $z_0$ (except perhaps at $z_0$ itself) and if the values of $f$ are "close" to $l$ for all $z$ "close" to $z_0$; in precise terms, if for every positive real $\epsilon$ we can find a positive real $\delta$ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$ (Fig. 330) we have

$$|f(z) - l| < \epsilon; \tag{2}$$

geometrically, if for every $z \neq z_0$ in that $\delta$-disk the value of $f$ lies in the disk (2).

Formally, this definition is similar to that in calculus, but there is a big difference. Whereas in the real case, $x$ can approach an $x_0$ only along the real line, here, by definition, $z$ may approach $z_0$ ***from any direction*** in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique. (See Team Project 26.)

A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{3}$$

Note that by definition of a limit this implies that $f(z)$ is defined in some neighborhood of $z_0$.

$f(z)$ is said to be *continuous in a domain* if it is continuous at each point of this domain.

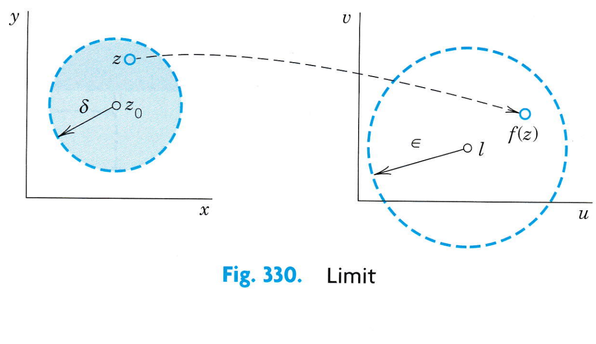



Fig. 330. Limit

## Derivative

The **derivative** of a complex function $f$ at a point $z_0$ is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{4}$$

provided this limit exists. Then $f$ is said to be **differentiable** at $z_0$. If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and (4) takes the form

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{4'}$$

Now comes an ***important point.*** Remember that, by the definition of limit, $f(z)$ is defined in a neighborhood of $z_0$ and $z$ in (4′) may approach $z_0$ from any direction in the complex plane. Hence differentiability at $z_0$ means that, along whatever path $z$ approaches $z_0$, the quotient in (4′) always approaches a certain value and all these values are equal. This is important and should be kept in mind.

**EXAMPLE 3 Differentiability. Derivative**

The function $f(z) = z^2$ is differentiable for all $z$ and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$ ■

*The* ***differentiation rules*** *are the same as in real calculus,* since their proofs are literally the same. Thus for any analytic functions $f$ and $g$ and constants $c$ we have

$$(cf)' = cf', \quad (f+g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

as well as the chain rule and the power rule $(z^n)' = nz^{n-1}$ ($n$ integer).

Also, if $f(z)$ is differentiable at $z_0$, it is continuous at $z_0$. (See Team Project 26.)

**EXAMPLE 4 $\bar{z}$ not Differentiable**

It may come as a surprise that there are many complex functions that do not have a derivative at any point. For instance, $f(z) = \bar{z} = x - iy$ is such a function. To see this, we write $\Delta z = \Delta x + i\Delta y$ and obtain

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{(z + \Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}. \tag{5}$$

If $\Delta y = 0$, this is $+1$. If $\Delta x = 0$, this is $-1$. Thus (5) approaches $+1$ along path I in Fig. 331 but $-1$ along path II. Hence, by definition, the limit of (5) as $\Delta z \to 0$ does not exist at any $z$. ■

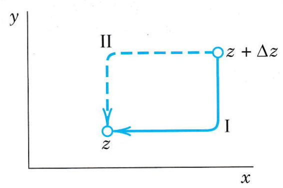



**Fig. 331.** Paths in (5)

Surprising as Example 4 may be, it merely illustrates that differentiability of a *complex* function is a rather severe requirement.

The idea of proof (approach of $z$ from different directions) is basic and will be used again as the crucial argument in the next section.

## Analytic Functions

Complex analysis is concerned with the theory and application of "analytic functions," that is, functions that are differentiable in some domain, so that we can do "calculus in complex." The definition is as follows.

**DEFINITION**

**Analyticity**

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of $D$. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in $D$ if $f(z)$ is analytic in a neighborhood of $z_0$.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

Hence analyticity of $f(z)$ at $z_0$ means that $f(z)$ has a derivative at every point in some neighborhood of $z_0$ (including $z_0$ itself since, by definition, $z_0$ is a point of all its neighborhoods). This concept is *motivated* by the fact that it is of no practical interest if a function is differentiable merely at a single point $z_0$ but not throughout some neighborhood of $z_0$. Team Project 26 gives an example.

A more modern term for *analytic in D* is *holomorphic in D.*

**EXAMPLE 5** **Polynomials, Rational Functions**

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials,** that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function.** This $f$ is analytic except at the points where $h(z) = 0$; here we assume that common factors of $g$ and $h$ have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

## PROBLEM SET 13.3

### 1–10 CURVES AND REGIONS OF PRACTICAL INTEREST

Find and sketch or graph the sets in the complex plane given by

1. $|z - 3 - 2i| = \frac{4}{3}$
2. $1 \leqq |z - 1 + 4i| \leqq 5$
3. $0 < |z - 1| < 1$
4. $-\pi < \text{Re } z < \pi$
5. $\text{Im } z^2 = 2$
6. $\text{Re } z > -1$
7. $|z + 1| = |z - 1|$
8. $|\text{Arg } z| \leqq \frac{1}{4}\pi$
9. $\text{Re } z \leqq \text{Im } z$
10. $\text{Re } (1/z) < 1$

**11. WRITING PROJECT. Sets in the Complex Plane.** Extend the part of the text on sets in the complex plane by formulating that part in your own words and including examples of your own and comparing with calculus when applicable.

### COMPLEX FUNCTIONS AND DERIVATIVES

**12–15** **Function Values.** Find Re $f$ and Im $f$. Also find their values at the given point $z$.

**12.** $f = 3z^2 - 6z + 3i$, $z = 2 + i$

**13.** $f = z/(z + 1)$, $z = 4 - 5i$

**14.** $f = 1/(1 - z)$, $z = \frac{1}{2} + \frac{1}{4}i$

**15.** $f = 1/z^2$, $z = 1 + i$

**16–19** **Continuity.** Find out (and give reason) whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function $f$ is equal to:

**16.** $[\text{Re}\,(z^2)]/|z|^2$

**17.** $[\text{Im}\,(z^2)]/|z|$

**18.** $|z|^2\,\text{Re}\,(1/z)$

**19.** $(\text{Im}\,z)/(1 - |z|)$

**20–24** **Derivative.** Differentiate

**20.** $(z^2 - 9)/(z^2 + 1)$

**21.** $(z^3 + i)^2$

**22.** $(3z + 4i)/(1.5iz - 2)$

**23.** $i/(1 - z)^2$

**24.** $z^2/(z + i)^2$

**25. CAS PROJECT. Graphing Functions.** Find and graph Re $f$, Im $f$, and $|f|$ as surfaces over the $z$-plane. Also graph the two families of curves Re $f(z) = const$ and Im $f(z) = const$ in the same figure, and the curves $|f(z)| = const$ in another figure, where (a) $f(z) = z^2$, (b) $f(z) = 1/z$, (c) $f(z) = z^4$.

**26. TEAM PROJECT. Limit, Continuity, Derivative**

**(a) Limit.** Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re}\, f(z) = \text{Re}\, l, \qquad \lim_{z \to z_0} \text{Im}\, f(z) = \text{Im}\, l.$$

**(b) Limit.** If $\lim_{z \to z_0} f(z)$ exists, show that this limit is unique.

**(c) Continuity.** If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

**(d) Continuity.** If $f(z)$ is differentiable at $z_0$, show that $f(z)$ is continuous at $z_0$.

**(e) Differentiability.** Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any $z$. Can you find other such functions?

**(f) Differentiability.** Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

# 13.4 Cauchy–Riemann Equations. Laplace's Equation

***The Cauchy–Riemann equations are the most important equations in this chapter*** and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, $f$ is analytic in a domain $D$ if and only if the first partial derivatives of $u$ and $v$ satisfy the two **Cauchy–Riemann equations**[4]

$$u_x = v_y, \qquad u_y = -v_x \tag{1}$$

[4]The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GR9] in App. 1.

everywhere in $D$; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for $v$) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

*Example:* $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all $z$ (see Example 3 in Sec. 13.3), and $u = x^2 - y^2$ and $v = 2xy$ satisfy (1), namely, $u_x = 2x = v_y$ as well as $u_y = -2y = -v_x$. More examples will follow.

**THEOREM 1**

**Cauchy–Riemann Equations**

*Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at $z$ itself. Then at that point, the first-order partial derivatives of $u$ and $v$ exist and satisfy the Cauchy–Riemann equations* (1).

*Hence if $f(z)$ is analytic in a domain $D$, those partial derivatives exist and satisfy* (1) *at all points of $D$.*

**PROOF** By assumption, the derivative $f'(z)$ at $z$ exists. It is given by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2}$$

The idea of the proof is very simple. By the definition of a limit in complex (Sec. 13.3) we can let $\Delta z$ approach zero along any path in a neighborhood of $z$. Thus we may choose the two paths I and II in Fig. 332 and equate the results. By comparing the real parts we shall obtain the first Cauchy–Riemann equation and by comparing the imaginary parts the second. The technical details are as follows.

We write $\Delta z = \Delta x + i\Delta y$. Then $z + \Delta z = x + \Delta x + i(y + \Delta y)$, and in terms of $u$ and $v$ the derivative in (2) becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y}. \tag{3}$$

We first choose path I in Fig. 332. Thus we let $\Delta y \to 0$ first and then $\Delta x \to 0$. After $\Delta y$ is zero, $\Delta z = \Delta x$. Then (3) becomes, if we first write the two $u$-terms and then the two $v$-terms,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}.$$

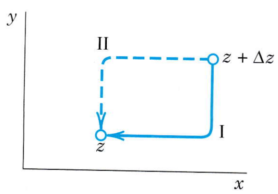



**Fig. 332.** Paths in (2)

Since $f'(z)$ exists, the two real limits on the right exist. By definition, they are the partial derivatives of $u$ and $v$ with respect to $x$. Hence the derivative $f'(z)$ of $f(z)$ can be written

$$f'(z) = u_x + iv_x. \tag{4}$$

Similarly, if we choose path II in Fig. 332, we let $\Delta x \to 0$ first and then $\Delta y \to 0$. After $\Delta x$ is zero, $\Delta z = i\Delta y$, so that from (3) we now obtain

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}.$$

Since $f'(z)$ exists, the limits on the right exist and give the partial derivatives of $u$ and $v$ with respect to $y$; noting that $1/i = -i$, we thus obtain

$$f'(z) = -iu_y + v_y. \tag{5}$$

The existence of the derivative $f'(z)$ thus implies the existence of the four partial derivatives in (4) and (5). By equating the real parts $u_x$ and $v_y$ in (4) and (5) we obtain the first Cauchy–Riemann equation (1). Equating the imaginary parts gives the other. This proves the first statement of the theorem and implies the second because of the definition of analyticity. ■

Formulas (4) and (5) are also quite practical for calculating derivatives $f'(z)$, as we shall see.

**EXAMPLE 1 Cauchy–Riemann Equations**

$f(z) = z^2$ is analytic for all $z$. It follows that the Cauchy–Riemann equations must be satisfied (as we have verified above).

For $f(z) = \bar{z} = x - iy$ we have $u = x, v = -y$ and see that the second Cauchy–Riemann equation is satisfied, $u_y = -v_x = 0$, but the first is not: $u_x = 1 \neq v_y = -1$. We conclude that $f(z) = \bar{z}$ is not analytic, confirming Example 4 of Sec. 13.3. Note the savings in calculation! ■

The Cauchy–Riemann equations are fundamental because they are not only necessary but also sufficient for a function to be analytic. More precisely, the following theorem holds.

**THEOREM 2 Cauchy–Riemann Equations**

*If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables $x$ and $y$ have* ***continuous*** *first partial derivatives that satisfy the Cauchy–Riemann equations in some domain $D$, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in $D$.*

The proof is more involved than that of Theorem 1 and we leave it optional (see App. 4).

Theorems 1 and 2 are of great practical importance, since by using the Cauchy–Riemann equations we can now easily find out whether or not a given complex function is analytic.

**EXAMPLE 2 Cauchy–Riemann Equations. Exponential Function**

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

***Solution.*** We have $u = e^x \cos y$, $v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$
$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

We see that the Cauchy–Riemann equations are satisfied and conclude that $f(z)$ is analytic for all $z$. ($f(z)$ will be the complex analog of $e^x$ known from calculus.) ■

**EXAMPLE 3 An Analytic Function of Constant Absolute Value Is Constant**

The Cauchy–Riemann equations also help in deriving general properties of analytic functions.

For instance, show that if $f(z)$ is analytic in a domain $D$ and $|f(z)| = k = const$ in $D$, then $f(z) = const$ in $D$. (We shall make crucial use of this in Sec. 18.6 in the proof of Theorem 3.)

***Solution.*** By assumption, $|f|^2 = |u + iv|^2 = u^2 + v^2 = k^2$. By differentiation,

$$uu_x + vv_x = 0,$$
$$uu_y + vv_y = 0.$$

Now use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, to get

$$\text{(6)} \qquad \begin{aligned} &\text{(a)} \quad uu_x - vu_y = 0, \\ &\text{(b)} \quad uu_y + vu_x = 0. \end{aligned}$$

To get rid of $u_y$, multiply (6a) by $u$ and (6b) by $v$ and add. Similarly, to eliminate $u_x$, multiply (6a) by $-v$ and (6b) by $u$ and add. This yields

$$(u^2 + v^2)u_x = 0.$$
$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $v_x = v_y = 0$. Together this implies $u = const$ and $v = const$; hence $f = const$. ■

We mention that if we use the polar form $z = r(\cos\theta + i \sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are (Prob. 11)

$$\textbf{(7)} \qquad \begin{aligned} u_r &= \frac{1}{r} v_\theta, \\ v_r &= -\frac{1}{r} u_\theta \end{aligned} \qquad (r > 0).$$

## Laplace's Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace's equation, the most important PDE of physics, which occurs in gravitation, electrostatics, fluid flow, heat conduction, and so on (see Chaps. 12 and 18).

**THEOREM 3**

**Laplace's Equation**

*If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $D$, then both $u$ and $v$ satisfy* **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{8}$$

($\nabla^2$ read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0, \tag{9}$$

*in $D$ and have continuous second partial derivatives in $D$.*

**PROOF** Differentiating $u_x = v_y$ with respect to $x$ and $u_y = -v_x$ with respect to $y$, we have

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}. \tag{10}$$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that $u$ and $v$ have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$. By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating $u_x = v_y$ with respect to $y$ and $u_y = -v_x$ with respect to $x$ and subtracting, using $u_{xy} = u_{yx}$. ■

Solutions of Laplace's equation having ***continuous*** second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory** (see also Sec. 12.10). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions $u$ and $v$ satisfy the Cauchy–Riemann equations in a domain $D$, they are the real and imaginary parts of an analytic function $f$ in $D$. Then $v$ is said to be a **harmonic conjugate function** of $u$ in $D$. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

**EXAMPLE 4** **How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations**

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function $v$ of $u$.

***Solution.*** $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate $v$ of $u$ must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to $y$ and differentiating the result with respect to $x$, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ ($c$ any real constant) is the most general harmonic conjugate of the given $u$. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$ ■

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shall show in Chap. 18.

## PROBLEM SET 13.4

### 1–10 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? [Use (1) or (7).]

**1.** $f(z) = z^4$

**2.** $f(z) = \text{Im}\,(z^2)$

**3.** $e^{2x}(\cos y + i \sin y)$

**4.** $f(z) = 1/(1 - z^4)$

**5.** $e^{-x}(\cos y - i \sin y)$

**6.** $f(z) = \text{Arg}\,\pi z$

**7.** $f(z) = \text{Re}\, z + \text{Im}\, z$

**8.** $f(z) = \ln |z| + i \,\text{Arg}\, z$

**9.** $f(z) = i/z^8$

**10.** $f(z) = z^2 + 1/z^2$

**11. (Cauchy–Riemann equations in polar form)** Derive (7) from (1).

### 12–21 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

**12.** $u = xy$

**13.** $v = xy$

**14.** $v = -y/(x^2 + y^2)$

**15.** $u = \ln |z|$

**16.** $v = \ln |z|$

**17.** $u = x^3 - 3xy^2$

**18.** $u = 1/(x^2 + y^2)$

**19.** $v = (x^2 - y^2)^2$

**20.** $u = \cos x \cosh y$

**21.** $u = e^{-x} \sin 2y$

**22–24** Determine $a$, $b$, $c$ such that the given functions are harmonic and find a harmonic conjugate.

**22.** $u = e^{3x} \cos ay$

**23.** $u = \sin x \cosh cy$

**24.** $u = ax^3 + by^3$

**25. (Harmonic conjugate)** Show that if $u$ is harmonic and $v$ is a harmonic conjugate of $u$, then $u$ is a harmonic conjugate of $-v$.

**26. TEAM PROJECT. Conditions for $f(z) = $ *const.*** Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = $ *const.*

**(a)** $\text{Re}\, f(z) = $ *const*

**(b)** $\text{Im}\, f(z) = $ *const*

**(c)** $f'(z) = 0$

**(d)** $|f(z)| = $ *const* (see Example 3)

**27. (Two further formulas for the derivative).** Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$(11) \qquad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

**28. CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = $ *const* of a harmonic function $u$ and of its conjugate $v$ on the same axes. Apply the program to (a) $u = x^2 - y^2$, $v = 2xy$, (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$, (c) $u = e^x \cos y$, $v = e^x \sin y$.

# 13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of $e^z$ in terms of the real functions $e^x$, $\cos y$, and $\sin y$ is

$$(1) \qquad e^z = e^x(\cos y + i \sin y).$$

This definition is motivated by the fact the $e^z$ ***extends*** the real exponential function $e^x$ of calculus in a natural fashion. Namely:

**(A)** $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.

**(B)** $e^z$ is analytic for all $z$. (Proved in Example 2 of Sec. 13.4.)

**(C)** The derivative of $e^z$ is $e^z$, that is,

$$(e^z)' = e^z. \tag{2}$$

This follows from (4) in Sec. 13.4,

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z.$$

REMARK. This definition provides for a relatively simple discussion. We could define $e^z$ by the familiar series $1 + x + x^2/2! + x^3/3! + \cdots$ with $x$ replaced by $z$, but we would then have to discuss complex series at this very early stage. (We will show the connection in Sec. 15.4.)

**Further Properties.** A function $f(z)$ that is analytic for all $z$ is called an **entire function.** Thus, $e^z$ is entire. Just as in calculus the ***functional relation***

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{3}$$

holds for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Indeed, by (1),

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2).$$

Since $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for these *real* functions, by an application of the addition formulas for the cosine and sine functions (similar to that in Sec. 13.2) we see that

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{z_1+z_2}$$

as asserted. An interesting special case of (3) is $z_1 = x$, $z_2 = iy$; then

$$e^z = e^x e^{iy}. \tag{4}$$

Furthermore, for $z = iy$ we have from (1) the so-called **Euler formula**

$$e^{iy} = \cos y + i \sin y. \tag{5}$$

Hence the **polar form** of a complex number, $z = r(\cos\theta + i \sin\theta)$, may now be written

$$z = re^{i\theta}. \tag{6}$$

From (5) we obtain

$$e^{2\pi i} = 1 \tag{7}$$

as well as the important formulas (verify!)

$$e^{\pi i/2} = i, \qquad e^{\pi i} = -1, \qquad e^{-\pi i/2} = -i, \qquad e^{-\pi i} = -1. \tag{8}$$

Another consequence of (5) is

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1. \tag{9}$$

That is, for pure imaginary exponents the exponential function has absolute value 1, a result you should remember. From (9) and (1),

$$|e^z| = e^x. \qquad \text{Hence} \qquad \arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2, \cdots), \tag{10}$$

since $|e^z| = e^x$ shows that (1) is actually $e^z$ in polar form.

From $|e^z| = e^x \neq 0$ in (10) we see that

$$e^z \neq 0 \qquad \text{for all } z. \tag{11}$$

So here we have an entire function that never vanishes, in contrast to (nonconstant) polynomials, which are also entire (Example 5 in Sec. 13.3) but always have a zero, as is proved in algebra.

**Periodicity of $e^z$ with period $2\pi i$,**

$$e^{z+2\pi i} = e^z \qquad \text{for all } z \tag{12}$$

is a basic property that follows from (1) and the periodicity of $\cos y$ and $\sin y$. Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width $2\pi$

$$-\pi < y \leqq \pi \qquad \text{(Fig. 333).} \tag{13}$$

This infinite strip is called a **fundamental region** of $e^z$.

**EXAMPLE 1 Function Values. Solution of Equations.**

Computation of values from (1) provides no problem. For instance, verify that

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$

$$|e^{1.4-0.6i}| = e^{1.4} = 4.055, \qquad \operatorname{Arg} e^{1.4-0.6i} = -0.6.$$

To illustrate (3), take the product of

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i \sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

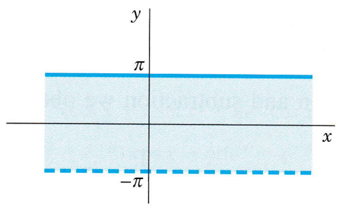



Fig. 333. Fundamental region of the exponential function $e^z$ in the $z$-plane

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

*Ans.* $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of $e^z$). They lie on the vertical line $x = 1.609$ at a distance $2\pi$ from their neighbors. ■

To summarize: many properties of $e^z = \exp z$ parallel those of $e^x$; an exception is the periodicity of $e^z$ with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that $e^z$ is an *entire function.* (Do you still remember what that means?)

## PROBLEM SET 13.5

**1.** Using the Cauchy–Riemann equations, show that $e^z$ is entire.

**2–8 Values of $e^z$.** Compute $e^z$ in the form $u + iv$ and $|e^z|$, where $z$ equals:

**2.** $3 + \pi i$
**3.** $1 + 2i$
**4.** $\sqrt{2} - \frac{1}{2}\pi i$
**5.** $7\pi i/2$
**6.** $(1 + i)\pi$
**7.** $0.8 - 5i$
**8.** $9\pi i/2$

**9–12 Real and Imaginary Parts.** Find Re and Im of:

**9.** $e^{-2z}$
**10.** $e^{z^3}$
**11.** $e^{z^2}$
**12.** $e^{1/z}$

**13–17 Polar Form.** Write in polar form:

**13.** $\sqrt{i}$
**14.** $1 + i$
**15.** $\sqrt[n]{z}$
**16.** $3 + 4i$
**17.** $-9$

**18–21 Equations.** Find all solutions and graph some of them in the complex plane.

**18.** $e^{3z} = 4$
**19.** $e^z = -2$
**20.** $e^z = 0$
**21.** $e^z = 4 - 3i$

**22. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity.** Show that $e^z$ is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

**(b) Special values.** Find all $z$ such that (i) $e^z$ is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

**(c) Harmonic function.** Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

**(d) Uniqueness.** It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where $f$ is assumed to be entire. Prove this using the Cauchy–Riemann equations.

# 13.6 Trigonometric and Hyperbolic Functions

Just as we extended the real $e^x$ to the complex $e^z$ in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions.* We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \tag{1}$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z} \tag{2}$$

and

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \tag{3}$$

Since $e^z$ is entire, $\cos z$ and $\sin z$ are entire functions. $\tan z$ and $\sec z$ are not entire; they are analytic except at the points where $\cos z$ is zero; and $\cot z$ and $\csc z$ are analytic except where $\sin z$ is zero. Formulas for the derivatives follow readily from $(e^z)' = e^z$ and (1)–(3); as in calculus,

$$(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z, \tag{4}$$

etc. Equation (1) also shows that **Euler's formula** ***is valid in complex:***

$$e^{iz} = \cos z + i \sin z \qquad \text{for all } z. \tag{5}$$

The real and imaginary parts of $\cos z$ and $\sin z$ are needed in computing values, and they also help in displaying properties of our functions. We illustrate this with a typical example.

**EXAMPLE 1 Real and Imaginary Parts. Absolute Value. Periodicity**

Show that

$$\text{(6)} \quad \begin{aligned} &\textbf{(a)} \quad \cos z = \cos x \cosh y - i \sin x \sinh y \\ &\textbf{(b)} \quad \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

and

$$\text{(7)} \quad \begin{aligned} &\textbf{(a)} \quad |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &\textbf{(b)} \quad |\sin z|^2 = \sin^2 x + \sinh^2 y \end{aligned}$$

and give some applications of these formulas.

***Solution.*** From (1),

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\ &= \tfrac{1}{2}e^{-y}(\cos x + i \sin x) + \tfrac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \tfrac{1}{2}(e^y + e^{-y}) \cos x - \tfrac{1}{2}i(e^y - e^{-y}) \sin x. \end{aligned}$$

This yields (6a) since, as is known form calculus,

$$\cosh y = \tfrac{1}{2}(e^y + e^{-y}), \qquad \sinh y = \tfrac{1}{2}(e^y - e^{-y}); \tag{8}$$

(6b) is obtained similarly. From (6a) and $\cosh^2 y = 1 + \sinh^2 y$ we obtain

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y.$$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

For instance, $\cos(2 + 3i) = \cos 2 \cosh 3 - i \sin 2 \sinh 3 = -4.190 - 9.109i$.

From (6) we see that $\cos z$ and $\sin z$ are ***periodic with period $2\pi$***, just as in real. Periodicity of $\tan z$ and $\cot z$ with period $\pi$ now follows.

Formula (7) points to an essential difference between the real and the complex cosine and sine; whereas $|\cos x| \leqq 1$ and $|\sin x| \leqq 1$, the complex cosine and sine functions are ***no longer bounded*** but approach infinity in absolute value as $y \to \infty$, since then $\sinh y \to \infty$ in (7). ∎

**EXAMPLE 2 Solutions of Equations. Zeros of cos z and sin z**

Solve (a) $\cos z = 5$ (which has no real solution!), (b) $\cos z = 0$, (c) $\sin z = 0$.

***Solution.*** (a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by $e^{iz}$. This is a quadratic equation in $e^{iz}$, with solutions (rounded off to 3 decimals)

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25 - 1} = 9.899 \quad \text{and} \quad 0.101.$$

Thus $e^{-y} = 9.899$ or $0.101$, $e^{ix} = 1$, $y = \pm 2.292$, $x = 2n\pi$. *Ans.* $z = \pm 2n\pi \pm 2.292i$ $(n = 0, 1, 2, \cdots)$.

Can you obtain this from (6a)?

(b) $\cos x = 0$, $\sinh y = 0$ by (7a), $y = 0$. *Ans.* $z = \pm\frac{1}{2}(2n + 1)\pi$ $(n = 0, 1, 2, \cdots)$.

(c) $\sin x = 0$, $\sinh y = 0$ by (7b). *Ans.* $z = \pm n\pi$ $(n = 0, 1, 2, \cdots)$. Hence the only zeros of $\cos z$ and $\sin z$ are those of the real cosine and sine functions. ∎

**General formulas** *for the real trigonometric functions continue to hold for complex values.* This follows immediately from the definitions. We mention in particular the addition rules

$$\begin{aligned} \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1 \end{aligned} \tag{9}$$

and the formula

$$\cos^2 z + \sin^2 z = 1. \tag{10}$$

Some further useful formulas are included in the problem set.

## Hyperbolic Functions

The complex **hyperbolic cosine** and **sine** are defined by the formulas

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}). \tag{11}$$

This is suggested by the familiar definitions for a real variable [see (8)]. These functions are entire, with derivatives

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z, \tag{12}$$

as in calculus. The other hyperbolic functions are defined by

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$
$$(13) \qquad \operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

***Complex Trigonometric and Hyperbolic Functions Are Related.*** If in (11), we replace $z$ by $iz$ and then use (1), we obtain

$$\textbf{(14)} \qquad \cosh iz = \cos z, \qquad \sinh iz = i \sin z.$$

Similarly, if in (1) we replace $z$ by $iz$ and then use (11), we obtain conversely

$$\textbf{(15)} \qquad \cos iz = \cosh z, \qquad \sin iz = i \sinh z.$$

Here we have another case of *unrelated* real functions that have *related* complex analogs, pointing again to the advantage of working in complex in order to get both a more unified formalism and a deeper understanding of special functions. This is one of the main reasons for the importance of complex analysis to the engineer and physicist.

## PROBLEM SET 13.6

**1.** Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.

**2.** Verify by differentiation that $\operatorname{Re} \cos z$ and $\operatorname{Im} \sin z$ are harmonic.

**3–6 FORMULAS FOR HYPERBOLIC FUNCTIONS**

Show that

**3.** $$\cosh z = \cosh x \cos y + i \sinh x \sin y$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

**4.** $$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

**5.** $\cosh^2 z - \sinh^2 z = 1$

**6.** $\cosh^2 z + \sinh^2 z = \cosh 2z$

**7–15 Function Values.** Compute (in the form $u + iv$)

**7.** $\cos (1 + i)$

**8.** $\sin (1 + i)$

**9.** $\sin 5i$, $\cos 5i$

**10.** $\cos 3\pi i$

**11.** $\cosh (-2 + 3i)$, $\cos (-3 - 2i)$

**12.** $-i \sinh (-\pi + 2i)$, $\sin (2 + \pi i)$

**13.** $\cosh (2n + 1)\pi i$, $n = 1, 2, \cdots$

**14.** $\sinh (4 - 3i)$

**15.** $\cosh (4 - 6\pi i)$

**16. (Real and imaginary parts)** Show that

$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$
$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

**17–21 Equations.** Find all solutions of the following equations.

**17.** $\cosh z = 0$

**18.** $\sin z = 100$

**19.** $\cos z = 2i$

**20.** $\cosh z = -1$

**21.** $\sinh z = 0$

**22.** Find all $z$ for which (a) $\cos z$, (b) $\sin z$ has real values.

**23–25 Equations and Inequalities.** Using the definitions, prove:

**23.** $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.

**24.** $|\sinh y| \leqq |\cos z| \leqq \cosh y$, $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.

**25.** $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

# 13.7 Logarithm. General Power

We finally introduce the *complex logarithm,* which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all $w$; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now from Sec. 13.5 we know that $e^{u+iv}$ has the absolute value $e^u$ and the argument $v$. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\text{(1)} \qquad \ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z).$$

Now comes an important point (without analog in real calculus). Since the argument of $z$ is determined only up to integer multiples of $2\pi$, ***the complex natural logarithm* $\ln z$ ($z \neq 0$) *is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\operatorname{Arg} z$ (see Sec. 13.2) is denoted by $\operatorname{Ln} z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\text{(2)} \qquad \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z \qquad (z \neq 0).$$

The uniqueness of $\operatorname{Arg} z$ for given $z$ ($\neq 0$) implies that $\operatorname{Ln} z$ is single-valued, that is, a function in the usual sense. Since the other values of $\arg z$ differ by integer multiples of $2\pi$, the other values of $\ln z$ are given by

$$\text{(3)} \qquad \ln z = \operatorname{Ln} z \pm 2n\pi i \qquad (n = 1, 2, \cdots).$$

They all have the same real part, and their imaginary parts differ by integer multiples of $2\pi$.

If $z$ is positive real, then $\operatorname{Arg} z = 0$, and $\operatorname{Ln} z$ becomes identical with the real natural logarithm known from calculus. If $z$ is negative real (so that the natural logarithm of calculus is not defined!), then $\operatorname{Arg} z = \pi$ and

$$\operatorname{Ln} z = \ln |z| + \pi i \qquad (z \text{ negative real}).$$

From (1) and $e^{\ln r} = r$ for positive real $r$ we obtain

$$e^{\ln z} = z \tag{4a}$$

as expected, but since $\arg(e^z) = y \pm 2n\pi$ is multivalued, so is

$$\ln(e^z) = z \pm 2n\pi i, \qquad n = 0, 1, \cdots. \tag{4b}$$

**EXAMPLE 1 Natural Logarithm. Principal Value**

$$\begin{aligned}
\ln 1 &= 0, \pm 2\pi i, \pm 4\pi i, \cdots & \mathrm{Ln}\, 1 &= 0 \\
\ln 4 &= 1.386\,294 \pm 2n\pi i & \mathrm{Ln}\, 4 &= 1.386\,294 \\
\ln(-1) &= \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots & \mathrm{Ln}(-1) &= \pi i \\
\ln(-4) &= 1.386\,294 \pm (2n+1)\pi i & \mathrm{Ln}(-4) &= 1.386\,294 + \pi i \\
\ln i &= \pi i/2, -3\pi/2, 5\pi i/2, \cdots & \mathrm{Ln}\, i &= \pi i/2 \\
\ln 4i &= 1.386\,294 + \pi i/2 \pm 2n\pi i & \mathrm{Ln}\, 4i &= 1.386\,294 + \pi i/2 \\
\ln(-4i) &= 1.386\,294 - \pi i/2 \pm 2n\pi i & \mathrm{Ln}(-4i) &= 1.386\,294 - \pi i/2 \\
\ln(3-4i) &= \ln 5 + i \arg(3-4i) & \mathrm{Ln}(3-4i) &= 1.609\,438 - 0.927\,295i \\
&= 1.609\,438 - 0.927\,295i \pm 2n\pi i & &\text{(Fig. 334)}
\end{aligned}$$

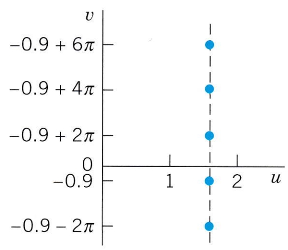



Fig. 334. Some values of ln (3 − 4i) in Example 1

The familiar relations for the natural logarithm continue to hold for complex values, that is,

$$\text{(a)}\quad \ln(z_1 z_2) = \ln z_1 + \ln z_2, \qquad \text{(b)}\quad \ln(z_1/z_2) = \ln z_1 - \ln z_2 \tag{5}$$

but these relations are to be understood in the sense that each value of one side is also contained among the values of the other side; see the next example.

**EXAMPLE 2 Illustration of the Functional Relation (5) in Complex**

Let

$$z_1 = z_2 = e^{\pi i} = -1.$$

If we take the principal values

$$\mathrm{Ln}\, z_1 = \mathrm{Ln}\, z_2 = \pi i,$$

then (5a) holds provided we write $\ln(z_1 z_2) = \ln 1 = 2\pi i$; however, it is not true for the principal value, $\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}\, 1 = 0$.

**THEOREM 1**

**Analyticity of the Logarithm**

*For every* $n = 0, \pm 1, \pm 2, \cdots$ *formula* (3) *defines a function, which is analytic, except at* 0 *and on the negative real axis, and has the derivative*

**(6)** $$(\ln z)' = \frac{1}{z} \qquad (z \text{ not } 0 \text{ or negative real}).$$

**PROOF** We show that the Cauchy–Riemann equations are satisfied. From (1)–(3) we have

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2} \ln (x^2 + y^2) + i\left(\arctan \frac{y}{x} + c\right)$$

where the constant $c$ is a multiple of $2\pi$. By differentiation,

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right).$$

Hence the Cauchy–Riemann equations hold. [Confirm this by using these equations in polar form, which we did not use since we proved them only in the problems (to Sec. 13.4).] Formula (4) in Sec. 13.4 now gives (6),

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2 + y^2} + i\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}. \quad \blacksquare$$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm. The negative real axis is known as a **branch cut** and is usually graphed as shown in Fig. 335. The branch for $n = 0$ is called the **principal branch** of $\ln z$.

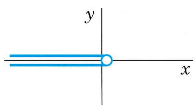



**Fig. 335.** Branch cut for ln z

## General Powers

General powers of a complex number $z = x + iy$ are defined by the formula

**(7)** $$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$

Since $\ln z$ is infinitely many-valued, $z^c$ will, in general, be multivalued. The particular value

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** *of* $z^c$.

If $c = n = 1, 2, \cdots$, then $z^n$ is single-valued and identical with the usual $n$th power of $z$. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n)\,\ln z} \qquad (z \neq 0),$$

the exponent is determined up to multiples of $2\pi i/n$ and we obtain the $n$ distinct values of the $n$th root, in agreement with the result in Sec. 13.2. If $c = p/q$, the quotient of two positive integers, the situation is similar, and $z^c$ has only finitely many distinct values. However, if $c$ is real irrational or genuinely complex, then $z^c$ is infinitely many-valued.

**EXAMPLE 3 General Power**

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2}i \pm 2n\pi i\right)\right] = e^{-(\pi/2)\mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$\begin{aligned}(1+i)^{2-i} &= \exp[(2-i)\ln(1+i)] = \exp\left[(2-i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}\right]\\ &= 2e^{\pi/4 \pm 2n\pi}\left[\sin\left(\tfrac{1}{2}\ln 2\right) + i\cos\left(\tfrac{1}{2}\ln 2\right)\right].\end{aligned}$$

It is a ***convention*** that for real positive $z = x$ the expression $z^c$ means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\operatorname{Ln} z$ $(z = x > 0)$ in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number $a$,

$$a^z = e^{z \ln a}. \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them ($e^z$, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

## PROBLEM SET 13.7

**1–9 Principal Value Ln $z$.** Find $\operatorname{Ln} z$ when $z$ equals:

1. $-10$
2. $2 + 2i$
3. $2 - 2i$
4. $-5 \pm 0.1i$
5. $-3 - 4i$
6. $-100$
7. $0.6 + 0.8i$
8. $-ei$
9. $1 - i$

**10–16 All Values of ln $z$.** Find all values and graph some of them in the complex plane.

10. $\ln 1$
11. $\ln(-1)$
12. $\ln e$
13. $\ln(-6)$
14. $\ln(4 + 3i)$
15. $\ln(-e^{-i})$
16. $\ln(e^{3i})$

17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

**18–21 Equations.** Solve for $z$:

18. $\ln z = (2 - \frac{1}{2}i)\pi$
19. $\ln z = 0.3 + 0.7i$
20. $\ln z = e - \pi i$
21. $\ln z = 2 + \frac{1}{4}\pi i$

**22–28** **General Powers.** Showing the details of your work, find the principal value of:

**22.** $i^{2i}$, $(2i)^i$

**23.** $4^{3+i}$

**24.** $(1 - i)^{1+i}$

**25.** $(1 + i)^{1-i}$

**26.** $(-1)^{1-2i}$

**27.** $i^{1/2}$

**28.** $(3 - 4i)^{1/3}$

**29.** How can you find the answer to Prob. 24 from the answer to Prob. 25?

**30. TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued.***) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

**(a)** $\arccos z = -i \ln (z + \sqrt{z^2 - 1})$

**(b)** $\arcsin z = -i \ln (iz + \sqrt{1 - z^2})$

**(c)** $\operatorname{arccosh} z = \ln (z + \sqrt{z^2 - 1})$

**(d)** $\operatorname{arcsinh} z = \ln (z + \sqrt{z^2 + 1})$

**(e)** $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

**(f)** $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

**(g)** Show that $w = \arcsin z$ is infinitely many-valued, and if $w_1$ is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

## CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

**1.** Add, subtract, multiply, and divide $26 - 7i$ and $3 + 4i$ as well as their complex conjugates.

**2.** Write the two given numbers in Prob. 1 in polar form. Find the principal value of their arguments.

**3.** What is the triangle inequality? Its geometric meaning? Its significance?

**4.** If you know the values of $\sqrt[6]{1}$, how do you get from them the values of $\sqrt[6]{z}$ for any $z$?

**5.** State the definition of the derivative from memory. It looks similar to that in calculus. But what is the big difference?

**6.** What is an analytic function? How would you test for analyticity?

**7.** Can a function be differentiable at a point without being analytic there? If yes, give an example.

**8.** Are $|z|$, $\bar{z}$, Re $z$, Im $z$ analytic? Give reason.

**9.** State the definitions of $e^z$, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ and the relations between these functions. Do these relations have analogs in real?

**10.** What properties of $e^z$ are similar to those of $e^x$ ? Which one is different?

**11.** What is the fundamental region of $e^z$ ? Its significance?

**12.** What is an entire function? Give examples.

**13.** Why is $\ln z$ much more complicated than $\ln x$? Explain from memory.

**14.** What is the principal value of $\ln z$?

**15.** How is the general power $z^c$ defined? Give examples.

**16–21** **Complex Numbers.** Find, in the form $x + iy$, showing the details:

**16.** $(1 + i)^{12}$

**17.** $(-2 + 6i)^2$

**18.** $1/(3 - 7i)$

**19.** $(1 - i)/(1 + i)^2$

**20.** $\sqrt{-5 - 12i}$

**21.** $(43 - 19i)/(8 + i)$

**22–26** **Polar Form.** Represent in polar form, with the principal argument:

**22.** $1 - 3i$

**23.** $-6 + 6i$

**24.** $\sqrt{20}/(4 + 2i)$

**25.** $-12i$

**26.** $2 + 2i$

**27–30** **Roots.** Find and graph all values of

**27.** $\sqrt{8i}$

**28.** $\sqrt[4]{256}$

**29.** $\sqrt[4]{-1}$

**30.** $\sqrt{32 - 24i}$

**31–35** **Analytic Functions.** Find $f(z) = u(x, y) + iv(x, y)$ with $u$ or $v$ as given. Check for analyticity.

**31.** $u = x/(x^2 + y^2)$

**32.** $v = e^{-3x} \sin 3y$

**33.** $u = x^2 - 2xy - y^2$

**34.** $u = \cos 2x \cosh 2y$

**35.** $v = e^{x^2-y^2} \sin 2xy$

**36–39** **Harmonic Functions.** Are the following functions harmonic? If so, find a harmonic conjugate.

**36.** $x^2y^2$

**37.** $xy$

**38.** $e^{-x/2} \cos \frac{1}{2}y$

**39.** $x^2 + y^2$

**40–45** **Special Function Values.** Find the values of

**40.** $\sin (3 + 4\pi i)$

**41.** $\sinh 4\pi i$

**42.** $\cos (5\pi + 2i)$

**43.** $\operatorname{Ln} (0.8 + 0.6i)$

**44.** $\tan (1 + i)$

**45.** $\cosh (1 + \pi i)$

## SUMMARY OF CHAPTER 13
# Complex Numbers and Functions

For arithmetic operations with **complex numbers**

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta), \tag{1}$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain $D$ if it has a **derivative** (Sec. 13.3)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{2}$$

everywhere in $D$. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of $z_0$ (not merely at $z_0$ itself).

If $f(z)$ is analytic in $D$, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

everywhere in $D$. Then $u$ and $v$ also satisfy **Laplace's equation**

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0 \tag{4}$$

everywhere in $D$. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in $D$ that satisfy (3) in $D$, then $f(z) = u(x, y) + iv(x, y)$ is analytic in $D$. See Sec. 13.4. (More on Laplace's equation and complex analysis follows in Chap. 18.)

The complex **exponential function** (Sec. 13.5)

$$e^z = \exp z = e^x(\cos y + i\sin y) \tag{5}$$

reduces to $e^x$ if $z = x$ $(y = 0)$. It is periodic with $2\pi i$ and has the derivative $e^z$.

The **trigonometric functions** are (Sec. 13.6)

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \end{aligned} \tag{6}$$

and, furthermore,

$$\tan z = (\sin z)/\cos z, \qquad \cot z = 1/\tan z, \qquad \text{etc.}$$

The **hyperbolic functions** are (Sec. 13.6)

$$\text{(7)} \qquad \cosh z = \tfrac{1}{2}(e^z + e^{-z}) = \cos iz, \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}) = -i \sin iz$$

etc. The functions (5)–(7) are **entire,** that is, analytic everywhere in the complex plane.

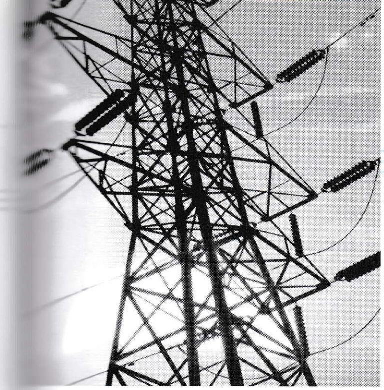
The **natural logarithm** is (Sec. 13.7)

$$\text{(8)} \qquad \ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z \pm 2n\pi i$$

where $z \neq 0$ and $n = 0, 1, \cdots$ . Arg $z$ is the **principal value** of arg $z$, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** Ln $z$ of ln $z$; thus $\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z$.

**General powers** are defined by (Sec. 13.7)

$$\text{(9)} \qquad z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$



CHAPTER 14

# Complex Integration

Two main reasons account for the importance of integration in the complex plane. The practical reason is that complex integration can evaluate certain real integrals appearing in applications that are not accessible by real integral calculus. The theoretical reason is that some basic properties of analytic functions are difficult to prove by other methods. A striking property of this type is the existence of higher derivatives of an analytic function.

Complex integration also plays a role in connection with special functions, such as the gamma function (see [GR1], p. 255), the error function, various polynomials (see [GR10]) and others, and the application of these functions in physics.

In this chapter we define and explain complex integrals. The most important result in the chapter is **Cauchy's integral theorem** or the *Cauchy–Goursat theorem,* as it is also called (Sec. 14.2). It implies **Cauchy's integral formula** (Sec. 14.3), which in turn implies the existence of all higher derivatives of an analytic function. Hence in this respect, complex analytic functions behave much more simply than real-valued functions of real variables, which may have derivatives only up to a certain order.

A further method of complex integration, known as integration by residues, and its application to real integrals will need complex series and follows in Chap. 16.

*Prerequisite:* Chap. 13
*References and Answers to Problems:* App. 1 Part D, App. 2.

## 14.1 Line Integral in the Complex Plane

As in calculus we distinguish between definite integrals and indefinite integrals or antiderivatives. An **indefinite integral** is a function whose derivative equals a given analytic function in a region. By inverting known differentiation formulas we may find many types of indefinite integrals.

*Complex* definite integrals are called (complex) **line integrals.** They are written

$$\int_C f(z)\, dz.$$

Here the **integrand** $f(z)$ is integrated over a given curve $C$ or a portion of it (an *arc,* but we shall say ***"curve"*** in either case, for simplicity). This curve $C$ in the complex plane is called the **path of integration.** We may represent $C$ by a parametric representation

$$z(t) = x(t) + iy(t) \qquad (a \leqq t \leqq b). \tag{1}$$

The sense of increasing $t$ is called the **positive sense** on $C$, and we say that $C$ is **oriented** by (1).

For instance, $z(t) = t + 3it$ $(0 \leqq t \leqq 2)$ gives a portion (a segment) of the line $y = 3x$. The function $z(t) = 4 \cos t + 4i \sin t$ $(-\pi \leqq t \leqq \pi)$ represents the circle $|z| = 4$, and so on. More examples follow below.

We assume $C$ to be a **smooth curve,** that is, $C$ has a continuous and nonzero derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

at each point. Geometrically this means that $C$ has everywhere a continuously turning tangent, as follows directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} \qquad \text{(Fig. 336).}$$

Here we use a dot since a prime $'$ denotes the derivative with respect to $z$.

## Definition of the Complex Line Integral

This is similar to the method in calculus. Let $C$ be a smooth curve in the complex plane given by (1), and let $f(z)$ be a continuous function given (at least) at each point of $C$. We now subdivide (we ***"partition"***) the interval $a \leqq t \leqq b$ in (1) by points

$$t_0\ (= a), \quad t_1, \quad \cdots, \quad t_{n-1}, \quad t_n\ (= b)$$

where $t_0 < t_1 < \cdots < t_n$. To this subdivision there corresponds a subdivision of $C$ by points

$$z_0, \quad z_1, \quad \cdots, \quad z_{n-1}, \quad z_n\ (= Z) \qquad \text{(Fig. 337),}$$

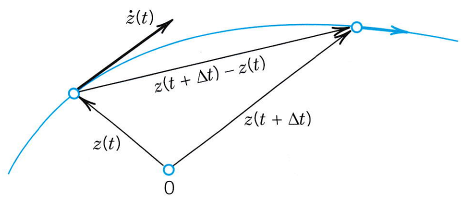



**Fig. 336.** Tangent vector $\dot{z}(t)$ of a curve $C$ in the complex plane given by $z(t)$. The arrowhead on the curve indicates the *positive sense* (sense of increasing $t$).

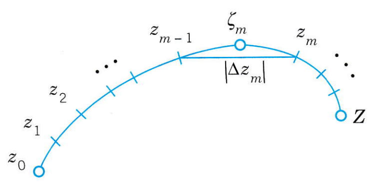



**Fig. 337.** Complex line integral

where $z_j = z(t_j)$. On each portion of subdivision of $C$ we choose an arbitrary point, say, a point $\zeta_1$ between $z_0$ and $z_1$ (that is, $\zeta_1 = z(t)$ where $t$ satisfies $t_0 \leqq t \leqq t_1$), a point $\zeta_2$ between $z_1$ and $z_2$, etc. Then we form the sum

$$\text{(2)} \qquad S_n = \sum_{m=1}^{n} f(\zeta_m)\, \Delta z_m \qquad \text{where} \qquad \Delta z_m = z_m - z_{m-1}.$$

We do this for each $n = 2, 3, \cdots$ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest

$|\Delta z_m|$ also approaches zero. Indeed, it cannot exceed the length of the arc of $C$ from $z_{m-1}$ to $z_m$ and the latter goes to zero since the arc length of the smooth curve $C$ is a continuous function of $t$. The limit of the sequence of complex numbers $S_2, S_3, \cdots$ thus obtained is called the **line integral** (or simply the *integral*) of $f(z)$ over the path of integration $C$ with the orientation given by (1). This line integral is denoted by

$$\int_C f(z)\,dz, \qquad \text{or by} \qquad \oint_C f(z)\,dz \tag{3}$$

if $C$ is a **closed path** (one whose terminal point $Z$ coincides with its initial point $z_0$, as for a circle or for a curve shaped like an 8).

**General Assumption.** *All paths of integration for complex line integrals are assumed to be* **piecewise smooth,** *that is, they consist of finitely many smooth curves joined end to end.*

## Basic Properties Directly Implied by the Definition

1. **Linearity.** Integration is a **linear operation,** that is, we can integrate sums term by term and can take out constant factors from under the integral sign. This means that if the integrals of $f_1$ and $f_2$ over a path $C$ exist, so does the integral of $k_1 f_1 + k_2 f_2$ over the same path and

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz. \tag{4}$$

2. **Sense reversal** in integrating over the ***same*** path, from $z_0$ to $Z$ (left) and from $Z$ to $z_0$ (right), introduces a minus sign as shown,

$$\int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz. \tag{5}$$

3. **Partitioning of path** (see Fig. 338)

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{6}$$

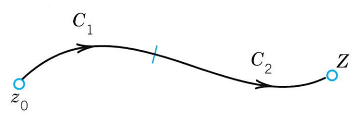



**Fig. 338.** Partitioning of path [formula (6)]

## Existence of the Complex Line Integral

Our assumptions that $f(z)$ is continuous and $C$ is piecewise smooth imply the existence of the line integral (3). This can be seen as follows.

As in the preceding chapter let us write $f(z) = u(x, y) + iv(x, y)$. We also set

$$\zeta_m = \xi_m + i\eta_m \qquad \text{and} \qquad \Delta z_m = \Delta x_m + i\Delta y_m.$$

Then (2) may be written

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m) \tag{7}$$

where $u = u(\zeta_m, \eta_m)$, $v = v(\zeta_m, \eta_m)$ and we sum over $m$ from 1 to $n$. Performing the multiplication, we may now split up $S_n$ into four sums:

$$S_n = \sum u\,\Delta x_m - \sum v\,\Delta y_m + i\left[\sum u\,\Delta y_m + \sum v\,\Delta x_m\right].$$

These sums are real. Since $f$ is continuous, $u$ and $v$ are continuous. Hence, if we let $n$ approach infinity in the aforementioned way, then the greatest $\Delta x_m$ and $\Delta y_m$ will approach zero and each sum on the right becomes a real line integral:

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz = \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right]. \tag{8}$$

This shows that under our assumptions on $f$ and $C$ the line integral (3) exists and its value is independent of the choice of subdivisions and intermediate points $\zeta_m$. ■

## First Evaluation Method: Indefinite Integration and Substitution of Limits

This method is the analog of the evaluation of definite integrals in calculus by the well-known formula

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad [F'(x) = f(x)].$$

It is simpler than the next method, but it is suitable for analytic functions only. To formulate it, we need the following concept of general interest.

A domain $D$ is called **simply connected** if every **simple closed curve** (closed curve without self-intersections) encloses only points of $D$.

For instance, a circular disk is simply connected, whereas an annulus (Sec. 13.3) is not simply connected. (Explain!)

**THEOREM 1**

**Indefinite Integration of Analytic Functions**

*Let $f(z)$ be analytic in a simply connected domain $D$. Then there exists an indefinite integral of $f(z)$ in the domain $D$, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in $D$, and for all paths in $D$ joining two points $z_0$ and $z_1$ in $D$ we have*

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]. \tag{9}$$

*(Note that we can write $z_0$ and $z_1$ instead of $C$, since we get the same value for all those $C$ from $z_0$ to $z_1$.)*

This theorem will be proved in the next section.

***Simple connectedness is quite essential*** in Theorem 1, as we shall see in Example 5.

Since analytic functions are our main concern, and since differentiation formulas will often help in finding $F(z)$ for a given $f(z) = F'(z)$, the present method is of great practical interest.

If $f(z)$ is entire (Sec. 13.5), we can take for $D$ the complex plane (which is certainly simply connected).

**EXAMPLE 1** $\displaystyle\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$ ■

**EXAMPLE 2** $\displaystyle\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i\sinh \pi = 23.097i$ ■

**EXAMPLE 3** $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$

since $e^z$ is periodic with period $2\pi i$. ■

**EXAMPLE 4** $\displaystyle\int_{-i}^{i} \frac{dz}{z} = \operatorname{Ln} i - \operatorname{Ln}(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi$. Here $D$ is the complex plane without 0 and the negative real axis (where $\operatorname{Ln} z$ is not analytic). Obviously, $D$ is a simply connected domain. ■

## Second Evaluation Method: Use of a Representation of a Path

This method is not restricted to analytic functions but applies to any continuous complex function.

**THEOREM 2**

**Integration by the Use of the Path**

*Let $C$ be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on $C$. Then*

**(10)** $$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

**PROOF** The left side of (10) is given by (8) in terms of real line integrals, and we show that the right side of (10) also equals (8). We have $z = x + iy$, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write $u$ for $u[x(t), y(t)]$ and $v$ for $v[x(t), y(t)]$. We also have $dx = \dot{x}\,dt$ *and* $dy = \dot{y}\,dt$. Consequently, in (10)

$$\begin{aligned}\int_a^b f[z(t)]\dot{z}(t)\,dt &= \int_a^b (u + iv)(\dot{x} + i\dot{y})\,dt \\ &= \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)] \\ &= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx).\end{aligned}$$ ■

COMMENT. In (7) and (8) of the existence proof of the complex line integral we referred to real line integrals. If one wants to avoid this, one can take (10) as a *definition* of the complex line integral.

## Steps in Applying Theorem 2

**(A)** Represent the path $C$ in the form $z(t)$ $(a \leqq t \leqq b)$.

**(B)** Calculate the derivative $\dot{z}(t) = dz/dt$.

**(C)** Substitute $z(t)$ for every $z$ in $f(z)$ (hence $x(t)$ for $x$ and $y(t)$ for $y$).

**(D)** Integrate $f[z(t)]\dot{z}(t)$ over $t$ from $a$ to $b$.

### EXAMPLE 5 A Basic Result: Integral of 1/z Around the Unit Circle

We show that by integrating $1/z$ counterclockwise around the unit circle (the circle of radius 1 and center 0; see Sec. 13.3) we obtain

$$\oint_C \frac{dz}{z} = 2\pi i \qquad (C \text{ the unit circle, counterclockwise}). \tag{11}$$

***This is a very important result*** that we shall need quite often.

***Solution.*** **(A)** We may represent the unit circle $C$ in Fig. 327 of Sec. 13.3 by

$$z(t) = \cos t + i \sin t = e^{it} \qquad (0 \leqq t \leqq 2\pi),$$

so that counterclockwise integration corresponds to an increase of $t$ from 0 to $2\pi$.

**(B)** Differentiation gives $\dot{z}(t) = ie^{it}$ (chain rule!).

**(C)** By substitution, $f(z(t)) = 1/z(t) = e^{-it}$.

**(D)** From (10) we thus obtain the result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

Check this result by using $z(t) = \cos t + i \sin t$.

***Simple connectedness is essential in Theorem 1.*** Equation (9) in Theorem 1 gives 0 for any closed path because then $z_1 = z_0$, so that $F(z_1) - F(z_0) = 0$. Now $1/z$ is not analytic at $z = 0$. But any *simply connected* domain containing the unit circle must contain $z = 0$, so that Theorem 1 does not apply—it is not enough that $1/z$ is analytic in an annulus, say, $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected! ■

### EXAMPLE 6 Integral of $1/z^m$ with Integer Power $m$

Let $f(z) = (z - z_0)^m$ where $m$ is the integer and $z_0$ a constant. Integrate counterclockwise around the circle $C$ of radius $\rho$ with center at $z_0$ (Fig. 339).

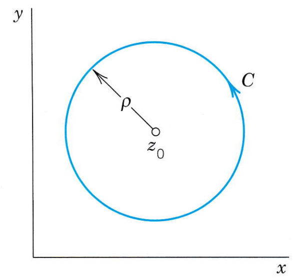



Fig. 339. Path in Example 6

***Solution.*** We may represent $C$ in the form

$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \qquad (0 \leqq t \leqq 2\pi).$$

Then we have

$$(z - z_0)^m = \rho^m e^{imt}, \qquad dz = i\rho e^{it}\, dt$$

and obtain

$$\oint_C (z - z_0)^m\, dz = \int_0^{2\pi} \rho^m e^{imt}\, i\rho e^{it}\, dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t}\, dt.$$

By the Euler formula (5) in Sec. 13.6 the right side equals

$$i\rho^{m+1} \left[ \int_0^{2\pi} \cos\,(m + 1)t\, dt + i \int_0^{2\pi} \sin\,(m + 1)t\, dt \right].$$

If $m = -1$, we have $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. We thus obtain $2\pi i$. For integer $m \neq 1$ each of the two integrals is zero because we integrate over an interval of length $2\pi$, equal to a period of sine and cosine. Hence the result is

$$\text{(12)} \qquad \oint_C (z - z_0)^m\, dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}). \end{cases}$$

**Dependence on path.** Now comes a very important fact. If we integrate a given function $f(z)$ from a point $z_0$ to a point $z_1$ along different paths, the integrals will in general have different values. In other words, ***a complex line integral depends not only on the endpoints of the path but in general also on the path itself.*** The next example gives a first impression of this, and a systematic discussion follows in the next section.

**EXAMPLE 7 Integral of a Nonanalytic Function. Dependence on Path**

Integrate $f(z) = \text{Re}\, z = x$ from 0 to $1 + 2i$ (a) along $C^*$ in Fig. 340, (b) along $C$ consisting of $C_1$ and $C_2$.

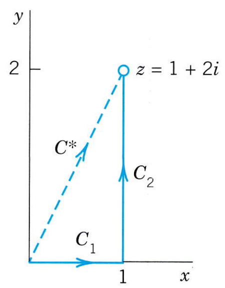



**Fig. 340.** Paths in Example 7

***Solution.*** **(a)** $C^*$ can be represented by $z(t) = t + 2it$ $(0 \leqq t \leqq 1)$. Hence $\dot{z}(t) = 1 + 2i$ and $f[z(t)] = x(t) = t$ on $C^*$. We now calculate

$$\int_{C^*} \text{Re}\, z\, dz = \int_0^1 t(1 + 2i)\, dt = \tfrac{1}{2}(1 + 2i) = \tfrac{1}{2} + i.$$

**(b)** We now have

$$C_1\colon z(t) = t, \qquad \dot{z}(t) = 1, \qquad f(z(t)) = x(t) = t \qquad (0 \leqq t \leqq 1)$$

$$C_2\colon z(t) = 1 + it, \qquad \dot{z}(t) = i, \qquad f(z(t)) = x(t) = 1 \qquad (0 \leqq t \leqq 2).$$

Using (6) we calculate

$$\int_C \operatorname{Re} z\, dz = \int_{C_1} \operatorname{Re} z\, dz + \int_{C_2} \operatorname{Re} z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \tfrac{1}{2} + 2i.$$

Note that this result differs from the result in (a). ■

## Bounds for Integrals. *ML*-Inequality

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

$$\left| \int_C f(z)\, dz \right| \leqq ML \qquad \textbf{(13)} \quad (\textbf{\textit{ML}-inequality});$$

$L$ is the length of $C$ and $M$ a constant such that $|f(z)| \leqq M$ everywhere on $C$.

**PROOF** Taking the absolute value in (2) and applying the generalized inequality (6*) in Sec. 13.2, we obtain

$$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m)\, \Delta z_m \right| \leqq \sum_{m=1}^{n} |f(\zeta_m)|\, |\Delta z_m| \leqq M \sum_{m=1}^{n} |\Delta z_m|.$$

Now $|\Delta z_m|$ is the length of the chord whose endpoints are $z_{m-1}$ and $z_m$ (see Fig. 337 on p. 638). Hence the sum on the right represents the length $L^*$ of the broken line of chords whose endpoints are $z_0, z_1, \cdots, z_n (= Z)$. If $n$ approaches infinity in such a way that the greatest $|\Delta t_m|$ and thus $|\Delta z_m|$ approach zero, then $L^*$ approaches the length $L$ of the curve $C$, by the definition of the length of a curve. From this the inequality (13) follows. ■

We cannot see from (13) how close to the bound $ML$ the actual absolute value of the integral is, but this will be no handicap in applying (13). For the time being we explain the practical use of (13) by a simple example.

**EXAMPLE 8** **Estimation of an Integral**

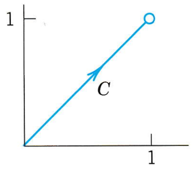


**Fig. 341.** Path in Example 8

Find an upper bound for the absolute value of the integral

$$\int_C z^2\, dz, \qquad C \text{ the straight-line segment from } 0 \text{ to } 1 + i, \text{ Fig. 341.}$$

***Solution.*** $L = \sqrt{2}$ and $|f(z)| = |z^2| \leqq 2$ on $C$ gives by (13)

$$\left| \int_C z^2\, dz \right| \leqq 2\sqrt{2} = 2.8284.$$

The absolute value of the integral is $\left| -\dfrac{2}{3} + \dfrac{2}{3} i \right| = \dfrac{2}{3}\sqrt{2} = 0.9428$ (see Example 1). ■

**Summary on Integration.** Line integrals of $f(z)$ can always be evaluated by (10), using a representation (1) of the path of integration. If $f(z)$ is analytic, indefinite integration by (9) as in calculus will be simpler.

## PROBLEM SET 14.1

### 1–9 PARAMETRIC REPRESENTATIONS

Find and sketch the path and its orientation given by:

**1.** $z(t) = (1 + 3i)t \ (1 \leqq t \leqq 4)$

**2.** $z(t) = 5 - 2it \ (-3 \leqq t \leqq 3)$

**3.** $z(t) = 4 + i + 3e^{it} \ (0 \leqq t \leqq 2\pi)$

**4.** $z(t) = 1 + i + e^{-\pi i t} \ (0 \leqq t \leqq 2)$

**5.** $z(t) = e^{it} \ (0 \leqq t \leqq \pi)$

**6.** $z(t) = 3 + 4i + 5e^{it} \ (\pi \leqq t \leqq 2\pi)$

**7.** $z(t) = 6 \cos 2t + 5i \sin 2t \ (0 \leqq t \leqq \pi)$

**8.** $z(t) = 1 + 2t + 8it^2 \ (-1 \leqq t \leqq 1)$

**9.** $z(t) = t + \frac{1}{2}it^3 \ (-1 \leqq t \leqq 2)$

### 10–18 PARAMETRIC REPRESENTATIONS

Sketch and represent parametrically:

**10.** Segment from $1 + i$ to $4 - 2i$

**11.** Unit circle (clockwise)

**12.** Segment from $a + ib$ to $c + id$

**13.** Hyperbola $xy = 1$ from $1 + i$ to $4 + \frac{1}{4}i$

**14.** Semi-ellipse $x^2/a^2 + y^2/b^2 = 1, \ y \geqq 0$

**15.** Parabola $y = 4 - 4x^2 \ (-1 \leqq x \leqq 1)$

**16.** $|z - 2 + 3i| = 4$ (counterclockwise)

**17.** $|z + a + ib| = r$ (clockwise)

**18.** Ellipse $4(x - 1)^2 + 9(y + 2)^2 = 36$

### 19–29 INTEGRATION

Integrate by the first method or state why it does not apply and then use the second method. (Show the details of your work.)

**19.** $\int_C \operatorname{Re} z \, dz$, $C$ the shortest path from 0 to $1 + i$

**20.** $\int_C \operatorname{Re} z \, dz$, $C$ the parabola $y = x^2$ from 0 to $1 + i$

**21.** $\int_C e^{2z} \, dz$, $C$ the shortest path from $\pi i$ to $2\pi i$

**22.** $\int_C \sin z \, dz$, $C$ any path from 0 to $2i$

**23.** $\int_C \cos^2 z \, dz$ from $-\pi i$ along $|z| = \pi$ to $\pi i$ in the right half-plane

**24.** $\int_C (z + z^{-1}) \, dz$, $C$ the unit circle (counterclockwise)

**25.** $\int_C \cosh 4z \, dz$, $C$ any path from $-\pi i/8$ to $\pi i/8$

**26.** $\int_C \bar{z} \, dz$, $C$ from $-1 + i$ along the parabola $y = x^2$ to $1 + i$

**27.** $\int_C \sec^2 z \, dz$, $C$ any path from $\pi/4$ to $\pi i/4$

**28.** $\int_C \operatorname{Im} z^2 \, dz$ counterclockwise around the triangle with vertices $z = 0, 1, i$

**29.** $\int_C z e^{z^2/2} \, dz$, $C$ from $i$ along the axes to 1

**30. (Sense reversal)** Verify (5) for $f(z) = z^2$, where $C$ is the segment from $-1 - i$ to $1 + i$.

**31. (Path partitioning)** Verify (6) for $f(z) = 1/z$ and $C_1$ and $C_2$ the upper and lower halfs of the unit circle

**32. (*ML*-inequality)** Find an upper bound of the absolute value of the integral in Prob. 19.

**33. (Linearity)** Illustrate (4) with an example of your own. Prove (4).

**34. TEAM PROJECT. Integration. (a) Comparison.** Write a short report comparing the essential points of the two integration methods.

**(b) Comparison.** Evaluate $\int_C f(z) \, dz$ by Theorem 1 and check the result by Theorem 2, where:

(i) $f(z) = z^4$ and $C$ is the semicircle $|z| = 2$ from $-2i$ to $2i$ in the right half-plane,

(ii) $f(z) = e^{2z}$ and $C$ is the shortest path from 0 to $1 + 2i$.

**(c) Continuous deformation of path.** Experiment with a family of paths with common endpoints, say, $z(t) = t + ia \sin t, \ 0 \leqq t \leqq \pi$, with real parameter $a$. Integrate nonanalytic functions ($\operatorname{Re} z$, $\operatorname{Re}(z^2)$, etc.) and explore how the result depends on $a$. Then take analytic functions of your choice. (Show the details of your work.) Compare and comment.

**(d) Continuous deformation of path.** Choose another family, for example, semi-ellipses $z(t) = a \cos t + i \sin t, \ -\pi/2 \leqq t \leqq \pi/2$, and experiment as in (c).

**35. CAS PROJECT. Integration.** Write programs for the two integration methods. Apply them to problems of your choice. Could you make them into a joint program that also decides which of the two methods to use in a given case?

## 14.2 Cauchy's Integral Theorem

We have just seen in Sec. 14.1 that a line integral of a function $f(z)$ generally depends not merely on the endpoints of the path, but also on the choice of the path itself. This dependence often complicates situations. Hence conditions under which this does ***not*** occur are of considerable importance. Namely, if $f(z)$ is analytic in a domain $D$ and $D$ is simply connected (see Sec. 14.1 and also below), then the integral will not depend on the choice of a path between given points. This result (Theorem 2) follows from Cauchy's integral theorem, along with other basic consequences that make ***Cauchy's integral theorem the most important theorem in this chapter*** and fundamental throughout complex analysis.

Let us begin by repeating and illustrating the definition of simple connectedness (Sec. 14.1) and adding some more details.

1. A **simple closed path** is a closed path (Sec. 14.1) that does not intersect or touch itself (Fig. 342). For example, a circle is simple, but a curve shaped like an 8 is not simple.

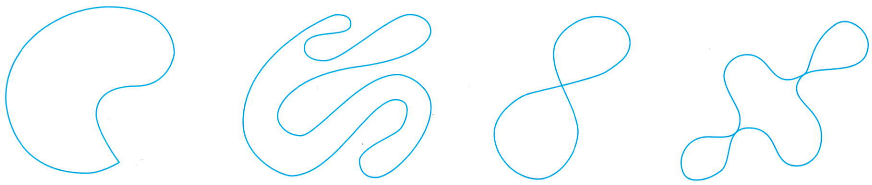



**Fig. 342.** Closed paths

2. A **simply connected domain** $D$ in the complex plane is a domain (Sec. 13.3) such that every simple closed path in $D$ encloses only points of $D$. *Examples:* The interior of a circle ("open disk"), ellipse, or any simple closed curve. A domain that is not simply connected is called **multiply connected.** *Examples:* An annulus (Sec. 13.3), a disk without the center, for example, $0 < |z| < 1$. See also Fig. 343.

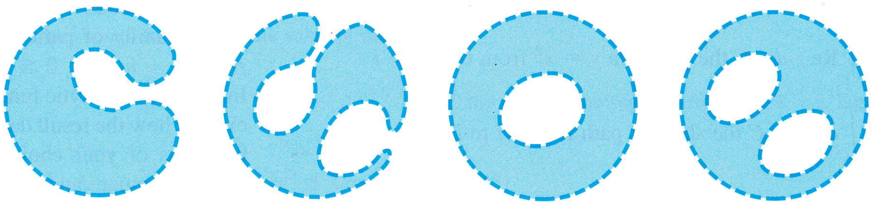



**Fig. 343.** Simply and multiply connected domains

More precisely, a **bounded domain** $D$ (that is, a domain that lies entirely in some circle about the origin) is called ***p*-fold connected** if its boundary consists of $p$ closed connected sets without common points. These sets can be curves, segments, or single points (such as $z = 0$ for $0 < |z| < 1$, for which $p = 2$). Thus, $D$ has $p - 1$ **"holes",** where "hole" may also mean a segment or even a single point. Hence an annulus is doubly connected ($p = 2$).

**THEOREM 1**

**Cauchy's Integral Theorem**

*If $f(z)$ is analytic in a simply connected domain $D$, then for every simple closed path $C$ in $D$,*

$$\oint_C f(z)\,dz = 0. \tag{1}$$

See Fig. 344.

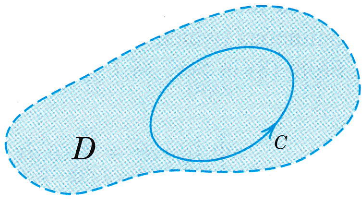



Fig. 344. Cauchy's integral theorem

Before we prove the theorem, let us consider some examples in order to really understand what is going on. A simple closed path is sometimes called a *contour* and an integral over such a path a **contour integral.** Thus, (1) and our examples involve contour integrals.

**EXAMPLE 1 No Singularities (Entire Functions)**

$$\oint_C e^z\,dz = 0, \qquad \oint_C \cos z\,dz = 0, \qquad \oint_C z^n\,dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these functions are entire (analytic for all $z$). ■

**EXAMPLE 2 Singularities Outside the Contour**

$$\oint_C \sec z\,dz = 0, \qquad \oint_C \frac{dz}{z^2+4} = 0$$

where $C$ is the unit circle, $\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside $C$; none lies on $C$ or inside $C$. Similarly for the second integral, whose integrand is not analytic at $z = \pm 2i$ outside $C$. ■

**EXAMPLE 3 Nonanalytic Function**

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i$$

where $C$: $z(t) = e^{it}$ is the unit circle. This does not contradict Cauchy's theorem because $f(z) = \bar{z}$ is not analytic. ■

**EXAMPLE 4 Analyticity Sufficient, Not Necessary**

$$\oint_C \frac{dz}{z^2} = 0$$

where $C$ is the unit circle. This result does *not* follow from Cauchy's theorem, because $f(z) = 1/z^2$ is not analytic at $z = 0$. Hence *the condition that $f$ be analytic in $D$ is sufficient rather than necessary for* (1) *to be true.* ■

**EXAMPLE 5 Simple Connectedness Essential**

$$\oint_C \frac{dz}{z} = 2\pi i$$

for counterclockwise integration around the unit circle (see Sec. 14.1). $C$ lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $1/z$ is analytic, but this domain is not simply connected, so that Cauchy's theorem cannot be applied. Hence ***the condition that the domain D be simply connected is essential.***

In other words, by Cauchy's theorem, if $f(z)$ is analytic on a simple closed path $C$ and everywhere inside $C$, with no exception, not even a single point, then (1) holds. The point that causes trouble here is $z = 0$ where $1/z$ is not analytic. ■

**PROOF** Cauchy proved his integral theorem under the additional assumption that the derivative $f'(z)$ is continuous (which is true, but would need an extra proof). His proof proceeds as follows. From (8) in Sec. 14.1 we have

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx).$$

Since $f(z)$ is analytic in $D$, its derivative $f'(z)$ exists in $D$. Since $f'(z)$ is assumed to be continuous, (4) and (5) in Sec. 13.4 imply that $u$ and $v$ have *continuous* partial derivatives in $D$. Hence Green's theorem (Sec. 10.4) (with $u$ and $-v$ instead of $F_1$ and $F_2$) is applicable and gives

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

where $R$ is the region bounded by $C$. The second Cauchy–Riemann equation (Sec. 13.4) shows that the integrand on the right is identically zero. Hence the integral on the left is zero. In the same fashion it follows by the use of the first Cauchy–Riemann equation that the last integral in the above formula is zero. This completes Cauchy's proof. ■

**Goursat's proof *without the condition that f′(z) is continuous***[1] is much more complicated. We leave it optional and include it in App. 4.

## Independence of Path

We know from the preceding section that the value of a line integral of a given function $f(z)$ from a point $z_1$ to a point $z_2$ will in general depend on the path $C$ over which we integrate, not merely on $z_1$ and $z_2$. It is important to characterize situations in which this difficulty of path dependence does not occur. This task suggests the following concept. We call an integral of $f(z)$ **independent of path in a domain *D*** if for every $z_1, z_2$ in $D$ its value depends (besides on $f(z)$, of course) only on the initial point $z_1$ and the terminal point $z_2$, but not on the choice of the path $C$ in $D$ [so that every path in $D$ from $z_1$ to $z_2$ gives the same value of the integral of $f(z)$].

[1]ÉDOUARD GOURSAT (1858–1936), French mathematician. Cauchy published the theorem in 1825. The removal of that condition by Goursat (see *Transactions Amer. Math. Soc.*, vol. 1, 1900) is quite important, for instance, in connection with the fact that derivatives of analytic functions are also analytic, as we shall prove soon. Goursat also made important contributions to PDEs.

**THEOREM 2**

**Independence of Path**

*If $f(z)$ is analytic in a simply connected domain $D$, then the integral of $f(z)$ is independent of path in $D$.*

**PROOF** Let $z_1$ and $z_2$ be any points in $D$. Consider two paths $C_1$ and $C_2$ in $D$ from $z_1$ to $z_2$ without further common points, as in Fig. 345. Denote by $C_2^*$ the path $C_2$ with the orientation reserved (Fig. 346). Integrate from $z_1$ over $C_1$ to $z_2$ and over $C_2^*$ back to $z_1$. This is a simple closed path, and Cauchy's theorem applies under our assumptions of the present theorem and gives zero:

$$(2') \qquad \int_{C_1} f\,dz + \int_{C_2^*} f\,dz = 0, \qquad \text{thus} \qquad \int_{C_1} f\,dz = -\int_{C_2^*} f\,dz.$$

But the minus sign on the right disappears if we integrate in the reverse direction, from $z_1$ to $z_2$, which shows that the integrals of $f(z)$ over $C_1$ and $C_2$ are equal,

$$(2) \qquad \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad \text{(Fig. 345).}$$

This proves the theorem for paths that have only the endpoints in common. For paths that have finitely many further common points, apply the present argument to each "loop" (portions of $C_1$ and $C_2$ between consecutive common points; four loops in Fig. 347). For paths with infinitely many common points we would need additional argumentation not to be presented here. ■

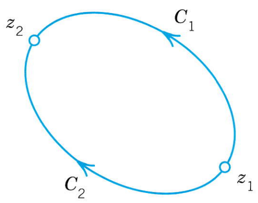



**Fig. 345.** Formula (2)

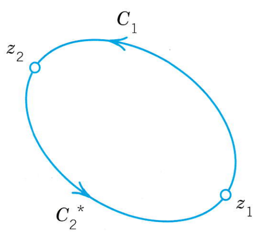



**Fig. 346.** Formula (2′)

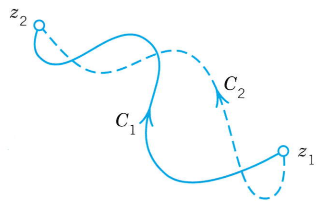



**Fig. 347.** Paths with more common points

## Principle of Deformation of Path

This idea is related to path independence. We may imagine that the path $C_2$ in (2) was obtained from $C_1$ by continuously moving $C_1$ (with ends fixed!) until it coincides with $C_2$. Figure 348 shows two of the infinitely many intermediate paths for which the integral always retains its value (because of Theorem 2). Hence we may impose a continuous deformation of the path of an integral, keeping the ends fixed. As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value. This is called the **principle of deformation of path.**

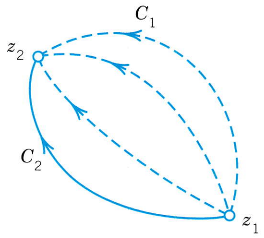



**Fig. 348.** Continuous deformation of path

**EXAMPLE 6** **A Basic Result: Integral of Integer Powers**

From Example 6 in Sec. 14.1 and the principle of deformation of path it follows that

$$\oint (z - z_0)^m \, dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1 \text{ and integer}) \end{cases} \tag{3}$$

for counterclockwise integration around ***any simple closed path containing $z_0$ in its interior.***

Indeed, the circle $|z - z_0| = \rho$ in Example 6 of Sec. 14.1 can be continuously deformed in two steps into a path as just indicated, namely, by first deforming, say, one semicircle and then the other one. (Make a sketch). ■

## Existence of Indefinite Integral

We shall now justify our indefinite integration method in the preceding section [formula (9) in Sec. 14.1]. The proof will need Cauchy's integral theorem.

**THEOREM 3** **Existence of Indefinite Integral**

*If $f(z)$ is analytic in a simply connected domain $D$, then there exists an indefinite integral $F(z)$ of $f(z)$ in $D$—thus, $F'(z) = f(z)$—which is analytic in $D$, and for all paths in $D$ joining any two points $z_0$ and $z_1$ in $D$, the integral of $f(z)$ from $z_0$ to $z_1$ can be evaluated by formula* (9) *in* Sec. 14.1.

**PROOF** The conditions of Cauchy's integral theorem are satisfied. Hence the line integral of $f(z)$ from any $z_0$ in $D$ to any $z$ in $D$ is independent of path in $D$. We keep $z_0$ fixed. Then this integral becomes a function of $z$, call if $F(z)$,

$$F(z) = \int_{z_0}^{z} f(z^*) \, dz^* \tag{4}$$

which is uniquely determined. We show that this $F(z)$ is analytic in $D$ and $F'(z) = f(z)$. The idea of doing this is as follows. Using (4) we form the difference quotient

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*) \, dz^* - \int_{z_0}^{z} f(z^*) \, dz^*\right] = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*) \, dz^*. \tag{5}$$

We now subtract $f(z)$ from (5) and show that the resulting expression approaches zero as $\Delta z \to 0$. The details are as follows.

We keep $z$ fixed. Then we choose $z + \Delta z$ in $D$ so that the whole segment with endpoints $z$ and $z + \Delta z$ is in $D$ (Fig. 349). This can be done because $D$ is a domain, hence it contains a neighborhood of $z$. We use this segment as the path of integration in (5). Now we subtract $f(z)$. This is a constant because $z$ is kept fixed. Hence we can write

$$\int_z^{z+\Delta z} f(z)\, dz^* = f(z) \int_z^{z+\Delta z} dz^* = f(z)\,\Delta z. \qquad \text{Thus} \qquad f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\, dz^*.$$

By this trick and from (5) we get a single integral:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(z^*) - f(z)]\, dz^*.$$

Since $f(z)$ is analytic, it is continuous. An $\epsilon > 0$ being given, we can thus find a $\delta > 0$ such that $|f(z^*) - f(z)| < \epsilon$ when $|z^* - z| < \delta$. Hence, letting $|\Delta z| < \delta$, we see that the *ML*-inequality (Sec. 14.1) yields

$$\left|\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_z^{z+\Delta z} [f(z^*) - f(z)]\, dz^*\right| \leqq \frac{1}{|\Delta z|}\,\epsilon|\Delta z| = \epsilon.$$

By the definition of limit and derivative, this proves that

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).$$

Since $z$ is any point in $D$, this implies that $F(z)$ is analytic in $D$ and is an indefinite integral or antiderivative of $f(z)$ in $D$, written

$$F(z) = \int f(z)\, dz.$$

Also, if $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in $D$; hence $F(z) - G(z)$ is constant in $D$ (see Team Project 26 in Problem Set 13.4). That is, two indefinite integrals of $f(z)$ can differ only by a constant. The latter drops out in (9) of Sec. 14.1, so that we can use any indefinite integral of $f(z)$. This proves Theorem 3. ■

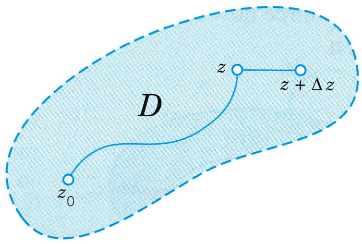



**Fig. 349.** Path of integration

## Cauchy's Integral Theorem for Multiply Connected Domains

Cauchy's theorem applies to multiply connected domains. We first explain this for a **doubly connected domain** $D$ with outer boundary curve $C_1$ and inner $C_2$ (Fig. 350). If a function $f(z)$ is analytic in any domain $D^*$ that contains $D$ and its boundary curves, we claim that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \qquad \text{(Fig. 350)} \tag{6}$$

both integrals being taken counterclockwise (or both clockwise, and regardless of whether or not the full interior of $C_2$ belongs to $D^*$).

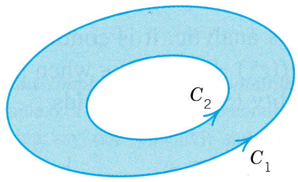



**Fig. 350.** Paths in (5)

**PROOF** By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ (Fig. 351) we cut $D$ into two simply connected domains $D_1$ and $D_2$ in which and on whose boundaries $f(z)$ is analytic. By Cauchy's integral theorem the integral over the entire boundary of $D_1$ (taken in the sense of the arrows in Fig. 351) is zero, and so is the integral over the boundary of $D_2$, and thus their sum. In this sum the integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel because we integrate over them in both directions—this is the key—and we are left with the integrals over $C_1$ (counterclockwise) and $C_2$ (clockwise; see Fig. 351); hence by reversing the integration over $C_2$ (to counterclockwise) we have

$$\oint_{C_1} f\,dz - \oint_{C_2} f\,dz = 0$$

and (6) follows. ■

For domains of higher connectivity the idea remains the same. Thus, for a **triply connected domain** we use three cuts $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ (Fig. 352). Adding integrals as before, the integrals over the cuts cancel and the sum of the integrals over $C_1$ (counterclockwise) and $C_2$, $C_3$ (clockwise) is zero. Hence the integral over $C_1$ equals the sum of the integrals over $C_2$ and $C_3$, all three now taken counterclockwise. Similarly for quadruply connected domains, and so on.

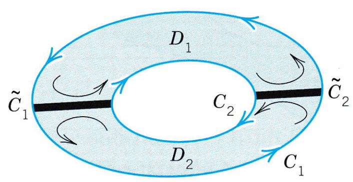



**Fig. 351.** Doubly connected domain

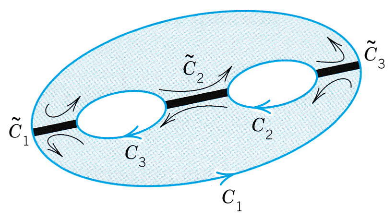



**Fig. 352.** Triply connected domain

## PROBLEM SET 14.2

### 1–11 CAUCHY'S INTEGRAL THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle, indicating whether Cauchy's integral theorem applies. (Show the details of your work.)

**1.** $f(z) = \operatorname{Re} z$

**2.** $f(z) = 1/(3z - \pi i)$

**3.** $f(z) = e^{z^2/2}$

**4.** $f(z) = 1/\bar{z}$

**5.** $f(z) = \tan z^2$

**6.** $f(z) = \sec(z/2)$

**7.** $f(z) = 1/(z^8 - 1.2)$

**8.** $f(z) = 1/(4z - 3)$

**9.** $f(z) = 1/(2|z|^3)$

**10.** $f(z) = \bar{z}^2$

**11.** $f(z) = z^2 \cot z$

### 12–17 COMMENTS ON TEXT AND EXAMPLES

**12. (Singularities)** Can we conclude in Example 2 that the integral of $1/(z^2 + 4)$ taken over (a) $|z - 2| = 2$, (b) $|z - 2| = 3$ is zero? Give reasons.

**13. (Cauchy's integral theorem)** Verify Theorem 1 for the integral of $z^2$ over the boundary of the square with vertices $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$ (counterclockwise).

**14. (Cauchy's integral theorem)** For what contours $C$ will it follow from Theorem 1 that

(a) $\displaystyle\oint_C \frac{dz}{z} = 0$, (b) $\displaystyle\oint_C \frac{\cos z}{z^6 - z^2}\, dz = 0$,

(c) $\displaystyle\oint_C \frac{e^{1/z}}{z^2 + 9}\, dz = 0$?

**15. (Deformation principle)** Can we conclude from Example 4 that the integral is also zero over the contour in Problem 13?

**16. (Deformation principle)** If the integral of a function $f(z)$ over the unit circle equals 3 and over the circle $|z| = 2$ equals 5, can we conclude that $f(z)$ is analytic everywhere in the annulus $1 < |z| < 2$?

**17. (Path independence)** Verify Theorem 2 for the integral of $\cos z$ from 0 to $(1 + i)\pi$ (a) over the shortest path, (b) over the $x$-axis to $\pi$ and then straight up to $(1 + i)\pi$.

**18. TEAM PROJECT. Cauchy's Integral Theorem.**
**(a) Main Aspects.** Each of the problems in Examples 1–5 explains a basic fact in connection with Cauchy's theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of those facts.

**(b) Partial fractions.** Write $f(z)$ in terms of partial fractions and integrate it counterclockwise over the unit circle, where

(i) $f(z) = \dfrac{2z + 3i}{z^2 + \frac{1}{4}}$ (ii) $f(z) = \dfrac{z + 1}{z^2 + 2z}$.

**(c) Deformation of path.** Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, $z(t) = t + ia(t - t^2)$, $0 \leqq t \leqq 1$, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., $z$, $\operatorname{Im} z$, $z^2$, $\operatorname{Re} z^2$, $\operatorname{Im} z^2$, etc).

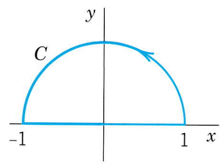
### 19–30 FURTHER CONTOUR INTEGRALS

Evaluate (showing the details and using partial fractions if necessary)

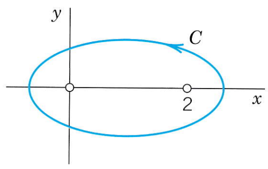
**19.** $\displaystyle\oint_C \frac{dz}{2z - i}$, $C$ the circle $|z| = 3$ (counterclockwise)

**20.** $\displaystyle\oint_C \tanh z\, dz$, $C$ the circle $|z - \frac{1}{4}\pi i| = \frac{1}{2}$ (clockwise)

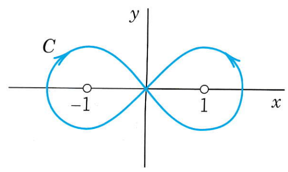
**21.** $\displaystyle\oint_C \operatorname{Re} 2z\, dz$, $C$ as shown




**22.** $\displaystyle\oint_C \frac{7z - 6}{z^2 - 2z}\, dz$, $C$ as shown




**23.** $\displaystyle\oint_C \frac{dz}{z^2 - 1}$, $C$ as shown




**24.** $\displaystyle\oint_C \frac{e^{2z}}{4z}\, dz$, $C$ consists of $|z| = 2$ (clockwise) and $|z| = \frac{1}{2}$ (counterclockwise)

**25.** $\oint_C \frac{\cos z}{z}\, dz$, $C$ consists of $|z| = 1$ (counterclockwise) and $|z| = 3$ (clockwise)

**26.** $\oint_C \operatorname{Ln}(2 + z)\, dz$, $C$ the boundary of the square with vertices $\pm 1, \pm i$

**27.** $\oint_C \frac{dz}{z^2 + 1}$, $C$: (a) $|z| = \frac{1}{2}$, (b) $|z - i| = \frac{3}{2}$ (counterclockwise)

**28.** $\oint_C \frac{dz}{z^2 + 1}$, $C$: (a) $|z + i| = 1$, (b) $|z - i| = 1$ (counterclockwise)

**29.** $\oint_C \frac{\sin z}{z + 2i}\, dz$, $C$: $|z - 4 - 2i| = 5.5$ (clockwise)

**30.** $\oint_C \frac{\tan(z/2)}{z^4 - 16}\, dz$, $C$ the boundary of the square with vertices $\pm 1, \pm i$ (clockwise)

## 14.3 Cauchy's Integral Formula

The most important consequence of Cauchy's integral theorem is Cauchy's integral formula. This formula is useful for evaluating integrals, as we show below. Even more important is its key role in proving the surprising fact that analytic functions have derivatives of all orders (Sec. 14.4), in establishing Taylor series representations (Sec. 15.4), and so on. Cauchy's integral formula and its conditions of validity may be stated as follows.

**THEOREM 1**

**Cauchy's Integral Formula**

*Let $f(z)$ be analytic in a simply connected domain D. Then for any point $z_0$ in D and any simple closed path C in D that encloses $z_0$* (Fig. 353),

(1) $$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$ **(Cauchy's integral formula)**

*the integration being taken counterclockwise. Alternatively* (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

(1*) $$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$ **(Cauchy's integral formula).**

**PROOF** By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. Inserting this into (1) on the left and taking the constant factor $f(z_0)$ out from under the integral sign, we have

(2) $$\oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0) \oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz.$$

The first term on the right equals $f(z_0) \cdot 2\pi i$ (see Example 6 in Sec. 14.2 with $m = -1$). This proves the theorem, provided the second integral on the right is zero. This is what we are now going to show. Its integrand is analytic, except at $z_0$. Hence by (6) in Sec. 14.2 we can replace $C$ by a small circle $K$ of radius $\rho$ and center $z_0$ (Fig. 354), without

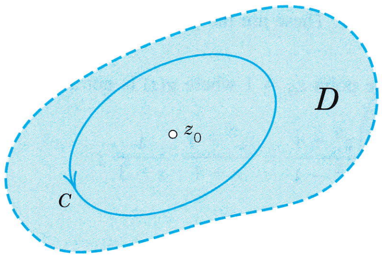



Fig. 353. Cauchy's integral formula

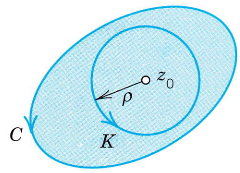



Fig. 354. Proof of Cauchy's integral formula

altering the value of the integral. Since $f(z)$ is analytic, it is continuous (Team Project 26, Sec. 13.3). Hence an $\epsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all $z$ in the disk $|z - z_0| < \delta$. Choosing the radius $\rho$ of $K$ smaller than $\delta$, we thus have the inequality

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \frac{\epsilon}{\rho}$$

at each point of $K$. The length of $K$ is $2\pi\rho$. Hence, by the $ML$-inequality in Sec. 14.1,

$$\left|\oint_K \frac{f(z) - f(z_0)}{z - z_0}\, dz\right| < \frac{\epsilon}{\rho}\, 2\pi\rho = 2\pi\epsilon.$$

Since $\epsilon$ ($> 0$) can be chosen arbitrarily small, it follows that the last integral in (2) must have the value zero, and the theorem is proved. ■

**EXAMPLE 1 Cauchy's Integral Formula**

$$\oint_C \frac{e^z}{z - 2}\, dz = 2\pi i e^z \Big|_{z=2} = 2\pi i e^2 = 46.4268i$$

for any contour enclosing $z_0 = 2$ (since $e^z$ is entire), and zero for any contour for which $z_0 = 2$ lies outside (by Cauchy's integral theorem). ■

**EXAMPLE 2 Cauchy's Integral Formula**

$$\begin{aligned}\oint_C \frac{z^3 - 6}{2z - i}\, dz &= \oint_C \frac{\frac{1}{2}z^3 - 3}{z - \frac{1}{2}i}\, dz \\ &= 2\pi i\left[\tfrac{1}{2}z^3 - 3\right]\Big|_{z=i/2} \\ &= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside } C).\end{aligned}$$ ■

**EXAMPLE 3 Integration Around Different Contours**

Integrate

$$g(z) = \frac{z^2 + 1}{z^2 - 1} = \frac{z^2 + 1}{(z + 1)(z - 1)}$$

counterclockwise around each of the four circles in Fig. 355.

***Solution.*** $g(z)$ is not analytic at $-1$ and 1. These are the points we have to watch for. We consider each circle separately.

**(a)** The circle $|z-1| = 1$ encloses the point $z_0 = 1$ where $g(z)$ is not analytic. Hence in (1) we have to write

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\,\frac{1}{z-1};$$

thus

$$f(z) = \frac{z^2+1}{z+1}$$

and (1) gives

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(1) = 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

**(b)** gives the same as (a) by the principle of deformation of path.

**(c)** The function $g(z)$ is as before, but $f(z)$ changes because we must take $z_0 = -1$ (instead of 1). This gives a factor $z - z_0 = z + 1$ in (1). Hence we must write

$$g(z) = \frac{z^2+1}{z-1}\,\frac{1}{z+1};$$

thus

$$f(z) = \frac{z^2+1}{z-1}.$$

Compare this for a minute with the previous expression and then go on:

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(-1) = 2\pi i \left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i.$$

**(d)** gives 0. Why? ■

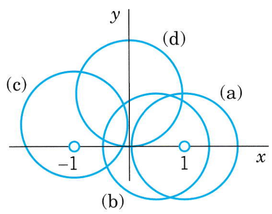



**Fig. 355.** Example 3

**Multiply connected domains** may be handled as in Sec. 14.2. For instance, if $f(z)$ is analytic on $C_1$ and $C_2$ and in the ring-shaped domain bounded by $C_1$ and $C_2$ (Fig. 356) and $z_0$ is any point in that domain, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz, \tag{3}$$

where the outer integral (over $C_1$) is taken counterclockwise and the inner clockwise, as indicated in Fig. 356.

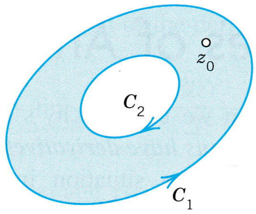



**Fig. 356.** Formula (3)

Our discussion in this section has illustrated the use of Cauchy's integral formula in integration. In the next section we show that the formula plays the key role in proving the surprising fact that an analytic function has derivatives of all orders, which are thus analytic functions themselves.

## PROBLEM SET 14.3

### 1–4 CONTOUR INTEGRATION

Integrate $(z^2 - 4)/(z^2 + 4)$ counterclockwise around the circle:

**1.** $|z - i| = 2$

**2.** $|z - 1| = 2$

**3.** $|z + 3i| = 2$

**4.** $|z| = \pi/2$

### 5–17 CONTOUR INTEGRATION

Using Cauchy's integral formula (and showing the details), integrate counterclockwise (or as indicated)

**5.** $\oint_C \frac{z+2}{z-2}\,dz, \quad C: |z - 1| = 2$

**6.** $\oint_C \frac{e^{3z}}{3z - i}\,dz, \quad C: |z| = 1$

**7.** $\oint_C \frac{\sinh \pi z}{z^2 - 3z}\,dz, \quad C: |z| = 1$

**8.** $\oint_C \frac{dz}{z^2 - 1}, \quad C: |z - 1| = \pi/2$

**9.** $\oint_C \frac{dz}{z^2 - 1}, \quad C: |z + 1| = 1$

**10.** $\oint_C \frac{e^z}{z - 2i}\,dz, \quad C: |z - 2i| = 4$

**11.** $\oint_C \frac{\cos z}{2z}\,dz, \quad C: |z| = \frac{1}{2}$

**12.** $\oint_C \frac{\tan z}{z - i}\,dz$, $C$ the boundary of the triangle with vertices 0 and $\pm 1 + 2i$

**13.** $\oint_C \frac{e^{-3\pi z}}{2z + i}\,dz$, $C$ the boundary of the square with vertices $\pm 1, \pm i$

**14.** $\oint_C \frac{\text{Ln}(z+1)}{z^2 + 1}\,dz$, $C$ consists of $|z - 2i| = 2$ (counterclockwise) and $|z - 2i| = \frac{1}{2}$ (clockwise)

**15.** $\oint_C \frac{\text{Ln}(z-1)}{z - 5}\,dz, \quad C: |z - 4| = 2$

**16.** $\oint_C \frac{\sin z}{z^2 - 2iz}\,dz$, $C$ consists of $|z| = 3$ (counterclockwise) and $|z| = 1$ (clockwise)

**17.** $\oint_C \frac{\cosh^2 z}{(z - 1 - i)z^2}\,dz$, $C$ as in Prob. 16

**18.** Show that $\oint_C (z - z_1)^{-1}(z - z_2)^{-1}\,dz = 0$ for a simple closed path $C$ enclosing $z_1$ and $z_2$, which are arbitrary.

**19. CAS PROJECT. Contour Integration.** Experiment to find out to what extent your CAS can do contour integration (a) by using the second method in Sec. 14.1, (b) by Cauchy's integral formula.

**20. TEAM PROJECT. Cauchy's Integral Theorem.** Gain additional insight into the proof of Cauchy's integral theorem by producing (2) with a contour enclosing $z_0$ (as in Fig. 353) and taking the limit as in the text. Choose

(a) $\oint_C \frac{z^3 - 6}{z - \frac{1}{2}i}\,dz$, (b) $\oint_C \frac{\sin z}{z - \frac{1}{2}\pi}\,dz$,

and (c) two other examples of your choice.

# 14.4 Derivatives of Analytic Functions

In this section we use Cauchy's integral formula to show the basic fact that ***complex analytic functions have derivatives of all orders.*** This is very surprising because it differs strikingly from the situation in real calculus. Indeed, if a *real* function is once differentiable, nothing follows about the existence of second or higher derivatives. Thus, in this respect, complex analytic functions behave much more simply than real functions that are once differentiable.

The existence of those derivatives will result from a general integral formula, as follows.

**THEOREM 1**

**Derivatives of an Analytic Function**

*If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point $z_0$ in D are given by the formulas*

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz \tag{1'}$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3}\, dz \tag{1''}$$

*and in general*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 1, 2, \cdots); \tag{1}$$

*here C is any simple closed path in D that encloses $z_0$ and whose full interior belongs to D; and we integrate counterclockwise around C* (Fig. 357).

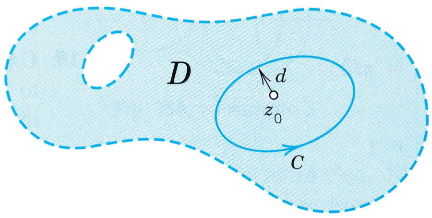



**Fig. 357.** Theorem 1 and its proof

**COMMENT.** For memorizing (1), it is useful to observe that these formulas are obtained formally by differentiating the Cauchy formula (1*), Sec. 14.3, under the integral sign *with respect to $z_0$*.

**PROOF** We prove (1′), starting from the definition of the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

On the right we represent $f(z_0 + \Delta z)$ and $f(z_0)$ by Cauchy's integral formula:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i \Delta z}\left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)}\, dz - \oint_C \frac{f(z)}{z - z_0}\, dz\right].$$

We now write the two integrals as a single integral. Taking the common denominator gives the numerator $f(z)\{z - z_0 - [z - (z_0 + \Delta z)]\} = f(z)\,\Delta z$, so that a factor $\Delta z$ drops out and we get

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz.$$

Clearly, we can now establish (1′) by showing that, as $\Delta z \to 0$, the integral on the right approaches the integral in (1′). To do this, we consider the difference between these two integrals. We can write this difference as a single integral by taking the common denominator and simplifying the numerator (as just before). This gives

$$\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz - \oint_C \frac{f(z)}{(z - z_0)^2}\, dz = \oint_C \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz.$$

We show by the *ML*-inequality (Sec. 14.1) that the integral on the right approaches zero as $\Delta z \to 0$.

Being analytic, the function $f(z)$ is continuous on $C$, hence bounded in absolute value, say, $|f(z)| \leqq K$. Let $d$ be the smallest distance from $z_0$ to the points of $C$ (see Fig. 357). Then for all $z$ on $C$,

$$|z - z_0|^2 \geqq d^2, \qquad \text{hence} \qquad \frac{1}{|z - z_0|^2} \leqq \frac{1}{d^2}.$$

Furthermore, by the triangle inequality for all $z$ on $C$ we then also have

$$d \leqq |z - z_0| = |z - z_0 - \Delta z + \Delta z| \leqq |z - z_0 - \Delta z| + |\Delta z|.$$

We now subtract $|\Delta z|$ on both sides and let $|\Delta z| \leqq d/2$, so that $-|\Delta z| \geqq -d/2$. Then

$$\tfrac{1}{2}d \leqq d - |\Delta z| \leqq |z - z_0 - \Delta z|. \qquad \text{Hence} \qquad \frac{1}{|z - z_0 - \Delta z|} \leqq \frac{2}{d}.$$

Let $L$ be the length of $C$. If $|\Delta z| \leqq d/2$, then by the *ML*-inequality

$$\left|\oint_C \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz\right| \leqq KL\,|\Delta z|\,\frac{2}{d} \cdot \frac{1}{d^2}.$$

This approaches zero as $\Delta z \to 0$. Formula $(1')$ is proved.

Note that we used Cauchy's integral formula (1*), Sec. 14.3, but if all we had known about $f(z_0)$ is the fact that it can be represented by (1*), Sec. 14.3, our argument would have established the existence of the derivative $f'(z_0)$ of $f(z)$. This is essential to the continuation and completion of this proof, because it implies that $(1'')$ can be proved by a similar argument, with $f$ replaced by $f'$, and that the general formula (1) follows by induction. ■

**EXAMPLE 1** **Evaluation of Line Integrals**

From $(1')$, for any contour enclosing the point $\pi i$ (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi.$$ ■

**EXAMPLE 2** From $(1'')$, for any contour enclosing the point $-i$ we obtain by counterclockwise integration

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz = \pi i(z^4-3z^2+6)''\Big|_{z=-i} = \pi i\left[12z^2-6\right]_{z=-i} = -18\pi i.$$ ■

**EXAMPLE 3** By $(1')$, for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Bigg|_{z=1}$$

$$= 2\pi i \frac{e^z(z^2+4)-e^z 2z}{(z^2+4)^2}\Bigg|_{z=1} = \frac{6e\pi}{25}\, i \approx 2.050i.$$ ■

## Cauchy's Inequality. Liouville's and Morera's Theorems

As a new aspect, let us now show that Cauchy's integral theorem is also fundamental in deriving general results on analytic functions.

**Cauchy's Inequality.** Theorem 1 yields a basic inequality that has many applications. To get it, all we have to do is to choose for $C$ in (1) a circle of radius $r$ and center $z_0$ and apply the $ML$-inequality (Sec. 14.1); with $|f(z)| \leqq M$ on $C$ we obtain from (1)

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leqq \frac{n!}{2\pi}\, M\, \frac{1}{r^{n+1}}\, 2\pi r.$$

This gives **Cauchy's inequality**

$$|f^{(n)}(z_0)| \leqq \frac{n!M}{r^n}. \tag{2}$$

To gain a first impression of the importance of this inequality, let us prove a famous theorem on entire functions (definition in Sec. 13.5). (For Liouville, see Sec. 5.7.)

**THEOREM 2**

**Liouville's Theorem**

*If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.*

**PROOF** By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all $z$. Using (2), we see that $|f'(z_0)| < K/r$. Since $f(z)$ is entire, this holds for every $r$, so that we can take $r$ as large as we please and conclude that $f'(z_0) = 0$. Since $z_0$ is arbitrary, $f'(z) = u_x + iv_x = 0$ for all $z$ (see (4) in Sec. 13.4), hence $u_x = v_x = 0$, and $u_y = v_y = 0$ by the Cauchy–Riemann equations. Thus $u = const$, $v = const$, and $f = u + iv = const$ for all $z$. This completes the proof. ■

Another very interesting consequence of Theorem 1 is

**THEOREM 3**

**Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)**

*If $f(z)$ is continuous in a simply connected domain $D$ and if*

$$\oint_C f(z)\,dz = 0 \qquad (3)$$

*for every closed path in $D$, then $f(z)$ is analytic in $D$.*

**PROOF** In Sec. 14.2 we showed that if $f(z)$ is analytic in a simply connected domain $D$, then

$$F(z) = \int_{z_0}^{z} f(z^*)\,dz^*$$

is analytic in $D$ and $F'(z) = f(z)$. In the proof we used only the continuity of $f(z)$ and the property that its integral around every closed path in $D$ is zero; from these assumptions we concluded that $F(z)$ is analytic. By Theorem 1, the derivative of $F(z)$ is analytic, that is, $f(z)$ is analytic in $D$, and Morera's theorem is proved. ■

## PROBLEM SET 14.4

### 1–8 CONTOUR INTEGRATION

Integrate counterclockwise around the circle $|z| = 2$. ($n$ is a positive integer, $a$ is arbitrary.) Show the details of your work.

1. $\dfrac{\cosh 3z}{z^5}$
2. $\dfrac{\sin z}{(z - \pi i/2)^4}$
3. $\dfrac{e^z \cos z}{(z - \pi/2)^2}$
4. $\dfrac{\cos z}{z^{2n+1}}$
5. $\dfrac{\sinh az}{z^4}$
6. $\dfrac{\operatorname{Ln}(z + 3) + \cos z}{(z + 1)^2}$
7. $\dfrac{z^n}{(z - a)^{n+1}}$
8. $\dfrac{e^z}{(z - a)^n}$

[2]GIACINTO MORERA (1856–1909), Italian mathematician who worked in Genoa and Turin.

### 9–13 INTEGRATION AROUND DIFFERENT CONTOURS

Integrate around $C$. Show the details.

**9.** $\dfrac{(1+2z)\cos z}{(2z-1)^2}$, $C$ the unit circle, counterclockwise

**10.** $\dfrac{\sin 4z}{(z-4)^3}$, $C$ consists of $|z| = 5$ (counterclockwise) and $|z-3| = \frac{3}{2}$ (clockwise)

**11.** $\dfrac{\tan \pi z}{z^2}$, $C$ the ellipse $16x^2 + y^2 = 1$, counterclockwise

**12.** $\dfrac{e^{2z}}{z(z-2i)^2}$, $C$ consists of $|z-i| = 3$ (counterclockwise) and $|z| = 1$ (clockwise)

**13.** $\dfrac{e^{z/2}}{(z-a)^4}$, $C$ the circle $|z-2-i| = 3$, counterclockwise

**14. TEAM PROJECT. Theory on Growth**

**(a) Growth of entire functions.** If $f(z)$ is not a constant and is analytic for all (finite) $z$, and $R$ and $M$ are any positive real numbers (no matter how large), show that there exist values of $z$ for which $|z| > R$ and $|f(z)| > M$.

**(b) Growth of polynomials.** If $f(z)$ is a polynomial of degree $n > 0$ and $M$ is an arbitrary positive real number (no matter how large), show that there exists a positive real number $R$ such that $|f(z)| > M$ for all $|z| > R$.

**(c) Exponential function.** Show that $f(z) = e^z$ has the property characterized in (a) but does not have that characterized in (b).

**(d) Fundamental theorem of algebra.** *If $f(z)$ is a polynomial in $z$, not a constant, then $f(z) = 0$ for at least one value of $z$.* Prove this, using (a).

**15. (Proof of Theorem 1)** Complete the proof of Theorem 1 by performing the induction mentioned at the end.

## CHAPTER 14 REVIEW QUESTIONS AND PROBLEMS

1. What is a path of integration? What did we assume about paths?
2. State the definition of a complex line integral from memory.
3. What do we mean by saying that complex integration is a linear operation?
4. Make a list of integration methods discussed. Illustrate each with a simple example.
5. Which integration methods apply to analytic functions only?
6. What value do you get if you integrate $1/z$ counterclockwise around the unit circle? (You should memorize this basic result.) If you integrate $1/z^2$, $1/z^3, \cdots$?
7. Which theorem in this chapter do you regard as most important? State it from memory.
8. What is independence of path? What is the principle of deformation of path? Why is this important?
9. Do not confuse Cauchy's integral theorem and Cauchy's integral formula. State both. How are they related?
10. How can you extend Cauchy's integral theorem to doubly and triply connected domains?
11. If integrating $f(z)$ over the boundary circles of an annulus $D$ gives different values, can $f(z)$ be analytic in $D$? (Give reason.)
12. Is $\left|\int_C f(z)\,dz\right| = \int_C |f(z)|\,dz$? How would you find a bound for the integral on the left?
13. Is $\operatorname{Re}\int_C f(z)\,dz = \int_C \operatorname{Re} f(z)\,dz$? Give examples.
14. How did we use integral formulas for derivatives in integration?
15. What is Liouville's theorem? Give examples. State consequences.

### 16–30 INTEGRATION

Integrate by a suitable method:

16. $4z^3 + 2z$ from $-i$ to $2 + i$ along any path
17. $5z - 3/z$ counterclockwise around the unit circle
18. $z + 1/z$ counterclockwise around $|z + 3i| = 2$
19. $e^{2z}$ from $-2 + 3\pi i$ along the straight segment to $-2 + 5\pi i$
20. $e^{z^2}/(z-1)^2$ counterclockwise around $|z| = 2$
21. $z/(z^2+1)$ clockwise around $|z + i| = 1$
22. $\operatorname{Re} z$ from 0 to 4 and then vertically up to $4 + 3i$
23. $\cosh 4z$ from 0 to $2i$ along the imaginary axis
24. $e^z/z$ over $C$ consisting of $|z| = 1$ (counterclockwise) and $|z| = \frac{1}{2}$ (clockwise)
25. $(\sin z)/z$ clockwise around a circle containing $z = 0$ in its interior
26. $\operatorname{Im} z$ counterclockwise around $|z| = r$
27. $(\operatorname{Ln} z)/(z-2i)^2$ counterclockwise around $|z - 2i| = 1$
28. $(\tan \pi z)/(z-1)^2$ counterclockwise around $|z - 1| = 0.2$
29. $|z| + z$ clockwise around the unit circle
30. $(z - i)^{-3}(z^3 + \sin z)$ counterclockwise around any circle with center $i$

## SUMMARY OF CHAPTER 14
# Complex Integration

The **complex line integral** of a function $f(z)$ taken over a path $C$ is denoted by

(1) $$\int_C f(z)\,dz \qquad \text{or, if } C \text{ is closed, also by} \qquad \oint_C f(z) \qquad \text{(Sec. 14.1).}$$

If $f(z)$ is analytic in a simply connected domain $D$, then we can evaluate (1) as in calculus by indefinite integration and substitution of limits, that is,

(2) $$\int_C f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

for every path $C$ in $D$ from a point $z_0$ to a point $z_1$ (see Sec. 14.1). These assumptions imply **independence of path,** that is, (2) depends only on $z_0$ and $z_1$ (and on $f(z)$, of course) but not on the choice of $C$ (Sec. 14.2). The existence of an $F(z)$ such that $F'(z) = f(z)$ is proved in Sec. 14.2 by Cauchy's integral theorem (see below).

A general method of integration, not restricted to analytic functions, uses the equation $z = z(t)$ of $C$, where $a \leqq t \leqq b$,

(3) $$\int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

**Cauchy's integral theorem** is the most important theorem in this chapter. It states that if $f(z)$ is analytic in a simply connected domain $D$, then for every closed path $C$ in $D$ (Sec. 14.2),

(4) $$\oint_C f(z)\,dz = 0.$$

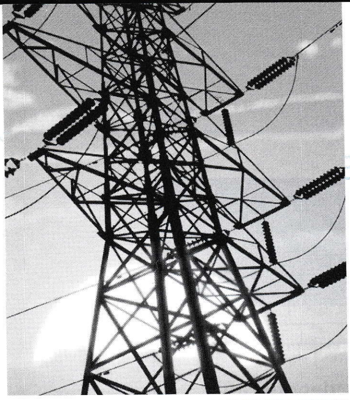
Under the same assumptions and for any $z_0$ in $D$ and closed path $C$ in $D$ containing $z_0$ in its interior we also have **Cauchy's integral formula**

(5) $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz.$$

Furthermore, under these assumptions $f(z)$ has derivatives of all orders in $D$ that are themselves analytic functions in $D$ and (Sec. 14.4)

(6) $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots).$$

This implies *Morera's theorem* (the converse of Cauchy's integral theorem) and *Cauchy's inequality* (Sec. 14.4), which in turn implies *Liouville's theorem* that an entire function that is bounded in the whole complex plane must be constant.



# CHAPTER 15

# Power Series, Taylor Series

Complex power series, in particular, Taylor series, are analogs of real power and Taylor series in calculus. However, they are much more fundamental in complex analysis than their real counterparts in calculus. The reason is that power series represent analytic functions (Sec. 15.3) and, conversely, every analytic function can be represented by power series, called Taylor series (Sec. 15.4).

Use Sec. 15.1 for reference if you are familiar with convergence tests for *real* series—in complex this is quite similar. The last section (15.5) on uniform convergence is *optional.*

*Prerequisite:* Chaps. 13, 14.
*Sections that may be omitted in a shorter course:* 14.1, 14.5.
*References and Answers to Problems:* App. 1 Part D, App. 2.

## 15.1 Sequences, Series, Convergence Tests

In this section we define the basic concepts for ***complex*** sequences and series and discuss tests for convergence and divergence. This is very similar to *real* sequences and series in calculus. ***If you feel at home with the latter and want to take for granted that the ratio test also holds in complex, skip this section and go to Sec. 15.2.***

### Sequences

The basic definitions are as in calculus. An *infinite sequence* or, briefly, a **sequence,** is obtained by assigning to each positive integer $n$ a number $z_n$, called a **term** of the sequence, and is written

$$z_1, z_2, \cdots \qquad \text{or} \qquad \{z_1, z_2, \cdots\} \qquad \text{or briefly} \qquad \{z_n\}.$$

We may also write $z_0, z_1, \cdots$ or $z_2, z_3, \cdots$ or start with some other integer if convenient.

A **real sequence** is one whose terms are real.

**Convergence.** A **convergent sequence** $z_1, z_2, \cdots$ is one that has a limit $c$, written

$$\lim_{n\to\infty} z_n = c \qquad \text{or simply} \qquad z_n \to c.$$

By definition of **limit** this means that for every $\epsilon > 0$ we can find an $N$ such that

$$(1) \qquad |z_n - c| < \epsilon \qquad \text{for all } n > N;$$

geometrically, all terms $z_n$ with $n > N$ lie in the open disk of radius $\epsilon$ and center $c$ (Fig. 358) and only finitely many terms do not lie in that disk. [For a *real* sequence, (1) gives an open interval of length $2\epsilon$ and real midpoint $c$ on the real line; see Fig. 359.]

A **divergent sequence** is one that does not converge.

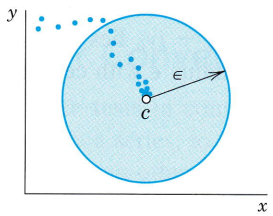



**Fig. 358.** Convergent complex sequence

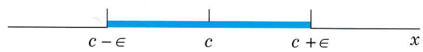



**Fig. 359.** Convergent real sequence

**EXAMPLE 1 Convergent and Divergent Sequences**

The sequence $\{i^n/n\} = \{i, -1/2, -i/3, 1/4, \cdots\}$ is convergent with limit 0.

The sequence $\{i^n\} = \{i, -1, -i, 1, \cdots\}$ is divergent, and so is $\{z_n\}$ with $z_n = (1 + i)^n$. ■

**EXAMPLE 2 Sequences of the Real and the Imaginary Parts**

The sequence $\{z_n\}$ with $z_n = x_n + iy_n = 1 - 1/n^2 + i(2 + 4/n)$ is $6i,\ 3/4 + 4i,\ 8/9 + 10i/3,\ 15/16 + 3i, \cdots$. (Sketch it.) It converges with the limit $c = 1 + 2i$. Observe that $\{x_n\}$ has the limit $1 = \operatorname{Re} c$ and $\{y_n\}$ has the limit $2 = \operatorname{Im} c$. This is typical. It illustrates the following theorem by which the convergence of a *complex* sequence can be referred back to that of the two *real* sequences of the real parts and the imaginary parts. ■

**THEOREM 1**

**Sequences of the Real and the Imaginary Parts**

*A sequence $z_1, z_2, \cdots, z_n, \cdots$ of complex numbers $z_n = x_n + iy_n$ (where $n = 1, 2, \cdots$) converges to $c = a + ib$ if and only if the sequence of the real parts $x_1, x_2, \cdots$ converges to $a$ and the sequence of the imaginary parts $y_1, y_2, \cdots$ converges to $b$.*

**PROOF** Convergence $z_n \to c = a + ib$ implies convergence $x_n \to a$ and $y_n \to b$ because if $|z_n - c| < \epsilon$, then $z_n$ lies within the circle of radius $\epsilon$ about $c = a + ib$, so that (Fig. 360a)

$$|x_n - a| < \epsilon, \qquad |y_n - b| < \epsilon.$$

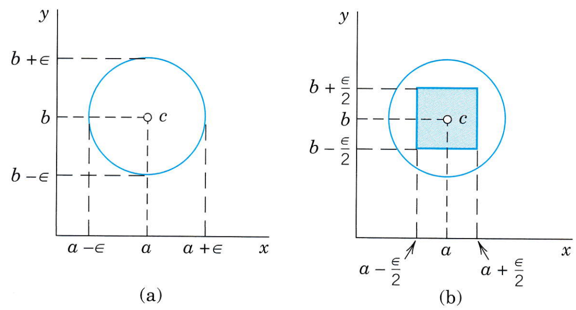



**Fig. 360.** Proof of Theorem 1

Conversely, if $x_n \to a$ and $y_n \to b$ as $n \to \infty$, then for a given $\epsilon > 0$ we can choose $N$ so large that, for every $n > N$,

$$|x_n - a| < \frac{\epsilon}{2}, \qquad |y_n - b| < \frac{\epsilon}{2}.$$

These two inequalities imply that $z_n = x_n + iy_n$ lies in a square with center $c$ and side $\epsilon$. Hence, $z_n$ must lie within a circle of radius $\epsilon$ with center $c$ (Fig. 360b). ■

## Series

Given a sequence $z_1, z_2, \cdots, z_m, \cdots$, we may form the sequence of the sums

$$s_1 = z_1, \qquad s_2 = z_1 + z_2, \qquad s_3 = z_1 + z_2 + z_3, \qquad \cdots$$

and in general

$$s_n = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \cdots). \tag{2}$$

$s_n$ is called the **$n$th partial sum** of the *infinite series* or **series**

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots. \tag{3}$$

The $z_1, z_2, \cdots$ are called the **terms** of the series. (Our usual ***summation letter*** is $n$, unless we need $n$ for another purpose, as here, and we then use $m$ as the summation letter.)

A **convergent series** is one whose sequence of partial sums converges, say,

$$\lim_{n\to\infty} s_n = s. \qquad \text{Then we write} \qquad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

and call $s$ the **sum** or *value* of the series. A series that is not convergent is called a **divergent series.**

If we omit the terms of $s_n$ from (3), there remains

$$R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots. \tag{4}$$

This is called the **remainder** *of the series* (3) *after the term* $z_n$. Clearly, if (3) converges and has the sum $s$, then

$$s = s_n + R_n, \qquad \text{thus} \qquad R_n = s - s_n.$$

Now $s_n \to s$ by the definition of convergence; hence $R_n \to 0$. In applications, when $s$ is unknown and we compute an approximation $s_n$ of $s$, then $|R_n|$ is the error, and $R_n \to 0$ means that we can make $|R_n|$ as small as we please, by choosing $n$ large enough.

An application of Theorem 1 to the partial sums immediately relates the convergence of a complex series to that of the two series of its real parts and of its imaginary parts:

**THEOREM 2**

**Real and Imaginary Parts**

*A series* (3) *with* $z_m = x_m + iy_m$ *converges and has the sum* $s = u + iv$ *if and only if* $x_1 + x_2 + \cdots$ *converges and has the sum* $u$ *and* $y_1 + y_2 + \cdots$ *converges and has the sum* $v$.

## Tests for Convergence and Divergence of Series

**Convergence tests** in complex are practically the same as in calculus. We apply them before we use a series, to make sure that the series converges.

Divergence can often be shown very simply as follows.

**THEOREM 3**

**Divergence**

*If a series* $z_1 + z_2 + \cdots$ *converges, then* $\lim_{m\to\infty} z_m = 0$. *Hence if this does not hold, the series diverges.*

**PROOF** If $z_1 + z_2 + \cdots$ converges, with the sum $s$, then, since $z_m = s_m - s_{m-1}$,

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} (s_m - s_{m-1}) = \lim_{m\to\infty} s_m - \lim_{m\to\infty} s_{m-1} = s - s = 0. \qquad \blacksquare$$

**CAUTION!** $z_m \to 0$ is *necessary* for convergence but *not sufficient*, as we see from the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which satisfies this condition but diverges, as is shown in calculus (see, for example, Ref. [GR11] in App. 1).

The practical difficulty in proving convergence is that in most cases the sum of a series is unknown. Cauchy overcame this by showing that a series converges if and only if its partial sums eventually get close to each other:

**THEOREM 4**

**Cauchy's Convergence Principle for Series**

*A series* $z_1 + z_2 + \cdots$ *is convergent if and only if for every given* $\epsilon > 0$ (*no matter how small*) *we can find an* $N$ (*which depends on* $\epsilon$, *in general*) *such that*

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots \tag{5}$$

The somewhat involved proof is left optional (see App. 4).

**Absolute Convergence.** A series $z_1 + z_2 + \cdots$ is called **absolutely convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$$

is convergent.

If $z_1 + z_2 + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \cdots$ is called, more precisely, **conditionally convergent.**

**EXAMPLE 3 A Conditionally Convergent Series**

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots$ converges, but only conditionally since the harmonic series diverges, as mentioned above (after Theorem 3). ■

*If a series is absolutely convergent, it is convergent.*

This follows readily from Cauchy's principle (see Team Project 30). This principle also yields the following general convergence test.

**THEOREM 5 Comparison Test**

*If a series $z_1 + z_2 + \cdots$ is given and we can find a convergent series $b_1 + b_2 + \cdots$ with nonnegative real terms such that $|z_1| \leqq b_1$, $|z_2| \leqq b_2, \cdots$, then the given series converges, even absolutely.*

**PROOF** By Cauchy's principle, since $b_1 + b_2 + \cdots$ converges, for any given $\epsilon > 0$ we can find an $N$ such that

$$b_{n+1} + \cdots + b_{n+p} < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots.$$

From this and $|z_1| \leqq b_1$, $|z_2| \leqq b_2, \cdots$ we conclude that for those $n$ and $p$,

$$|z_{n+1}| + \cdots + |z_{n+p}| \leqq b_{n+1} + \cdots + b_{n+p} < \epsilon.$$

Hence, again by Cauchy's principle, $|z_1| + |z_2| + \cdots$ converges, so that $z_1 + z_2 + \cdots$ is absolutely convergent. ■

A good comparison series is the geometric series, which behaves as follows.

**THEOREM 6 Geometric Series**

*The* ***geometric series***

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots \tag{6*}$$

*converges with the sum* $1/(1-q)$ *if* $|q| < 1$ *and diverges if* $|q| \geqq 1$.

**PROOF** If $|q| \geqq 1$, then $|q^m| \geqq 1$ and Theorem 3 implies divergence.

Now let $|q| < 1$. The $n$th partial sum is

$$s_n = 1 + q + \cdots + q^n.$$

From this,

$$qs_n = \qquad q + \cdots + q^n + q^{n+1}.$$

On subtraction, most terms on the right cancel in pairs, and we are left with

$$s_n - qs_n = (1-q)s_n = 1 - q^{n+1}.$$

Now $1 - q \neq 0$ since $q \neq 1$, and we may solve for $s_n$, finding

$$s_n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q}. \tag{6}$$

Since $|q| < 1$, the last term approaches zero as $n \to \infty$. Hence if $|q| < 1$, the series is convergent and has the sum $1/(1 - q)$. This completes the proof. ■

## Ratio Test

This is the most important test in our further work. We get it by taking the geometric series as comparison series $b_1 + b_2 + \cdots$ in Theorem 5:

**THEOREM 7**

**Ratio Test**

*If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, \cdots)$ has the property that for every $n$ greater than some $N$,*

$$\left|\frac{z_{n+1}}{z_n}\right| \leqq q < 1 \qquad (n > N) \tag{7}$$

*(where $q < 1$ is fixed), this series converges absolutely. If for every $n > N$,*

$$\left|\frac{z_{n+1}}{z_n}\right| \geqq 1 \qquad (n > N), \tag{8}$$

*the series diverges.*

**PROOF** If (8) holds, then $|z_{n+1}| \geqq |z_n|$ for $n > N$, so that divergence of the series follows from Theorem 3.

If (7) holds, then $|z_{n+1}| \leqq |z_n|\, q$ for $n > N$, in particular,

$$|z_{N+2}| \leqq |z_{N+1}|q, \qquad |z_{N+3}| \leqq |z_{N+2}|q \leqq |z_{N+1}|q^2, \qquad \text{etc.,}$$

and in general, $|z_{N+p}| \leqq |z_{N+1}|q^{p-1}$. Since $q < 1$, we obtain from this and Theorem 6

$$|z_{N+1}| + |z_{N+2}| + |z_{N+3}| + \cdots \leqq |z_{N+1}|\,(1 + q + q^2 + \cdots) \leqq |z_{N+1}|\,\frac{1}{1 - q}.$$

Absolute convergence of $z_1 + z_2 + \cdots$ now follows from Theorem 5. ■

**CAUTION!** The inequality (7) implies $|z_{n+1}/z_n| < 1$, but this does ***not*** imply convergence, as we see from the harmonic series, which satisfies $z_{n+1}/z_n = n/(n + 1) < 1$ for all $n$ but diverges.

If the sequence of the ratios in (7) and (8) converges, we get the more convenient

**THEOREM 8** **Ratio Test**

*If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, \cdots)$ is such that* $\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L$, *then:*

**(a)** *If $L < 1$, the series converges absolutely.*

**(b)** *If $L > 1$, the series diverges.*

**(c)** *If $L = 1$, the series may converge or diverge, so that the test fails and permits no conclusion.*

**PROOF** **(a)** We write $k_n = |z_{n+1}/z_n|$ and let $L = 1 - b < 1$. Then by the definition of limit, the $k_n$ must eventually get close to $1 - b$, say, $k_n \leqq q = 1 - \frac{1}{2}b < 1$ for all $n$ greater than some $N$. Convergence of $z_1 + z_2 + \cdots$ now follows from Theorem 7.

**(b)** Similarly, for $L = 1 + c > 1$ we have $k_n \geqq 1 + \frac{1}{2}c > 1$ for all $n > N^*$ (sufficiently large), which implies divergence of $z_1 + z_2 + \cdots$ by Theorem 7.

**(c)** The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ has $z_{n+1}/z_n = n/(n + 1)$, hence $L = 1$, and diverges. The series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \qquad \text{has} \qquad \frac{z_{n+1}}{z_n} = \frac{n^2}{(n+1)^2},$$

hence also $L = 1$, but it converges. Convergence follows from (Fig. 361)

$$s_n = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leqq 1 + \int_1^n \frac{dx}{x^2} = 2 - \frac{1}{n},$$

so that $s_1, s_2, \cdots$ is a bounded sequence and is monotone increasing (since the terms of the series are all positive); both properties together are sufficient for the convergence of the real sequence $s_1, s_2, \cdots$. (In calculus this is proved by the so-called *integral test,* whose idea we have used.) ■

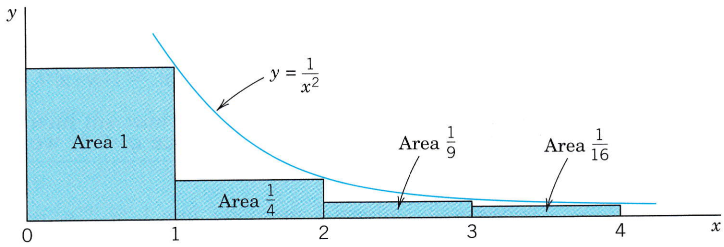



**Fig. 361.** Convergence of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

**EXAMPLE 4** **Ratio Test**

Is the following series convergent or divergent? (First guess, then calculate.)

$$\sum_{n=0}^{\infty} \frac{(100 + 75i)^n}{n!} = 1 + (100 + 75i) + \frac{1}{2!}(100 + 75i)^2 + \cdots$$

***Solution.*** By Theorem 8, the series is convergent, since

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100 + 75i|^{n+1}/(n+1)!}{|100 + 75i|^n/n!} = \frac{|100 + 75i|}{n+1} = \frac{125}{n+1} \quad \to \quad L = 0. \qquad \blacksquare$$

**EXAMPLE 5** **Theorem 7 More General than Theorem 8**

Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$. Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

***Solution.*** The ratios of the absolute values of successive terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$. Hence convergence follows from Theorem 7. Since the sequence of these ratios has no limit, Theorem 8 is not applicable. $\blacksquare$

## Root Test

The ratio test and the root test are the two practically most important tests. The ratio test is usually simpler, but the root test is somewhat more general.

**THEOREM 9** **Root Test**

*If a series $z_1 + z_2 + \cdots$ is such that for every $n$ greater than some $N$,*

$$\sqrt[n]{|z_n|} \leqq q < 1 \qquad (n > N) \tag{9}$$

*(where $q < 1$ is fixed), this series converges absolutely. If for infinitely many $n$,*

$$\sqrt[n]{|z_n|} \geqq 1, \tag{10}$$

*the series diverges.*

**PROOF** If (9) holds, then $|z_n| \leqq q^n < 1$ for all $n > N$. Hence the series $|z_1| + |z_2| + \cdots$ converges by comparison with the geometric series, so that the series $z_1 + z_2 + \cdots$ converges absolutely. If (10) holds, then $|z_n| \geqq 1$ for infinitely many $n$. Divergence of $z_1 + z_2 + \cdots$ now follows from Theorem 3. $\blacksquare$

**CAUTION!** Equation (9) implies $\sqrt[n]{|z_n|} < 1$, but this does not imply convergence, as we see from the harmonic series, which satisfies $\sqrt[n]{1/n} < 1$ (for $n > 1$) but diverges.

If the sequence of the roots in (9) and (10) converges, we more conveniently have

**THEOREM 10** **Root Test**

*If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then:*

**(a)** *The series converges absolutely if $L < 1$.*

**(b)** *The series diverges if $L > 1$.*

**(c)** *If $L = 1$, the test fails; that is, no conclusion is possible.*

**PROOF** The proof parallels that of Theorem 8.

**(a)** Let $L = 1 - a^* < 1$. Then by the definition of a limit we have $\sqrt[n]{|z_n|} < q = 1 - \frac{1}{2}a^* < 1$ for all $n$ greater than some (sufficiently large) $N^*$. Hence $|z_n| < q^n < 1$ for all $n > N^*$. Absolute convergence of the series $z_1 + z_2 + \cdots$ now follows by the comparison with the geometric series.

**(b)** If $L > 1$, then we also have $\sqrt[n]{|z_n|} > 1$ for all sufficiently large $n$. Hence $|z_n| > 1$ for those $n$. Theorem 3 now implies that $z_1 + z_2 + \cdots$ diverges.

**(c)** Both the *divergent* harmonic series and the *convergent* series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ give $L = 1$. This can be seen from $(\ln n)/n \to 0$ and

$$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{1/n}} = \frac{1}{e^{(1/n)\ln n}} \to \frac{1}{e^0}, \qquad \sqrt[n]{\frac{1}{n^2}} = \frac{1}{n^{2/n}} = \frac{1}{e^{(2/n)\ln n}} \to \frac{1}{e^0}. \quad \blacksquare$$

## PROBLEM SET 15.1

### 1–10 SEQUENCES

Are the following sequences $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find their limit points. (Show the details of your work.)

**1.** $z_n = (-1)^n + i/2^n$

**2.** $z_n = e^{-n\pi i/4}$

**3.** $z_n = (-1)^n/(n + i)$

**4.** $z_n = (1 + i)^n$

**5.** $z_n = \text{Ln}\,((2 + i)^n)$

**6.** $z_n = (3 + 4i)^n/n!$

**7.** $z_n = \sin(n\pi/4) + i^n$

**8.** $z_n = [(1 + 3i)/\sqrt{10}]^n$

**9.** $z_n = (0.9 + 0.1i)^{2n}$

**10.** $z_n = (5 + 5i)^{-n}$

**11.** Illustrate Theorem 1 by an example of your own.

**12. (Uniqueness of limit)** Show that if a sequence converges, its limit is unique.

**13. (Addition)** If $z_1, z_2, \cdots$ converges with the limit $l$ and $z_1^*, z_2^*, \cdots$ converges with the limit $l^*$, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ converges with the limit $l + l^*$.

**14. (Multiplication)** Show that under the assumptions of Prob. 13 the sequence $z_1z_1^*, z_2z_2^*, \cdots$ converges with the limit $ll^*$.

**15. (Boundedness)** Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.

### 16–24 SERIES

Are the following series convergent or divergent? (Give a reason.)

**16.** $\displaystyle\sum_{n=0}^{\infty} \frac{(10 - 15i)^n}{n!}$

**17.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1 + 2i)^{2n+1}}{(2n + 1)!}$

**18.** $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - 2i}$

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

**20.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}(1 + i)^n$

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{n - i}{3n + 2i}$

**24.** $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{i}{3}\right)^n$

**25.** What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?

**26.** Illustrate Theorem 2 by an example of your choice.

**27.** For what $n$ do we obtain the term of greatest absolute value of the series in Example 4? About how big is it? First guess, then calculate it by the Stirling formula in Sec. 24.4.

**28.** Give another example showing that Theorem 7 is more general than Theorem 8.

**29. CAS PROJECT. Sequences and Series. (a)** Write a program for graphing complex **sequences.** Apply it to sequences of your choice that have interesting "geometrical" properties (e.g., lying on an ellipse, spiraling toward its limit, etc.).

**(b)** Write a program for computing and graphing numeric values of the first $n$ partial sums of a **series** of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.

**30. TEAM PROJECT. Series. (a) Absolute convergence.** Show that if a series converges absolutely, it is convergent.

**(b)** Write a short report on the basic concepts and properties of series of numbers, explaining in each case whether or not they carry over from real series (discussed in calculus) to complex series, with reasons given.

**(c) Estimate of the remainder.** Let $|z_{n+1}/z_n| \leqq q < 1$, so that the series $z_1 + z_2 + \cdots$ converges by the ratio test. Show that the remainder $R_n = z_{n+1} + z_{n+2} + \cdots$ satisfies the inequality $|R_n| \leqq |z_{n+1}|/(1 - q)$.

**(d)** Using (c), find how many terms suffice for computing the sum $s$ of the series

$$\sum_{n=1}^{\infty} \frac{n + i}{2^n n}$$

with an error not exceeding 0.05 and compute $s$ to this accuracy.

**(e)** Find other applications of the estimate in (c).

# 15.2 Power Series

Power series are the most important series in complex analysis because we shall see that their sums are analytic functions, and every analytic function can be represented by power series (Theorem 5 in Sec. 15.3 and Theorem 1 in Sec. 15.4).

A **power series** *in powers of* $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \tag{1}$$

where $z$ is a complex variable, $a_0, a_1, \cdots$ are complex (or real) constants, called the **coefficients** of the series, and $z_0$ is a complex (or real) constant, called the **center** of the series. This generalizes real power series of calculus.

If $z_0 = 0$, we obtain as a particular case a *power series in powers of* $z$:

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots . \tag{2}$$

## Convergence Behavior of Power Series

Power series have variable terms (functions of $z$), but ***if we fix z, then all the concepts for series with constant terms in the last section apply.*** Usually a series with variable terms will converge for some $z$ and diverge for others. For a power series the situation is simple. The series (1) may converge in a disk with center $z_0$ or in the whole $z$-plane or only at $z_0$. We illustrate this with typical examples and then prove it.

**EXAMPLE 1 Convergence in a Disk. Geometric Series**

The *geometric series*

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges absolutely if $|z| < 1$ and diverges if $|z| \geqq 1$ (see Theorem 6 in Sec. 15.1). ■

**EXAMPLE 2 Convergence for Every z**

The power series (which will be the Maclaurin series of $e^z$ in Sec. 15.4)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is absolutely convergent for every $z$, In fact, by the ratio test, for any fixed $z$,

$$\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

**EXAMPLE 3** **Convergence Only at the Center. (Useless Series)**

The following power series converges only at $z = 0$, but diverges for every $z \neq 0$, as we shall show.

$$\sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^3 + \cdots$$

In fact, from the ratio test we have

$$\left|\frac{(n+1)! z^{n+1}}{n! z^n}\right| = (n+1)\,|z| \to \infty \quad \text{as} \quad n \to \infty \quad (z \text{ fixed and } \neq 0).$$

**THEOREM 1** **Convergence of a Power Series**

**(a)** *Every power series* (1) *converges at the center* $z_0$.

**(b)** *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* $z$ *closer to* $z_0$ *than* $z_1$, *that is,* $|z - z_0| < |z_1 - z_0|$. See Fig. 362.

**(c)** *If* (1) *diverges at a* $z = z_2$, *it diverges for every* $z$ *farther away from* $z_0$ *than* $z_2$. See Fig. 362.

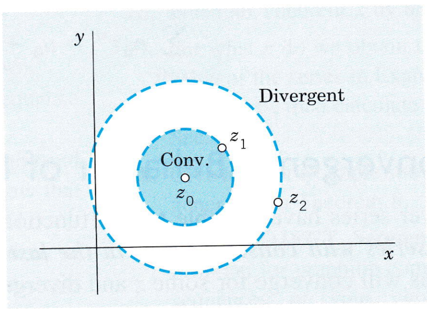



Fig. 362. Theroem 1

**PROOF** **(a)** For $z = z_0$ the series reduces to the single term $a_0$.

**(b)** Convergence at $z = z_1$ gives by Theorem 3 in Sec. 15.1 $a_n(z_1 - z_0)^n \to 0$ as $n \to \infty$. This implies boundedness in absolute value,

$$|a_n(z_1 - z_0)^n| < M \qquad \text{for every } n = 0, 1, \cdots.$$

Multiplying and dividing $a_n(z - z_0)^n$ by $(z_1 - z_0)^n$ we obtain from this

$$|a_n(z - z_0)^n| = \left|a_n(z_1 - z_0)^n \left(\frac{z - z_0}{z_1 - z_0}\right)^n\right| \leqq M \left|\frac{z - z_0}{z_1 - z_0}\right|^n.$$

Summation over $n$ gives

$$\sum_{n=1}^{\infty} |a_n(z - z_0)^n| \leqq M \sum_{n=1}^{\infty} \left| \frac{z - z_0}{z_1 - z_0} \right|^n. \tag{3}$$

Now our assumption $|z - z_0| < |z_1 - z_0|$ implies that $|(z - z_0)/(z_1 - z_0)| < 1$. Hence the series on the right side of (3) is a converging geometric series (see Theorem 6 in Sec. 15.1). Absolute convergence of (1) as stated in (b) now follows by the comparison test in Sec. 15.1.

**(c)** If this were false, we would have convergence at a $z_3$ farther away from $z_0$ than $z_2$. This would imply convergence at $z_2$, by (b), a contradiction to our assumption of divergence at $z_2$. ■

## Radius of Convergence of a Power Series

Convergence for every $z$ (the nicest case, Example 2) or for no $z \neq z_0$ (the useless case, Example 3) needs no further discussion, and we put these cases aside for a moment. We consider the ***smallest*** circle with center $z_0$ that includes all the points at which a given power series (1) converges. Let $R$ denote its radius. The circle

$$|z - z_0| = R \qquad \text{(Fig. 363)}$$

is called the **circle of convergence** and its radius $R$ the **radius of convergence** of (1). Theorem 1 then implies convergence everywhere within that circle, that is, for all $z$ for which

$$|z - z_0| < R \tag{4}$$

(the open disk with center $z_0$ and radius $R$). Also, since $R$ is as *small* as possible, the series (1) diverges for all $z$ for which

$$|z - z_0| > R. \tag{5}$$

No general statements can be made about the convergence of a power series (1) ***on the circle of convergence*** itself. The series (1) may converge at some or all or none of these points. Details will not be essential to us. Hence a simple example may just give us the idea.

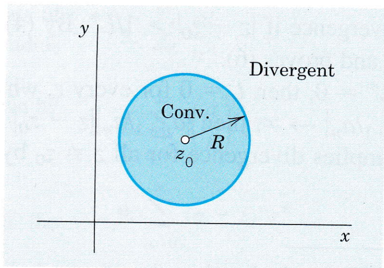



Fig. 363. Circle of convergence

**EXAMPLE 4** **Behavior on the Circle of Convergence**

On the circle of convergence (radius $R = 1$ in all three series),

$\Sigma\, z^n/n^2$ converges everywhere since $\Sigma\, 1/n^2$ converges,

$\Sigma\, z^n/n$ converges at $-1$ (by Leibniz's test) but diverges at 1,

$\Sigma\, z^n$ diverges everywhere. ■

**Notations $R = \infty$ and $R = 0$.** To incorporate these two excluded cases in the present notation, we write

$R = \infty$ if the series (1) converges for all $z$ (as in Example 2),

$R = 0$ if (1) converges only at the center $z = z_0$ (as in Example 3).

These are convenient notations, but nothing else.

**Real Power Series.** In this case in which powers, coefficients, and center are real, formula (4) gives the **convergence interval** $|x - x_0| < R$ of length $2R$ on the real line.

**Determination of the Radius of Convergence from the Coefficients.** For this important practical task we can use

**THEOREM 2** **Radius of Convergence $R$**

*Suppose that the sequence $|a_{n+1}/a_n|$, $n = 1, 2, \cdots$, converges with limit $L^*$. If $L^* = 0$, then $R = \infty$; that is, the power series (1) converges for all $z$. If $L^* \neq 0$ (hence $L^* > 0$), then*

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{6}$$

**(Cauchy–Hadamard formula[1]).**

*If $|a_{n+1}/a_n| \to \infty$, then $R = 0$ (convergence only at the center $z_0$).*

**PROOF** For (1) the ratio of the terms in the ratio test (Sec. 15.1) is

$$\left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n(z - z_0)^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |z - z_0|. \qquad \text{The limit is} \qquad L = L^*|z - z_0|.$$

Let $L^* \neq 0$, thus $L^* > 0$. We have convergence if $L = L^*|z - z_0| < 1$, thus $|z - z_0| < 1/L^*$, and divergence if $|z - z_0| > 1/L^*$. By (4) and (5) this shows that $1/L^*$ is the convergence radius and proves (6).

If $L^* = 0$, then $L = 0$ for every $z$, which gives convergence for all $z$ by the ratio test. If $|a_{n+1}/a_n| \to \infty$, then $|a_{n+1}/a_n||z - z_0| > 1$ for any $z \neq z_0$ and all sufficiently large $n$. This implies divergence for all $z \neq z_0$ by the ratio test (Theorem 7, Sec. 15.1). ■

[1]Named after the French mathematicians A. L. CAUCHY (see Sec. 2.5) and JACQUES HADAMARD (1865–1963). Hadamard made basic contributions to the theory of power series and devoted his lifework to partial differential equations.

Formula (6) will not help if $L^*$ does not exist, but extensions of Theorem 2 are still possible, as we discuss in Example 6 below.

**EXAMPLE 5 Radius of Convergence**

By (6) the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ is

$$R = \lim_{n\to\infty}\left[\frac{(2n)!}{(n!)^2}\Big/\frac{(2n+2)!}{((n+1)!)^2}\right] = \lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)!}\cdot\frac{((n+1)!)^2}{(n!)^2}\right] = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$

The series converges in the open disk $|z-3i| < \frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$. ■

**EXAMPLE 6 Extension of Theorem 2**

Find the radius of convergence $R$ of the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4+\cdots.$$

***Solution.*** The sequence of the ratios $1/6$, $2(2+\frac{1}{4})$, $1/(8(2+\frac{1}{4}))$, $\cdots$ does not converge, so that Theorem 2 is of no help. It can be shown that

$$\text{(6*)} \qquad R = 1/\tilde{L}, \qquad \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

This still does not help here, since $\{\sqrt[n]{|a_n|}\}$ does not converge because $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = 1/2$ for odd $n$, whereas for even $n$ we have

$$\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1 \quad \text{as} \quad n\to\infty,$$

so that $\sqrt[n]{|a_n|}$ has the two limit points $1/2$ and $1$. It can further be shown that

$$\text{(6**)} \qquad R = 1/\tilde{l}, \qquad \tilde{l} \text{ the greatest limit point of the sequence } \left\{\sqrt[n]{|a_n|}\right\}.$$

Here $\tilde{l} = 1$, so that $R = 1$. *Answer.* The series converges for $|z| < 1$. ■

**Summary.** Power series converge in an open circular disk or some even for every $z$ (or some only at the center, but they are useless); for the radius of convergence, see (6) or Example 6.

Except for the useless ones, power series have sums that are analytic functions (as we show in the next section); this accounts for their importance in complex analysis.

## PROBLEM SET 15.2

1. **(Powers missing)** Show that if $\Sigma\, a_n z^n$ has radius of convergence $R$ (assumed finite), then $\Sigma\, a_n z^{2n}$ has radius of convergence $\sqrt{R}$. Give examples.
2. **(Convergence behavior)** Illustrate the facts shown by Examples 1–3 by further examples of your own.

### 3–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence of the following power series. (Show the details.)

3. $\sum_{n=1}^{\infty}\frac{(z+i)^n}{n^2}$
4. $\sum_{n=0}^{\infty}\frac{n^n}{n!}(z+2i)^n$
5. $\sum_{n=0}^{\infty}\frac{n!}{n^n}(z+1)^n$
6. $\sum_{n=0}^{\infty}\frac{2^{100n}}{n!}z^n$
7. $\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n z^n$
8. $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}z^{2n}$
9. $\sum_{n=0}^{\infty}(n-i)^n z^n$
10. $\sum_{n=0}^{\infty}\frac{(2z)^{2n}}{(2n)!}$
11. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n$
12. $\sum_{n=0}^{\infty}\frac{4^n}{(1+i)^n}(z-5)^n$

**13.** $\sum_{n=2}^{\infty} n(n-1)(z-3+2i)^n$

**14.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$

**15.** $\sum_{n=0}^{\infty} 2^n (z-i)^{4n}$

**16.** $\sum_{n=0}^{\infty} \left(\frac{2+3i}{5-i}\right)^n (z-\pi)^n$

**17.** $\sum_{n=0}^{\infty} \frac{n^4}{2^n} z^{2n}$

**18.** $\sum_{n=0}^{\infty} \frac{(4n)!}{2^n (n!)^4} (z+\pi i)^n$

**19. CAS PROJECT. Radius of Convergence.** Write a program for computing $R$ from (6), (6*), or (6**), in this order, depending on the existence of the limits needed. Test the program on series of your choice and such that all three formulas (6), (6*), and (6**) will come up.

**20. TEAM PROJECT. Radius of Convergence. (a)** Formula (6) for $R$ contains $|a_n/a_{n+1}|$, not $|a_{n+1}/a_n|$. How could you memorize this by using a qualitative argument?

**(b) Change of coefficients.** What happens to $R$ $(0 < R < \infty)$ if you (i) multiply all $a_n$ by $k \neq 0$, (ii) multiply $a_n$ by $k^n \neq 0$, (iii) replace $a_n$ by $1/a_n$?

**(c) Example 6** extends Theorem 2 to nonconvergent cases of $a_n/a_{n+1}$. Do you understand the principle of "mixing" by which Example 6 was obtained? Use this principle for making up further examples.

**(d)** Does there exist a power series in powers of $z$ that converges at $z = 30 + 10i$ and diverges at $z = 31 - 6i$? (Give reason.)

# 15.3 Functions Given by Power Series

The main goal of this section is to show that power series represent analytic functions (Theorem 5). Along our way we shall see that power series behave nicely under addition, multiplication, differentiation, and integration, which makes these series very useful in complex analysis.

To simplify the formulas in this section, we take $z_0 = 0$ and write

$$\sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

This is no restriction because a series in powers of $\hat{z} - z_0$ with any $z_0$ can always be reduced to the form (1) if we set $\hat{z} - z_0 = z$.

**Terminology and Notation.** If any given power series (1) has a nonzero radius of convergence $R$ (thus $R > 0$), its sum is a function of $z$, say $f(z)$. Then we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R). \tag{2}$$

We say that $f(z)$ *is* **represented** *by the power series* or that *it is* **developed** *in the power series.* For instance, the geometric series *represents* the function $f(z) = 1/(1-z)$ in the interior of the unit circle $|z| = 1$. (See Theorem 6 in Sec. 15.1.)

**Uniqueness of a Power Series Representation.** This is our next goal. It means that *a function $f(z)$ cannot be represented by two different power series with the same center.* We claim that if $f(z)$ can at all be developed in a power series with center $z_0$, the development is unique. This important fact is frequently used in complex analysis (as well as in calculus). We shall prove it in Theorem 2. The proof will follow from

**THEOREM 1**

**Continuity of the Sum of a Power Series**

*If a function $f(z)$ can be represented by a power series (2) with radius of convergence $R > 0$, then $f(z)$ is continuous at $z = 0$.*

**PROOF** From (2) with $z = 0$ we have $f(0) = a_0$. Hence by the definition of continuity we must show that $\lim_{z\to 0} f(z) = f(0) = a_0$. That is, we must show that for a given $\epsilon > 0$ there is a $\delta > 0$ such that $|z| < \delta$ implies $|f(z) - a_0| < \epsilon$. Now (2) converges absolutely for $|z| \leqq r$ with any $r$ such that $0 < r < R$, by Theorem 1 in Sec. 15.2. Hence the series

$$\sum_{n=1}^{\infty} |a_n| r^{n-1} = \frac{1}{r} \sum_{n=1}^{\infty} |a_n| r^n$$

converges. Let $S \neq 0$ be its sum. ($S = 0$ is trivial.) Then for $0 < |z| \leqq r$,

$$|f(z) - a_0| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \leqq |z| \sum_{n=1}^{\infty} |a_n|\, |z|^{n-1} \leqq |z| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z| S$$

and $|z|S < \epsilon$ when $|z| < \delta$, where $\delta > 0$ is less than $r$ and less than $\epsilon/S$. Hence $|z|S < \delta S < (\epsilon/S)S = \epsilon$. This proves the theorem. ■

From this theorem we can now readily obtain the desired uniqueness theorem (again assuming $z_0 = 0$ without loss of generality):

**THEOREM 2**

**Identity Theorem for Power Series. Uniqueness**

*Let the power series $a_0 + a_1 z + a_2 z^2 + \cdots$ and $b_0 + b_1 z + b_2 z^2 + \cdots$ both be convergent for $|z| < R$, where $R$ is positive, and let them both have the same sum for all these $z$. Then the series are identical, that is, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, $\cdots$.*

*Hence if a function $f(z)$ can be represented by a power series with any center $z_0$, this representation is* ***unique.***

**PROOF** We proceed by induction. By assumption,

$$a_0 + a_1 z + a_2 z^2 + \cdots = b_0 + b_1 z + b_2 z^2 + \cdots \qquad (|z| < R).$$

The sums of these two power series are continuous at $z = 0$, by Theorem 1. Hence if we consider $|z| > 0$ and let $z \to 0$ on both sides, we see that $a_0 = b_0$: the assertion is true for $n = 0$. Now assume that $a_n = b_n$ for $n = 0, 1, \cdots, m$. Then on both sides we may omit the terms that are equal and divide the result by $z^{m+1}$ ($\neq 0$); this gives

$$a_{m+1} + a_{m+2} z + a_{m+3} z^2 + \cdots = b_{m+1} + b_{m+2} z + b_{m+3} z^2 + \cdots.$$

Similarly as before by letting $z \to 0$ we conclude from this that $a_{m+1} = b_{m+1}$. This completes the proof. ■

## Operations on Power Series

Interesting in itself, this discussion will serve as a preparation for our main goal, namely, to show that functions represented by power series are analytic.

**Termwise addition or subtraction** of two power series with radii of convergence $R_1$ and $R_2$ yields a power series with radius of convergence at least equal to the smaller of $R_1$ and $R_2$. *Proof.* Add (or subtract) the partial sums $s_n$ and $s_n^*$ term by term and use $\lim (s_n \pm s_n^*) = \lim s_n \pm \lim s_n^*$.

**Termwise multiplication** of two power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \cdots$$

and

$$g(z) = \sum_{m=0}^{\infty} b_m z^m = b_0 + b_1 z + \cdots$$

means the multiplication of each term of the first series by each term of the second series and the collection of like powers of $z$. This gives a power series, which is called the **Cauchy product** of the two series and is given by

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots$$
$$= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) z^n.$$

We mention without proof that this power series converges absolutely for each $z$ within the circle of convergence of each of the two given series and has the sum $s(z) = f(z)g(z)$. For a proof, see [D5] listed in App. 1.

**Termwise differentiation and integration** of power series is permissible, as we show next. We call **derived series** *of the power series* (1) the power series obtained from (1) by termwise differentiation, that is,

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots. \tag{3}$$

**THEOREM 3**

**Termwise Differentiation of a Power Series**

*The derived series of a power series has the same radius of convergence as the original series.*

**PROOF** This follows from (6) in Sec. 15.2 because

$$\lim_{n\to\infty} \frac{n|a_n|}{(n+1)\,|a_{n+1}|} = \lim_{n\to\infty} \frac{n}{n+1} \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

or, if the limit does not exist, from (6**) in Sec. 15.2 by noting that $\sqrt[n]{n} \to 1$ as $n \to \infty$. ■

**EXAMPLE 1** **Application of Theorem 3**

Find the radius of convergence $R$ of the following series by applying Theorem 3.

$$\sum_{n=2}^{\infty} \binom{n}{2} z^n = z^2 + 3z^3 + 6z^4 + 10z^5 + \cdots .$$

***Solution.*** Differentiate the geometric series twice term by term and multiply the result by $z^2/2$. This yields the given series. Hence $R = 1$ by Theorem 3. ■

**THEOREM 4** **Termwise Integration of Power Series**

*The power series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

*obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \cdots$ term by term has the same radius of convergence as the original series.*

The proof is similar to that of Theorem 3.

With Theorem 3 as a tool, we are now ready to establish our main result in this section.

## Power Series Represent Analytic Functions

**THEOREM 5** **Analytic Functions. Their Derivatives**

*A power series with a nonzero radius of convergence $R$ represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, by the first statement, each of them represents an analytic function.*

**PROOF** **(a)** We consider any power series (1) with positive radius of convergence $R$. Let $f(z)$ be its sum and $f_1(z)$ the sum of its derived series; thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{4}$$

We show that $f(z)$ is analytic and has the derivative $f_1(z)$ in the interior of the circle of convergence. We do this by proving that for any fixed $z$ with $|z| < R$ and $\Delta z \to 0$ the difference quotient $[f(z + \Delta z) - f(z)]/\Delta z$ approaches $f_1(z)$. By termwise addition we first have from (4)

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \left[ \frac{(z + \Delta z)^n - z^n}{\Delta z} - n z^{n-1} \right]. \tag{5}$$

Note that the summation starts with 2, since the constant term drops out in taking the difference $f(z + \Delta z) - f(z)$, and so does the linear term when we subtract $f_1(z)$ from the difference quotient.

**(b)** We claim that the series in (5) can be written

$$\sum_{n=2}^{\infty} a_n \Delta z[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n-2)z^{n-3}(z + \Delta z) + (n-1)z^{n-2}]. \tag{6}$$

The somewhat technical proof of this is given in App. 4.

**(c)** We consider (6). The brackets contain $n - 1$ terms, and the largest coefficient is $n - 1$. Since $(n-1)^2 \leqq n(n-1)$, we see that for $|z| \leqq R_0$ and $|z + \Delta z| \leqq R_0$, $R_0 < R$, the absolute value of this series (6) cannot exceed

$$|\Delta z| \sum_{n=2}^{\infty} |a_n| n(n-1) R_0^{n-2}. \tag{7}$$

This series with $a_n$ instead of $|a_n|$ is the second derived series of (2) at $z = R_0$ and converges absolutely by Theorem 3 of this section and Theorem 1 of Sec. 15.2. Hence our present series (7) converges. Let the sum of (7) (without the factor $|\Delta z|$) be $K(R_0)$. Since (6) is the right side of (5), our present result is

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) \right| \leqq |\Delta z| \, K(R_0).$$

Letting $\Delta z \to 0$ and noting that $R_0$ $(< R)$ is arbitrary, we conclude that $f(z)$ is analytic at any point interior to the circle of convergence and its derivative is represented by the derived series. From this the statements about the higher derivatives follow by induction. ■

**Summary.** The results in this section show that power series are about as nice as we could hope for: we can differentiate and integrate them term by term (Theorems 3 and 4). Theorem 5 accounts for the great importance of power series in complex analysis: the sum of such a series (with a positive radius of convergence) is an analytic function and has derivatives of all orders, which thus in turn are analytic functions. But this is only part of the story. In the next section we show that, conversely, *every* given analytic function $f(z)$ can be represented by power series, called **Taylor series** and being the complex analog of the real Taylor series of calculus.

## PROBLEM SET 15.3

### 1–10 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION

Find the radius of convergence in two ways: (a) directly by the Cauchy–Hadamard formula in Sec. 15.2, (b) from a series of simpler terms by using Theorem 3 or Theorem 4.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n} (z - 2i)^n$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n(n+1)} z^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} (z + i)^{2n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{z}{\pi}\right)^{2n+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n(n+1)}{5^n} (z - 1)^{2n}$

6. $\displaystyle\sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{z}{4}\right)^n$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^n}{n(n+1)(n+2)} z^{2n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$

**9.** $\sum_{n=0}^{\infty} \left[\binom{n+k}{k}\right]^{-1} z^{n+k}$

**10.** $\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

**11. (Addition and subtraction)** Write out the details of the proof on termwise addition and subtraction of power series.

**12. (Cauchy product)** Show that $(1 - z)^{-2} = \sum_{n=0}^{\infty} (n + 1)z^n$ (a) by using the Cauchy product, (b) by differentiating a suitable series.

**13. (Cauchy product)** Show that the Cauchy product of $\sum_{n=0}^{\infty} z^n/n!$ multiplied by itself gives $\sum_{n=0}^{\infty} (2z)^n/n!$.

**14. (On Theorem 3)** Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$ (as claimed in the proof of Theorem 3).

**15. (On Theorems 3 and 4)** Find further examples of your own.

### 16–20 APPLICATIONS OF THE IDENTITY THEOREM

State clearly and explicitly where and how you are using Theorem 2.

**16. (Bionomial coefficients)** Using $(1 + z)^p(1 + z)^q = (1 + z)^{p+q}$, obtain the basic relation

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

**17. (Odd function)** If $f(z)$ in (1) is *odd* (i.e., $f(-z) = -f(z)$), show that $a_n = 0$ for even $n$. Give examples.

**18. (Even functions)** If $f(z)$ in (1) is *even* (i.e., $f(-z) = f(z)$), show that $a_n = 0$ for odd $n$. Give examples.

**19.** Find applications of Theorem 2 in differential equations and elsewhere

**20. TEAM PROJECT. Fibonacci numbers.**[2] **(a)** The Fibonacci numbers are recursively defined by $a_0 = a_1 = 1$, $a_{n+1} = a_n + a_{n-1}$ if $n = 1, 2, \cdots$. Find the limit of the sequence $(a_{n+1}/a_n)$.

**(b) Fibonacci's rabbit problem.** Compute a list of $a_1, \cdots, a_{12}$. Show that $a_{12} = 233$ is the number of pairs of rabbits after 12 months if initially there is 1 pair and each pair generates 1 pair per month, beginning in the second month of existence (no deaths occurring).

**(c) Generating function.** Show that the *generating function* of the **Fibonacci numbers** is $f(z) = 1/(1 - z - z^2)$; that is, if a power series (1) represents this $f(z)$, its coefficients must be the Fibonacci numbers and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

# 15.4 Taylor and Maclaurin Series

The **Taylor series**[3] of a function $f(z)$, the complex analog of the real Taylor series is

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0) \tag{1}$$

or, by (1), Sec. 14.4,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^*. \tag{2}$$

In (2) we integrate counterclockwise around a simple closed path $C$ that contains $z_0$ in its interior and is such that $f(z)$ is analytic in a domain containing $C$ and every point inside $C$.

A **Maclaurin series**[3] is a Taylor series with center $z_0 = 0$.

[2] LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180–1250, Italian mathematician, credited with the first renaissance of mathematics on Christian soil.

[3] BROOK TAYLOR (1685–1731), English mathematician who introduced real Taylor series. COLIN MACLAURIN (1698–1746), Scots mathematician, professor at Edinburgh.

The **remainder** of the Taylor series (1) after the term $a_n(z - z_0)^n$ is

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \, dz^* \tag{3}$$

(proof below). Writing out the corresponding partial sum of (1), we thus have

$$\begin{aligned} f(z) = f(z_0) &+ \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots \\ &+ \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z). \end{aligned} \tag{4}$$

This is called **Taylor's formula** *with remainder.*

We see that ***Taylor series are power series.*** From the last section we know that power series represent analytic functions. And we now show that *every* analytic function can be represented by power series, namely, by Taylor series (with various centers). This makes Taylor series very important in complex analysis. Indeed, they are more fundamental in complex analysis than their real counterparts are in calculus.

**THEOREM 1**

**Taylor's Theorem**

*Let $f(z)$ be analytic in a domain $D$, and let $z = z_0$ be any point in $D$. Then there exists precisely one Taylor series (1) with center $z_0$ that represents $f(z)$. This representation is valid in the largest open disk with center $z_0$ in which $f(z)$ is analytic. The remainders $R_n(z)$ of (1) can be represented in the form (3). The coefficients satisfy the inequality*

$$|a_n| \leqq \frac{M}{r^n} \tag{5}$$

*where $M$ is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in $D$ whose interior is also in $D$.*

**PROOF** The key tool is Cauchy's integral formula in Sec. 14.3; writing $z$ and $z^*$ instead of $z_0$ and $z$ (so that $z^*$ is the variable of integration), we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} \, dz^*. \tag{6}$$

$z$ lies inside $C$, for which we take a circle of radius $r$ with center $z_0$ and interior in $D$ (Fig. 364). We develop $1/(z^* - z)$ in (6) in powers of $z - z_0$. By a ***standard algebraic manipulation*** (worth remembering!) we first have

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \dfrac{z - z_0}{z^* - z_0}\right)}. \tag{7}$$

For later use we note that since $z^*$ is on $C$ while $z$ is inside $C$, we have

$$\left|\frac{z - z_0}{z^* - z_0}\right| < 1 \qquad \text{(Fig. 364).} \tag{7*}$$

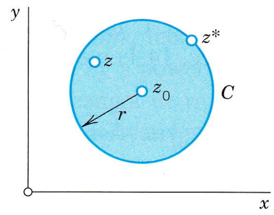



**Fig. 364.** Cauchy formula (6)

To (7) we now apply the sum formula for a finite geometric sum

$$1 + q + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q} \qquad (q \neq 1), \tag{8*}$$

which we use in the form (take the last term to the other side and interchange sides)

$$\frac{1}{1 - q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1 - q}. \tag{8}$$

Applying this with $q = (z - z_0)/(z^* - z_0)$ to the right side of (7), we get

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0}\left[1 + \frac{z - z_0}{z^* - z_0} + \left(\frac{z - z_0}{z^* - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{z^* - z_0}\right)^n\right] + \frac{1}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1}$$

We insert this into (6). Powers of $z - z_0$ do not depend on the variable of integration $z^*$, so that we may take them out from under the integral sign. This yields

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z_0}\,dz^* + \frac{z - z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^2}\,dz^* + \cdots + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\,dz^* + R_n(z)$$

with $R_n(z)$ given by (3). The integrals are those in (2) related to the derivatives, so that we have proved the Taylor formula (4).

Since analytic functions have derivatives of all orders, we can take $n$ in (4) as large as we please. If we let $n$ approach infinity, we obtain (1). Clearly, (1) will converge and represent $f(z)$ if and only if

$$\lim_{n\to\infty} R_n(z) = 0. \tag{9}$$

We prove (9) as follows. Since $z^*$ lies on $C$, whereas $z$ lies inside $C$ (Fig. 364), we have $|z^* - z| > 0$. Since $f(z)$ is analytic inside and on $C$, it is bounded, and so is the function $f(z^*)/(z^* - z)$, say,

$$\left| \frac{f(z^*)}{z^* - z} \right| \leqq \widetilde{M}$$

for all $z^*$ on $C$. Also, $C$ has the radius $r = |z^* - z_0|$ and the length $2\pi r$. Hence by the $ML$-inequality (Sec. 14.1) we obtain from (3)

$$\tag{10} \begin{aligned} |R_n| &= \frac{|z - z_0|^{n+1}}{2\pi} \left| \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \, dz^* \right| \\ &\leqq \frac{|z - z_0|^{n+1}}{2\pi} \widetilde{M} \frac{1}{r^{n+1}} 2\pi r = \widetilde{M} \left| \frac{z - z_0}{r} \right|^{n+1}. \end{aligned}$$

Now $|z - z_0| < r$ because $z$ lies *inside* $C$. Thus $|z - z_0|/r < 1$, so that the right side approaches 0 as $n \to \infty$. This proves the convergence of the Taylor series. Uniqueness follows from Theorem 2 in the last section. Finally, (5) follows from (1) and the Cauchy inequality in Sec. 14.4. This proves Taylor's theorem. ■

**Accuracy of Approximation.** We can achieve any preassinged accuracy in approximating $f(z)$ by a partial sum of (1) by choosing $n$ large enough. This is the practical aspect of formula (9).

**Singularity, Radius of Convergence.** On the circle of convergence of (1) there is at least one **singular point** of $f(z)$, that is, a point $z = c$ at which $f(z)$ is not analytic (but such that every disk with center $c$ contains points at which $f(z)$ *is* analytic). We also say that $f(z)$ **is singular** at $c$ or **has a singularity** at $c$. Hence the radius of convergence $R$ of (1) is usually equal to the distance from $z_0$ to the nearest singular point of $f(z)$.

(Sometimes $R$ can be greater than that distance: $\operatorname{Ln} z$ is singular on the negative real axis, whose distance from $z_0 = -1 + i$ is 1, but the Taylor series of $\operatorname{Ln} z$ with center $z_0 = -1 + i$ has radius of convergence $\sqrt{2}$.)

## Power Series as Taylor Series

Taylor series are power series—of course! Conversely, we have

**THEOREM 2**

**Relation to the Last Section**

*A power series with a nonzero radius of convergence is the Taylor series of its sum.*

**PROOF** Given the power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots.$$

Then $f(z_0) = a_0$. By Theorem 5 in Sec. 15.3 we obtain

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots, \qquad \text{thus} \qquad f'(z_0) = a_1$$

$$f''(z) = 2a_2 + 3 \cdot 2(z - z_0) + \cdots, \qquad \text{thus} \qquad f''(z_0) = 2!a_2$$

and in general $f^{(n)}(z_0) = n!\,a_n$. With these coefficients the given series becomes the Taylor series of $f(z)$ with center $z_0$. ■

**Comparison with Real Functions.** One surprising property of complex analytic functions is that they have derivatives of all orders, and now we have discovered the other surprising property that they can always be represented by power series of the form (1). This is not true in general for ***real functions;*** there are real functions that have derivatives of all orders but cannot be represented by a power series. (Example: $f(x) = \exp(-1/x^2)$ if $x \neq 0$ and $f(0) = 0$; this function cannot be represented by a Maclaurin series in an open disk with center 0 because all its derivatives at 0 are zero.)

## Important Special Taylor Series

These are as in calculus, with $x$ replaced by complex $z$. Can you see why? (*Answer.* The coefficient formulas are the same.)

**EXAMPLE 1 Geometric Series**

Let $f(z) = 1/(1 - z)$. Then we have $f^{(n)}(z) = n!/(1 - z)^{n+1}$, $f^{(n)}(0) = n!$. Hence the Maclaurin expansion of $1/(1 - z)$ is the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1). \tag{11}$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence. ■

**EXAMPLE 2 Exponential Function**

We know that the exponential function $e^z$ (Sec. 13.5) is analytic for all $z$, and $(e^z)' = e^z$. Hence from (1) with $z_0 = 0$ we obtain the Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots. \tag{12}$$

This series is also obtained if we replace $x$ in the familiar Maclaurin series of $e^x$ by $z$.

Furthermore, by setting $z = iy$ in (12) and separating the series into the real and imaginary parts (see Theorem 2, Sec. 15.1) we obtain

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}.$$

Since the series on the right are the familiar Maclaurin series of the real functions $\cos y$ and $\sin y$, this shows that we have rediscovered the **Euler formula**

$$e^{iy} = \cos y + i \sin y. \tag{13}$$

Indeed, one may use (12) for ***defining*** $e^z$ and derive from (12) the basic properties of $e^z$. For instance, the differentiation formula $(e^z)' = e^z$ follows readily from (12) by termwise differentiation. ■

**EXAMPLE 3 Trigonometric and Hyperbolic Functions**

By substituting (12) into (1) of Sec. 13.6 we obtain

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots \tag{14}$$
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots .$$

When $z = x$ these are the familiar Maclaurin series of the real functions $\cos x$ and $\sin x$. Similarly, by substituting (12) into (11), Sec. 13.6, we obtain

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \tag{15}$$
$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots .$$

■

**EXAMPLE 4 Logarithm**

From (1) it follows that

$$\text{Ln}\,(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots \qquad (|z| < 1). \tag{16}$$

Replacing $z$ by $-z$ and multiplying both sides by $-1$, we get

$$-\text{Ln}\,(1 - z) = \text{Ln}\,\frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (|z| < 1). \tag{17}$$

By adding both series we obtain

$$\text{Ln}\,\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1). \tag{18}$$

■

## Practical Methods

The following examples show ways of obtaining Taylor series more quickly than by the use of the coefficient formulas. Regardless of the method used, the result will be the same. This follows from the uniqueness (see Theorem 1).

**EXAMPLE 5 Substitution**

Find the Maclaurin series of $f(z) = 1/(1 + z^2)$.

***Solution.*** By substituting $-z^2$ for $z$ in (11) we obtain

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \qquad (|z| < 1). \tag{19}$$

■

**EXAMPLE 6** **Integration**

Find the Maclaurin series of $f(z) = \arctan z$.

***Solution.*** We have $f'(z) = 1/(1 + z^2)$. Integrating (19) term by term and using $f(0) = 0$ we get

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad (|z| < 1);$$

this series represents the principal value of $w = u + iv = \arctan z$ defined as that value for which $|u| < \pi/2$. ■

**EXAMPLE 7** **Development by Using the Geometric Series**

Develop $1/(c - z)$ in powers of $z - z_0$, where $c - z_0 \neq 0$.

***Solution.*** This was done in the proof of Theorem 1, where $c = z^*$. The beginning was simple algebra and then the use of (11) with $z$ replaced by $(z - z_0)/(c - z_0)$:

$$\frac{1}{c-z} = \frac{1}{c - z_0 - (z - z_0)} = \frac{1}{(c - z_0)\left(1 - \dfrac{z - z_0}{c - z_0}\right)} = \frac{1}{c - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n$$

$$= \frac{1}{c - z_0}\left(1 + \frac{z - z_0}{c - z_0} + \left(\frac{z - z_0}{c - z_0}\right)^2 + \cdots\right).$$

This series converges for

$$\left|\frac{z - z_0}{c - z_0}\right| < 1, \qquad \text{that is,} \qquad |z - z_0| < |c - z_0|.$$ ■

**EXAMPLE 8** **Binomial Series, Reduction by Partial Fractions**

Find the Taylor series of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

***Solution.*** We develop $f(z)$ in partial fractions and the first fraction in a **binomial series**

$$\textbf{(20)} \qquad \frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$
$$= 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

with $m = 2$ and the second fraction in a geometric series, and then add the two series term by term. This gives

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{[3 + (z-1)]^2} - \frac{2}{2 - (z-1)} = \frac{1}{9}\left(\frac{1}{[1 + \frac{1}{3}(z-1)]^2}\right) - \frac{1}{1 - \frac{1}{2}(z-1)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty} \binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 - \frac{275}{1944}(z-1)^3 - \cdots.$$

We see that the first series converges for $|z - 1| < 3$ and the second for $|z - 1| < 2$. This had to be expected because $1/(z + 2)^2$ is singular at $-2$ and $2/(z - 3)$ at 3, and these points have distance 3 and 2, respectively, from the center $z_0 = 1$. Hence the whole series converges for $|z - 1| < 2$. ■

## PROBLEM SET 15.4

### 1–12 TAYLOR AND MACLAURIN SERIES

Find the Taylor or Maclaurin series of the given function with the given point as center and determine the radius of convergence.

**1.** $e^{-2z}$, $0$ **2.** $1/(1 - z^3)$, $0$

**3.** $e^z$, $-2i$ **4.** $\cos^2 z$, $0$

**5.** $\sin z$, $\pi/2$ **6.** $1/z$, $1$

**7.** $1/(1 - z)$, $i$ **8.** $\operatorname{Ln}(1 - z)$, $i$

**9.** $e^{-z^2/2}$, $0$ **10.** $e^{z^2}\int_0^z e^{-t^2}\,dt$, $0$

**11.** $z^6 - z^4 + z^2 - 1$, $1$ **12.** $\sinh(z - 2i)$, $2i$

### 13–16 HIGHER TRANSCENDENTAL FUNCTIONS

Find the Maclaurin series by termwise integrating the integrand. (The integrals cannot be evaluated by the usual methods of calculus. They define the **error function** erf $z$, **sine integral** Si($z$), and **Fresnel integrals**[4] S($z$) and C($z$), which occur in statistics, heat conduction, optics, and other applications. These are special so-called higher transcendental functions.)

**13.** $\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-t^2}\,dt$ **14.** $\operatorname{Si}(z) = \displaystyle\int_0^z \frac{\sin t}{t}\,dt$

**15.** $S(z) = \displaystyle\int_0^z \sin t^2\,dt$ **16.** $C(z) = \displaystyle\int_0^z \cos t^2\,dt$

**17. CAS PROJECT. sec, tan, arcsin. (a) Euler numbers.** The Maclaurin series

$$\textbf{(21)}\quad \sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - + \cdots$$

defines the *Euler numbers* $E_{2n}$. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Write a program that computes the $E_{2n}$ from the coefficient formula in (1) or extracts them as a list from the series. (For tables see Ref. [GR1], p. 810, listed in App. 1.)

**(b) Bernoulli numbers.** The Maclaurin series

$$\textbf{(22)}\quad \frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots$$

defines the *Bernoulli numbers* $B_n$. Using undetermined coefficients, show that

$$(23)\quad \begin{aligned} &B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \\ &B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \cdots. \end{aligned}$$

Write a program for computing $B_n$.

**(c) Tangent.** Using (1), (2), Sec. 13.6, and (22), show that $\tan z$ has the following Maclaurin series and calculate from it a table of $B_0, \cdots, B_{20}$:

$$(24)\quad \begin{aligned} \tan z &= \frac{2i}{e^{2iz} - 1} - \frac{4i}{e^{4iz} - 1} - i \\ &= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^{2n}(2^{2n} - 1)}{(2n)!}B_{2n} z^{2n-1}. \end{aligned}$$

**18. (Inverse sine)** Developing $1/\sqrt{1 - z^2}$ and integrating, show that

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots \quad (|z| < 1).$$

Show that this series represents the principal value of $\arcsin z$ (defined in Team Project 30, Sec. 13.7).

**19. (Undetermined coefficients)** Using the relation $\sin z = \tan z \cos z$ and the Maclaurin series of $\sin z$ and $\cos z$, find the first four nonzero terms of the Maclaurin series of $\tan z$. (Show the details.)

**20. TEAM PROJECT. Properties from Maclaurin Series.** Clearly, from series we can compute function values. In this project we show that properties of functions can often be discovered from their Taylor or Maclaurin series. Using suitable series, prove the following.

**(a)** The formulas for the derivatives of $e^z$, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, and $\operatorname{Ln}(1 + z)$

**(b)** $\frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

**(c)** $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$

[4]AUGUSTIN FRESNEL (1788–1827), French physicist and engineer, known for his work in optics.

# 15.5 Uniform Convergence. *Optional*

We know that power series are *absolutely convergent* (Sec. 15.2, Theorem 1) and, as another basic property, we now show that they are *uniformly convergent.* Since uniform convergence is of general importance, for instance, in connection with termwise integration of series, we shall discuss it quite thoroughly.

To define uniform convergence, we consider a series whose terms are any complex functions $f_0(z), f_1(z), \cdots$:

$$\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + f_2(z) + \cdots. \tag{1}$$

(This includes power series as a special case in which $f_m(z) = a_m (z - z_0)^m$.) We assume that the series (1) converges for all $z$ in some region $G$. We call its sum $s(z)$ and its $n$th partial sum $s_n(z)$; thus

$$s_n(z) = f_0(z) + f_1(z) + \cdots + f_n(z).$$

Convergence in $G$ means the following. If we pick a $z = z_1$ in $G$, then, by the definition of convergence at $z_1$, for given $\epsilon > 0$ we can find an $N_1(\epsilon)$ such that

$$|s(z_1) - s_n(z_1)| < \epsilon \qquad \text{for all } n > N_1(\epsilon).$$

If we pick a $z_2$ in $G$, keeping $\epsilon$ as before, we can find an $N_2(\epsilon)$ such that

$$|s(z_2) - s_n(z_2)| < \epsilon \qquad \text{for all } n > N_2(\epsilon),$$

and so on. Hence, given an $\epsilon > 0$, to each $z$ in $G$ there corresponds a number $N_z(\epsilon)$. This number tells us how many terms we need (what $s_n$ we need) at a $z$ to make $|s(z) - s_n(z)|$ smaller than $\epsilon$. Thus this number $N_z(\epsilon)$ measures the speed of convergence.

Small $N_z(\epsilon)$ means rapid convergence, large $N_z(\epsilon)$ means slow convergence at the point $z$ considered. Now, if we can find an $N(\epsilon)$ larger than all these $N_z(\epsilon)$ for all $z$ in $G$, we say that the convergence of the series (1) in $G$ is *uniform.* Hence this basic concept is defined as follows.

**DEFINITION**

**Uniform Convergence**

A series (1) with sum $s(z)$ is called **uniformly convergent** in a region $G$ if for every $\epsilon > 0$ we can find an $N = N(\epsilon)$, ***not depending on z,*** such that

$$|s(z) - s_n(z)| < \epsilon \qquad \text{for all } n > N(\epsilon) \textit{ and all } z \textit{ in } G.$$

Uniformity of convergence is thus a property that always refers to an *infinite set* in the $z$-plane, that is, a set consisting of infinitely many points.

**EXAMPLE 1** **Geometric Series**

Show that the geometric series $1 + z + z^2 + \cdots$ is (a) uniformly convergent in any closed disk $|z| \leqq r < 1$, (b) not uniformly convergent in its whole disk of convergence $|z| < 1$.

***Solution.*** **(a)** For $z$ in that closed disk we have $|1 - z| \geqq 1 - r$ (sketch it). This implies that $1/|1 - z| \leqq 1/(1 - r)$. Hence (remember (8) in Sec. 15.4 with $q = z$)

$$|s(z) - s_n(z)| = \left| \sum_{m=n+1}^{\infty} z^m \right| = \left| \frac{z^{n+1}}{1 - z} \right| \leqq \frac{r^{n+1}}{1 - r} .$$

Since $r < 1$, we can make the right side as small as we want by choosing $n$ large enough, and since the right side does not depend on $z$ (in the closed disk considered), this means that the convergence is uniform.

**(b)** For given real $K$ (no matter how large) and $n$ we can always find a $z$ in the disk $|z| < 1$ such that

$$\left| \frac{z^{n+1}}{1 - z} \right| = \frac{|z|^{n+1}}{|1 - z|} > K,$$

simply by taking $z$ close enough to 1. Hence no single $N(\epsilon)$ will suffice to make $|s(z) - s_n(z)|$ smaller than a given $\epsilon > 0$ *throughout the whole disk.* By definition, this shows that the convergence of the geometric series in $|z| < 1$ is not uniform. ■

This example suggests that *for a power series, the uniformity of convergence may at most be disturbed near the circle of convergence.* This is true:

**THEOREM 1** **Uniform Convergence of Power Series**

*A power series*

$$\sum_{m=0}^{\infty} a_m (z - z_0)^m \tag{2}$$

*with a nonzero radius of convergence $R$ is uniformly convergent in every circular disk $|z - z_0| \leqq r$ of radius $r < R$.*

**PROOF** For $|z - z_0| \leqq r$ and any positive integers $n$ and $p$ we have

$$|a_{n+1}(z - z_0)^{n+1} + \cdots + a_{n+p}(z - z_0)^{n+p}| \leqq |a_{n+1}|r^{n+1} + \cdots + |a_{n+p}|r^{n+p}. \tag{3}$$

Now (2) converges absolutely if $|z - z_0| = r < R$ (by Theorem 1 in Sec. 15.2). Hence it follows from the Cauchy convergence principle (Sec. 15.1) that, an $\epsilon > 0$ being given, we can find an $N(\epsilon)$ such that

$$|a_{n+1}|r^{n+1} + \cdots + |a_{n+p}|r^{n+p} < \epsilon \qquad \text{for } n > N(\epsilon) \quad \text{and} \quad p = 1, 2, \cdots .$$

From this and (3) we obtain

$$|a_{n+1}(z - z_0)^{n+1} + \cdots + a_{n+p}(z - z_0)^{n+p}| < \epsilon$$

for all $z$ in the disk $|z - z_0| \leqq r$, every $n > N(\epsilon)$, and every $p = 1, 2, \cdots$. Since $N(\epsilon)$ is independent of $z$, this shows uniform convergence, and the theorem is proved. ■

Theorem 1 meets with our immediate need and concern, which is power series. The remainder of this section should provide a deeper understanding of the concept of uniform convergence in connection with arbitrary series of variable terms.

## Properties of Uniformly Convergent Series

Uniform convergence derives its main importance from two facts:

**1.** If a series of *continuous* terms is uniformly convergent, its sum is also continuous (Theorem 2, below).

**2.** Under the same assumptions, termwise integration is permissible (Theorem 3).

This raises two questions:

**1.** How can a converging series of continuous terms manage to have a discontinuous sum? (Example 2)

**2.** How can something go wrong in termwise integration? (Example 3)

Another natural question is:

**3.** What is the relation between absolute convergence and uniform convergence? The surprising answer: none. (Example 5)

These are the ideas we shall discuss.

If we add *finitely many* continuous functions, we get a continuous function as their sum. Example 2 will show that this is no longer true for an infinite series, even if it converges absolutely. However, if it converges *uniformly,* this cannot happen, as follows.

**THEOREM 2**

**Continuity of the Sum**

*Let the series*

$$\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$$

*be uniformly convergent in a region G. Let F(z) be its sum. Then if each term $f_m(z)$ is continuous at a point $z_1$ in G, the function F(z) is continuous at $z_1$.*

**PROOF** Let $s_n(z)$ be the $n$th partial sum of the series and $R_n(z)$ the corresponding remainder:

$$s_n = f_0 + f_1 + \cdots + f_n, \qquad R_n = f_{n+1} + f_{n+2} + \cdots .$$

Since the series converges uniformly, for a given $\epsilon > 0$ we can find an $N = N(\epsilon)$ such that

$$|R_N(z)| < \frac{\epsilon}{3} \qquad \text{for all } z \text{ in } G.$$

Since $s_N(z)$ is a sum of finitely many functions that are continuous at $z_1$, this sum is continuous at $z_1$. Therefore, we can find a $\delta > 0$ such that

$$|s_N(z) - s_N(z_1)| < \frac{\epsilon}{3} \quad \text{for all } z \text{ in } G \text{ for which } |z - z_1| < \delta.$$

Using $F = s_N + R_N$ and the triangle inequality (Sec. 13.2), for these $z$ we thus obtain

$$\begin{aligned} |F(z) - F(z_1)| &= |s_N(z) + R_N(z) - [s_N(z_1) + R_N(z_1)]| \\ &\leqq |s_N(z) - s_N(z_1)| + |R_N(z)| + |R_N(z_1)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This implies that $F(z)$ is continuous at $z_1$, and the theorem is proved. ■

**EXAMPLE 2** **Series of Continuous Terms with a Discontinuous Sum**

Consider the series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \cdots \qquad (x \text{ real}).$$

This is a geometric series with $q = 1/(1 + x^2)$ times a factor $x^2$. Its $n$th partial sum is

$$s_n(x) = x^2 \left[ 1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^n} \right].$$

We now use the trick by which one finds the sum of a geometric series, namely, we multiply $s_n(x)$ by $-q = -1/(1 + x^2)$,

$$-\frac{1}{1+x^2}\, s_n(x) = -x^2 \left[ \frac{1}{1+x^2} + \cdots + \frac{1}{(1+x^2)^n} + \frac{1}{(1+x^2)^{n+1}} \right].$$

Adding this to the previous formula, simplifying on the left, and canceling most terms on the right, we obtain

$$\frac{x^2}{1+x^2}\, s_n(x) = x^2 \left[ 1 - \frac{1}{(1+x^2)^{n+1}} \right],$$

thus

$$s_n(x) = 1 + x^2 - \frac{1}{(1+x^2)^n}.$$

The exciting Fig. 365 "explains" what is going on. We see that if $x \neq 0$, the sum is

$$s(x) = \lim_{n\to\infty} s_n(x) = 1 + x^2,$$

but for $x = 0$ we have $s_n(0) = 1 - 1 = 0$ for all $n$, hence $s(0) = 0$. So we have the surprising fact that the sum is discontinuous (at $x = 0$), although all the terms are continuous and the series converges even absolutely (its terms are nonnegative, thus equal to their absolute value!).

Theorem 2 now tells us that the convergence cannot be uniform in an interval containing $x = 0$. We can also verify this directly. Indeed, for $x \neq 0$ the remainder has the absolute value

$$|R_n(x)| = |s(x) - s_n(x)| = \frac{1}{(1+x^2)^n}$$

and we see that for a given $\epsilon$ ($< 1$) we cannot find an $N$ depending only on $\epsilon$ such that $|R_n| < \epsilon$ for all $n > N(\epsilon)$ and all $x$, say, in the interval $0 \leqq x \leqq 1$. ■

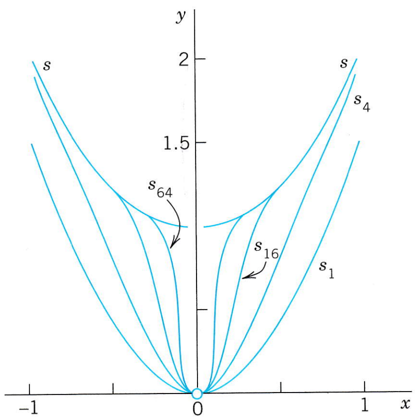



**Fig. 365.** Partial sums in Example 2

## Termwise Integration

This is our second topic in connection with uniform convergence, and we begin with an example to become aware of the danger of just blindly integrating term-by-term.

**EXAMPLE 3 Series for which Termwise Integration is Not Permissible**

Let $u_m(x) = mxe^{-mx^2}$ and consider the series

$$\sum_{m=0}^{\infty} f_m(x) \qquad \text{where} \qquad f_m(x) = u_m(x) - u_{m-1}(x)$$

in the interval $0 \leqq x \leqq 1$. The $n$th partial sum is

$$s_n = u_1 - u_0 + u_2 - u_1 + \cdots + u_n - u_{n-1} = u_n - u_0 = u_n.$$

Hence the series has the sum $F(x) = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} u_n(x) = 0 \quad (0 \leqq x \leqq 1)$. From this we obtain

$$\int_0^1 F(x)\,dx = 0.$$

On the other hand, by integrating term by term and using $f_1 + f_2 + \cdots + f_n = s_n$, we have

$$\sum_{m=1}^{\infty} \int_0^1 f_m(x)\,dx = \lim_{n\to\infty} \sum_{m=1}^{n} \int_0^1 f_m(x)\,dx = \lim_{n\to\infty} \int_0^1 s_n(x)\,dx.$$

Now $s_n = u_n$ and the expression on the right becomes

$$\lim_{n\to\infty} \int_0^1 u_n(x)\,dx = \lim_{n\to\infty} \int_0^1 nxe^{-nx^2}\,dx = \lim_{n\to\infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2},$$

but not 0. This shows that the series under consideration cannot be integrated term by term from $x = 0$ to $x = 1$. ■

The series in Example 3 is not uniformly convergent in the interval of integration, and we shall now prove that in the case of a uniformly convergent series of continuous functions we may integrate term by term.

**THEOREM 3 Termwise Integration**

*Let*

$$F(z) = \sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$$

*be a uniformly convergent series of continuous functions in a region G. Let C be any path in G. Then the series*

$$\sum_{m=0}^{\infty} \int_C f_m(z)\,dz = \int_C f_0(z)\,dz + \int_C f_1(z)\,dz + \cdots \tag{4}$$

*is convergent and has the sum* $\int_C F(z)\,dz$.

**PROOF** From Theorem 2 it follows that $F(z)$ is continuous. Let $s_n(z)$ be the $n$th partial sum of the given series and $R_n(z)$ the corresponding remainder. Then $F = s_n + R_n$ and by integration,

$$\int_C F(z)\, dz = \int_C s_n(z)\, dz + \int_C R_n(z)\, dz.$$

Let $L$ be the length of $C$. Since the given series converges uniformly, for every given $\epsilon > 0$ we can find a number $N$ such that $|R_n(z)| < \epsilon/L$ for all $n > N$ and all $z$ in $G$. By applying the $ML$-inequality (Sec. 14.1) we thus obtain

$$\left| \int_C R_n(z)\, dz \right| < \frac{\epsilon}{L} L = \epsilon \qquad \text{for all } n > N.$$

Since $R_n = F - s_n$, this means that

$$\left| \int_C F(z)\, dz - \int_C s_n(z)\, dz \right| < \epsilon \qquad \text{for all } n > N.$$

Hence, the series (4) converges and has the sum indicated in the theorem. ■

Theorems 2 and 3 characterize the two most important properties of uniformly convergent series. Also, since differentiation and integration are inverse processes, Theorem 3 implies

**THEOREM 4**

**Termwise Differentiation**

*Let the series $f_0(z) + f_1(z) + f_2(z) + \cdots$ be convergent in a region $G$ and let $F(z)$ be its sum. Suppose that the series $f_0'(z) + f_1'(z) + f_2'(z) + \cdots$ converges uniformly in $G$ and its terms are continuous in $G$. Then*

$$F'(z) = f_0'(z) + f_1'(z) + f_2'(z) + \cdots \qquad \textit{for all } z \textit{ in } G.$$

## Test for Uniform Convergence

Uniform convergence is usually proved by the following comparison test.

**THEOREM 5**

**Weierstrass[5] M-Test for Uniform Convergence**

*Consider a series of the form* (1) *in a region $G$ of the $z$-plane. Suppose that one can find a convergent series of constant terms,*

$$M_0 + M_1 + M_2 + \cdots, \tag{5}$$

*such that $|f_m(z)| \leqq M_m$ for all $z$ in $G$ and every $m = 0, 1, \cdots$. Then* (1) *is uniformly convergent in $G$.*

[5]KARL WEIERSTRASS (1815–1897), great German mathematician, whose lifework was the development of complex analysis based on the concept of power series (see the footnote in Sec. 13.4). He also made basic contributions to the calculus, the calculus of variations, approximation theory, and differential geometry. He obtained the concept of uniform convergence in 1841 (published 1894, *sic!*); the first publication on the concept was by G. G. STOKES (see Sec 10.9) in 1847.

The simple proof is left to the student (Team Project 18).

**EXAMPLE 4 Weierstrass *M*-Test**

Does the following series converge uniformly in the disk $|z| \leqq 1$?

$$\sum_{m=1}^{\infty} \frac{z^m + 1}{m^2 + \cosh m|z|}$$

***Solution.*** Uniform convergence follows by the Weierstrass $M$-test and the convergence of $\Sigma 1/m^2$ (see Sec. 15.1, in the proof of Theorem 8) because

$$\left| \frac{z^m + 1}{m^2 + \cosh m|z|} \right| \leqq \frac{|z|^m + 1}{m^2}$$
$$\leqq \frac{2}{m^2}.$$

## No Relation Between Absolute and Uniform Convergence

We finally show the surprising fact that there are series that converge absolutely but not uniformly, and others that converge uniformly but not absolutely, so that there is no relation between the two concepts.

**EXAMPLE 5 No Relation Between Absolute and Uniform Convergence**

The series in Example 2 converges absolutely but not uniformly, as we have shown. On the other hand, the series

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{x^2 + m} = \frac{1}{x^2 + 1} - \frac{1}{x^2 + 2} + \frac{1}{x^2 + 3} - + \cdots \qquad (x \text{ real})$$

converges uniformly on the whole real line but not absolutely.

*Proof.* By the familiar Leibniz test of calculus (see App. A3.3) the remainder $R_n$ does not exceed its first term in absolute value, since we have a series of alternating terms whose absolute values form a monotone decreasing sequence with limit zero. Hence given $\epsilon > 0$, for all $x$ we have

$$|R_n(x)| \leqq \frac{1}{x^2 + n + 1} < \frac{1}{n} < \epsilon \qquad \text{if } n > N(\epsilon) \geqq \frac{1}{\epsilon}.$$

This proves uniform convergence, since $N(\epsilon)$ does not depend on $x$.

The convergence is not absolute because for any *fixed* $x$ we have

$$\left| \frac{(-1)^{m-1}}{x^2 + m} \right| = \frac{1}{x^2 + m}$$
$$> \frac{k}{m}$$

where $k$ is a suitable constant, and $k\Sigma 1/m$ diverges.

## PROBLEM SET 15.5

### 1–8 UNIFORM CONVERGENCE

Prove that the given series converges uniformly in the indicated region.

1. $\sum_{n=0}^{\infty} (z - 2i)^{2n}, \quad |z - 2i| \leqq 0.999$

2. $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!}, \quad |z| \leqq 10^{10}$

3. $\sum_{n=0}^{\infty} \frac{\pi^n}{n^4} z^{2n}, \quad |z| \leqq 0.56$

**4.** $\sum_{n=1}^{\infty} \frac{\sin^n |\pi z|}{n(n+1)}, \quad |z| \leqq 10^5$

**5.** $\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| \leqq 1$

**6.** $\sum_{n=1}^{\infty} \frac{z^n}{n^2 \cosh n|z|}, \quad |z| \leqq 1$

**7.** $\sum_{n=0}^{\infty} \frac{\tanh^n |z|}{n^2+1} \quad |z| \leqq 10^{10}$

**8.** $\sum_{n=1}^{\infty} \frac{\cos n|z|}{n^2}, \quad |z| \leqq 10^{20}$

## 9–16 POWER SERIES

Find the region of uniform convergence. (Give reason.)

**9.** $\sum_{n=0}^{\infty} \frac{(z+1-2i)^n}{4^n}$

**10.** $\sum_{n=0}^{\infty} \frac{(z-i)^{2n}}{(2n)!}$

**11.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n} z^n$

**12.** $\sum_{n=2}^{\infty} \binom{n}{2} (2z-i)^n$

**13.** $\sum_{n=1}^{\infty} \frac{n!}{n^2} z^n$

**14.** $\sum_{n=1}^{\infty} (3^n \tanh n) z^{2n}$

**15.** $\sum_{n=1}^{\infty} \frac{z^{2n}}{5^n n^2}$

**16.** $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

**17. CAS PROJECT. Graphs of Partial Sums. (a) Figure 365.** Produce this exciting figure using your software and adding further curves, say, those of $s_{256}$, $s_{1024}$, etc.

**(b) Power series.** Study the nonuniformity of convergence experimentally by plotting partial sums near the endpoints of the convergence interval for real $z = x$.

**18. TEAM PROJECT. Uniform Convergence. (a) Weierstrass *M*-test.** Give a proof.

**(b) Termwise differentiation.** Derive Theorem 4 from Theorem 3.

**(c) Subregions.** Prove that uniform convergence of a series in a region $G$ implies uniform convergence in any portion of $G$. Is the converse true?

**(d) Example 2.** Find the precise region of convergence of the series in Example 2 with $x$ replaced by a complex variable $z$.

**(e) Figure 366.** Show that $x^2 \sum_{m=1}^{\infty} (1+x^2)^{-m} = 1$ if $x \neq 0$ and 0 if $x = 0$. Verify by computation that the partial sums $s_1, s_2, s_3$, look as shown in Fig. 366.

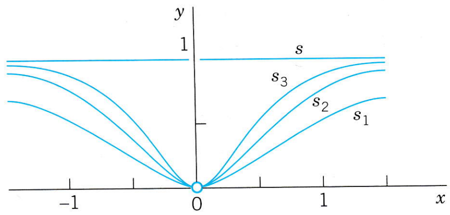



Fig. 366. Sum $s$ and partial sums in Team Project 18(e)

## 19–20 HEAT EQUATION

Show that (9) in Sec. 12.5 with coefficients (10) is a solution of the heat equation for $t > 0$, assuming that $f(x)$ is continuous on the interval $0 \leqq x \leqq L$ and has one-sided derivatives at all interior points of that interval. Proceed as follows.

**19.** Show that $|B_n|$ is bounded, say $|B_n| < K$ for all $n$. Conclude that

$$|u_n| < Ke^{-\lambda_n^2 t_0} \quad \text{if} \quad t \geqq t_0 > 0$$

and, by the Weierstrass test, the series (9) converges uniformly with respect to $x$ and $t$ for $t \geqq t_0$, $0 \leqq x \leqq L$. Using Theorem 2, show that $u(x, t)$ is continuous for $t \geqq t_0$ and thus satisfies the boundary conditions (2) for $t \geqq t_0$.

**20.** Show that $|\partial u_n / \partial t| < \lambda_n^2 K e^{-\lambda_n^2 t_0}$ if $t \geqq t_0$ and the series of the expressions on the right converges, by the ratio test. Conclude from this, the Weierstrass test, and Theorem 4 that the series (9) can be differentiated term by term with respect to $t$ and the resulting series has the sum $\partial u / \partial t$. Show that (9) can be differentiated twice with respect to $x$ and the resulting series has the sum $\partial^2 u / \partial x^2$. Conclude from this and the result to Prob. 19 that (9) is a solution of the heat equation for all $t \geqq t_0$. (The proof that (9) satisfies the given initial condition can be found in Ref. [C10] listed in App. 1.)

# CHAPTER 15 REVIEW QUESTIONS AND PROBLEMS

1. What are power series? Why are these series very important in complex analysis?
2. State from memory the ratio test, the root test, and the Cauchy–Hadamard formula for the radius of convergence.
3. What is absolute convergence? Conditional convergence? Uniform convergence?
4. What do you know about the convergence of power series?
5. What is a Taylor series? What was the idea of obtaining it from Cauchy's integral formula?
6. Give examples of practical methods for obtaining Taylor series.
7. What have power series to do with analytic functions?

8. Can properties of functions be discovered from their Maclaurin series? If so, give examples.
9. Make a list of Maclaurin series of $e^z$, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, $\text{Ln}(1-z)$ from memory.
10. What do you know about adding and multiplying power series?

### 11–20 RADIUS OF CONVERGENCE

Find the radius of convergence. Can you identify the sum as a familiar function in some of the problems? (Show the details of your work.)

11. $\sum_{n=0}^{\infty} \frac{(3z)^n}{n!}$
12. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{2n} z^n$
13. $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$
14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^n$
15. $\sum_{n=1}^{\infty} \frac{n^5}{n!} (z-3i)^{2n}$
16. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z-2)^{2n+1}$
17. $\sum_{n=0}^{\infty} \pi^n (z-2i)^{2n}$
18. $\sum_{n=0}^{\infty} \frac{(2z)^{2n}}{(2n)!}$
19. $\sum_{n=1}^{\infty} \frac{3^n}{n^{10}} z^n$
20. $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(3+4i)^n}$

### 21–30 TAYLOR AND MACLAURIN SERIES

Find the Taylor or Maclaurin series with the given point as center and determine the radius of convergence. (Show details.)

21. $e^z$, $\pi i$
22. $\text{Ln}\, z$, $2$
23. $1/(1-z)$, $-1$
24. $1/(4-3z)$, $1+i$
25. $1/(1-z)^3$, $0$
26. $1/z^2$, $i$
27. $1/z$, $-i$
28. $\int_0^z t^{-1}(e^t - 1)\, dt$, $0$
29. $\cos z$, $\frac{1}{2}\pi$
30. $\sin^2 z$, $0$

31. Does every function $f(z)$ have a Taylor series?
32. Does there exist a Taylor series in powers of $z - 1 - i$ that diverges at $5 + 5i$ but converges at $4 + 6i$?
33. Do we obtain an analytic function if we replace $x$ by $z$ in the Maclaurin series of a real function $f(x)$?
34. Using Maclaurin series, show that if $f(z)$ is even, its integral (with a suitable constant of integration) is odd.
35. Obtain the first few terms of the Maclaurin series of $\tan z$ by using the Cauchy product and

$$\sin z = \cos z \tan z.$$

## SUMMARY OF CHAPTER 15
## Power Series, Taylor Series

Sequences, series, and convergence tests are discussed in Sec. 15.1. A **power series** is of the form (Sec. 15.2)

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots; \tag{1}$$

$z_0$ is its *center*. The series (1) converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$, where $R$ is the **radius of convergence.** Some power series converge for all $z$ (then we write $R = \infty$). In exceptional cases a power series may converge only at the center; such a series is practically useless. Also, $R = \lim |a_n/a_{n+1}|$ if this limit exists. The series (1) converges absolutely (Sec. 15.2) and **uniformly** (Sec. 15.5) in every closed disk $|z - z_0| \leqq r < R$ $(R > 0)$. It represents an analytic function $f(z)$ for $|z - z_0| < R$. The derivatives $f'(z)$, $f''(z)$, $\cdots$ are obtained by termwise differentiation of (1), and these series have the same radius of convergence $R$ as (1). See Sec. 15.3.

Conversely, *every* analytic function $f(z)$ can be represented by power series. These **Taylor series** of $f(z)$ are of the form (Sec. 15.4)

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \qquad (|z - z_0| < R), \tag{2}$$

as in calculus. They converge for all $z$ in the open disk with center $z_0$ and radius generally equal to the distance from $z_0$ to the nearest **singularity** of $f(z)$ (point at which $f(z)$ ceases to be analytic as defined in Sec. 15.4). If $f(z)$ is **entire** (analytic for all $z$; see Sec. 13.5), then (2) converges for all $z$. The functions $e^z$, $\cos z$, $\sin z$, etc. have Maclaurin series, that is, Taylor series with center 0, similar to those in calculus (Sec. 15.4).

# CHAPTER 16

# Laurent Series. Residue Integration

Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disk, a **Laurent series** (Sec. 16.1) is a series of positive ***and negative*** integer powers of $z - z_0$ and converges in an annulus (a circular ring) with center $z_0$. Hence by a Laurent series we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.

We know that for a given function the Taylor series with a given center $z_0$ is unique. We shall see that, in contrast, a function $f(z)$ can have several Laurent series with the same center $z_0$ and valid in several concentric annuli. The most important of these series is that which converges for $0 < |z - z_0| < R$, that is, everywhere near the center $z_0$ except at $z_0$ itself, where $z_0$ is a singular point of $f(z)$. The series (or finite sum) of the negative powers of *this* Laurent series is called the **principal part** of the singularity of $f(z)$ at $z_0$, and is used to classify this singularity (Sec. 16.2). The coefficient of the power $1/(z - z_0)$ of *this* series is called the **residue** of $f(z)$ at $z_0$. Residues are used in an elegant and powerful integration method, called **residue integration,** for complex contour integrals (Sec. 16.3) as well as for certain complicated real integrals (Sec. 16.4).

*Prerequisite:* Chaps. 13, 14, Sec. 15.2.
*Sections that may be omitted in a shorter course:* 16.2, 16.4.
*References and Answers to Problems:* App. 1. Part D, App. 2.

## 16.1 Laurent Series

Laurent series generalize Taylor series. If in an application we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at $z_0$ (as defined in Sec. 15.4), we cannot use a Taylor series. Instead we may use a new kind of series, called **Laurent series,**[1] consisting of positive integer powers of $z - z_0$ (and a constant) as well as ***negative integer powers*** of $z - z_0$; this is the new feature.

Laurent series are also used for classifying singularities (Sec. 16.2) and in a powerful integration method ("residue integration", Sec. 16.3).

A Laurent series of $f(z)$ converges in an annulus (in the "hole" of which $f(z)$ may have singularities), as follows.

[1]PIERRE ALPHONSE LAURENT (1813–1854), French military engineer and mathematician, published the theorem in 1843.

**THEOREM 1**

**Laurent's Theorem**

*Let $f(z)$ be analytic in a domain containing two concentric circles $C_1$ and $C_2$ with center $z_0$ and the annulus between them* (blue in Fig. 367). *Then $f(z)$ can be represented by the Laurent series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$\textbf{(1)} \qquad = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$\cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

*consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals*

$$\textbf{(2)} \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*)\, dz^*,$$

*taken counterclockwise around any simple closed path $C$ that lies in the annulus and encircles the inner circle,* as in Fig. 367. [The variable of integration is denoted by $z^*$ since $z$ is used in (1).]

*This series converges and represents $f(z)$ in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle $C_1$ and decreasing $C_2$ until each of the two circles reaches a point where $f(z)$ is singular.*

*In the important special case that $z_0$ is the only singular point of $f(z)$ inside $C_2$, this circle can be shrunk to the point $z_0$, giving convergence in a disk except at the center. In this case the series (or finite sum) of the negative powers of* (1) *is called the* **principal part** *of the singularity of $f(z)$ at $z_0$.*

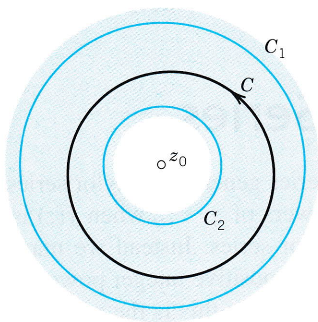



Fig. 367. Laurent's theorem

**COMMENT.** Obviously, instead of (1), (2) we may write (denoting $b_n$ by $a_{-n}$)

$$\textbf{(1')} \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

where all the coefficients are now given by a single integral formula, namely,

$$\text{(2')} \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^* \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

**PROOF** We prove Laurent's theorem. **(a)** ***The nonnegative powers*** are those of a Taylor series. To see this, we use Cauchy's integral formula (3) in Sec. 14.3 with $z^*$ (instead of $z$) as the variable of integration and $z$ instead of $z_0$. Let $g(z)$ and $h(z)$ denote the functions represented by the two terms in (3), Sec. 14.3. Then

$$\text{(3)} \qquad f(z) = g(z) + h(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z} \, dz^* - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z^*)}{z^* - z} \, dz^*.$$

Here $z$ is any point in the given annulus and we integrate counterclockwise over both $C_1$ and $C_2$, so that the minus sign appears since in (3) of Sec. 14.3 the integration over $C_2$ is taken clockwise. We transform each of these two integrals as in Sec. 15.4. The first integral is precisely as in Sec. 15.4. Hence we get precisely the same result, namely, the Taylor series of $g(z)$,

$$\text{(4)} \qquad g(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z} \, dz^* = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with coefficients [see (2), Sec. 15.4, counterclockwise integration]

$$\text{(5)} \qquad a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^*.$$

Here we can replace $C_1$ by $C$ (see Fig. 367), by the principle of deformation of path, since $z_0$, the point where the integrand in (5) is not analytic, is not a point of the annulus. This proves the formula for the $a_n$ in (2).

**(b)** ***The negative powers*** in (1) and the formula for $b_n$ in (2) are obtained if we consider $h(z)$ (the second integral times $-1/(2\pi i)$ in (3). Since $z$ lies in the annulus, it lies in the exterior of the path $C_2$. Hence the situation differs from that for the first integral. The essential point is that instead of [see (7*) in Sec. 15.4]

$$\text{(6)} \quad \text{(a)} \quad \left| \frac{z - z_0}{z^* - z_0} \right| < 1 \qquad \text{we now have} \qquad \text{(b)} \quad \left| \frac{z^* - z_0}{z - z_0} \right| < 1.$$

Consequently, we must develop the expression $1/(z^* - z)$ in the integrand of the second integral in (3) in powers of $(z^* - z_0)/(z - z_0)$ (instead of the reciprocal of this) to get a *convergent* series. We find

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \dfrac{z^* - z_0}{z - z_0}\right)}.$$

Compare this for a moment with (7) in Sec. 15.4, to really understand the difference. Then go on and apply formula (8), Sec. 15.4, for a finite geometric sum, obtaining

$$\frac{1}{z^*-z} = -\frac{1}{z-z_0}\left\{1+\frac{z^*-z_0}{z-z_0}+\left(\frac{z^*-z_0}{z-z_0}\right)^2+\cdots+\left(\frac{z^*-z_0}{z-z_0}\right)^n\right\}$$
$$-\frac{1}{z-z^*}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1}.$$

Multiplication by $-f(z^*)/2\pi i$ and integration over $C_2$ on both sides now yield

$$\begin{aligned} h(z) &= -\frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^*-z}\,dz^* \\ &= \frac{1}{2\pi i}\left\{\frac{1}{z-z_0}\oint_{C_2} f(z^*)\,dz^* + \frac{1}{(z-z_0)^2}\oint_{C_2}(z^*-z_0)f(z^*)\,dz^* + \cdots\right. \\ &\qquad + \frac{1}{(z-z_0)^n}\oint_{C_2}(z^*-z_0)^{n-1}f(z^*)\,dz^* \\ &\qquad\qquad \left. + \frac{1}{(z-z_0)^{n+1}}\oint_{C_2}(z^*-z_0)^n f(z^*)\,dz^*\right\} + R_n^*(z) \end{aligned}$$

with the last term on the right given by

$$R_n^*(z) = \frac{1}{2\pi i(z-z_0)^{n+1}}\oint_{C_2}\frac{(z^*-z_0)^{n+1}}{z-z^*}f(z^*)\,dz^*. \tag{7}$$

As before, we can integrate over $C$ instead of $C_2$ in the integrals on the right. We see that on the right, the power $1/(z-z_0)^n$ is multiplied by $b_n$ as given in (2). This establishes Laurent's theorem, provided

$$\lim_{n\to\infty} R_n^*(z) = 0. \tag{8}$$

**(c)** ***Convergence proof of*** **(8).** Very often (1) will have only finitely many negative powers. Then there is nothing to be proved. Otherwise, we begin by noting that $f(z^*)/(z-z^*)$ in (7) is bounded in absolute value, say,

$$\left|\frac{f(z^*)}{z-z^*}\right| < \widetilde{M} \qquad \text{for all } z^* \text{ on } C_2$$

because $f(z^*)$ is analytic in the annulus and on $C_2$, and $z^*$ lies on $C_2$ and $z$ outside, so that $z - z^* \neq 0$. From this and the *ML*-inequality (Sec. 14.1) applied to (7) we get the inequality ($L$ = length of $C_2$, $|z^*-z_0|$ = radius of $C_2$ = *const*)

$$|R_n^*(z)| \leqq \frac{1}{2\pi|z-z_0|^{n+1}}|z^*-z_0|^{n+1}\widetilde{M}L = \frac{\widetilde{M}L}{2\pi}\left|\frac{z^*-z_0}{z-z_0}\right|^{n+1}.$$

From (6b) we see that the expression on the right approaches zero as $n$ approaches infinity. This proves (8). The representation (1) with coefficients (2) is now established in the given annulus.

**(d)** ***Convergence of*** **(1)** ***in the enlarged annulus.*** The first series in (1) is a Taylor series [representing $g(z)$]; hence it converges in the disk $D$ with center $z_0$ whose radius equals the distance of the singularity (or singularities) closest to $z_0$. Also, $g(z)$ must be singular at all points outside $C_1$ where $f(z)$ is singular.

The second series in (1), representing $h(z)$, is a power series in $Z = 1/(z - z_0)$. Let the given annulus be $r_2 < |z - z_0| < r_1$, where $r_1$ and $r_2$ are the radii of $C_1$ and $C_2$, respectively (Fig. 367). This corresponds to $1/r_2 > |Z| > 1/r_1$. Hence this power series in $Z$ must converge at least in the disk $|Z| < 1/r_2$. This corresponds to the exterior $|z - z_0| > r_2$ of $C_2$, so that $h(z)$ is analytic for all $z$ outside $C_2$. Also, $h(z)$ must be singular inside $C_2$ where $f(z)$ is singular, and the series of the negative powers of (1) converges for all $z$ in the exterior $E$ of the circle with center $z_0$ and radius equal to the maximum distance from $z_0$ to the singularities of $f(z)$ inside $C_2$. The domain common to $D$ and $E$ is the enlarged open annulus characterized near the end of Laurent's theorem, whose proof is now complete. ■

**Uniqueness.** *The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique* (see Team Project 24). *However, $f(z)$ may have different Laurent series in two annuli with the same center;* see the examples below. The uniqueness is essential. As for a Taylor series, to obtain the coefficients of Laurent series, we do not generally use the integral formulas (2); instead, we use various other methods, some of which we shall illustrate in our examples. If a Laurent series has been found by any such process, the uniqueness guarantees that it must be ***the*** Laurent series of the given function in the given annulus.

**EXAMPLE 1 Use of Maclaurin Series**

Find the Laurent series of $z^{-5} \sin z$ with center 0.

***Solution.*** By (14), Sec. 15.4, we obtain

$$z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040} z^2 + - \cdots \qquad (|z| > 0).$$

Here the "annulus" of convergence is the whole complex plane without the origin and the principal part of the series at 0 is $z^{-4} - \frac{1}{6}z^{-2}$. ■

**EXAMPLE 2 Substitution**

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

***Solution.*** From (12) in Sec. 15.4 with $z$ replaced by $1/z$ we obtain a Laurent series whose principal part is an infinite series,

$$z^2 e^{1/z} = z^2 \left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots \qquad (|z| > 0). \ ■$$

**EXAMPLE 3 Development of $1/(1 - z)$**

Develop $1/(1 - z)$ (a) in nonnegative powers of $z$, (b) in negative powers of $z$.

***Solution.***

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1).$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1). \ ■$$

**EXAMPLE 4** **Laurent Expansions in Different Concentric Annuli**

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

***Solution.*** Multiplying by $1/z^3$, we get from Example 3

$$\text{(I)} \qquad \frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1),$$

$$\text{(II)} \qquad \frac{1}{z^3 - z^4} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \qquad (|z| > 1).$$

**EXAMPLE 5** **Use of Partial Fractions**

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z + 3}{z^2 - 3z + 2}$ with center 0.

***Solution.*** In terms of partial fractions,

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}.$$

(a) and (b) in Example 3 take care of the first fraction. For the second fraction,

$$\text{(c)} \qquad -\frac{1}{z-2} = \frac{1}{2\left(1 - \frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

$$\text{(d)} \qquad -\frac{1}{z-2} = -\frac{1}{z\left(1 - \frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$

(I) From (a) and (c), valid for $|z| < 1$ (see Fig. 368),

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4} z + \frac{9}{8} z^2 + \cdots.$$

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots.$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots.$$

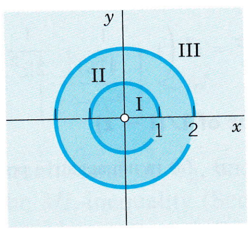



Fig. 368. Regions of convergence in Example 5

If $f(z)$ in Laurent's theorem is analytic inside $C_2$, the coefficients $b_n$ in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

## PROBLEM SET 16.1

### 1–6 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the given function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. (Show the details of your work.)

**1.** $\dfrac{1}{z^4 - z^5}$ **2.** $z \cos \dfrac{1}{z}$

**3.** $\dfrac{e^{-z}}{z^3}$ **4.** $\dfrac{\cosh 2z}{z^2}$

**5.** $z^{-3}e^{1/z^2}$ **6.** $\dfrac{e^z}{z^2 - z^3}$

### 7–14 LAURENT SERIES NEAR A SINGULARITY AT $z_0$

Expand the given function in a Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. (Show details.)

**7.** $\dfrac{e^z}{z - 1}, \quad z_0 = 1$ **8.** $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}, \quad z_0 = \frac{1}{4}\pi$

**9.** $\dfrac{1}{z^2 + 1}, \quad z_0 = i$ **10.** $\dfrac{\cos z}{(z - \pi)^4}, \quad z_0 = \pi$

**11.** $\dfrac{1}{(z + i)^2 - (z + i)}, \quad z_0 = -i$

**12.** $\dfrac{z^3}{(z + i)^2}, \quad z_0 = -i$ **13.** $\dfrac{z^2 - 4}{z - 1}, \quad z_0 = 1$

**14.** $z^2 \sinh \dfrac{1}{z}, \quad z_0 = 0$

### 15–23 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center $z = z_0$ and determine the precise regions of convergence.

**15.** $\dfrac{1}{1 - z^3}, \quad z_0 = 0$ **16.** $\dfrac{1}{1 - z^2}, \quad z_0 = 1$

**17.** $\dfrac{z^2}{1 - z^4}, \quad z_0 = 0$ **18.** $\dfrac{1}{z}, \quad z_0 = 1$

**19.** $\dfrac{z^3 - 2iz^2}{(z - i)^2}, \quad z_0 = i$ **20.** $\dfrac{\sinh z}{(z - 1)^4}, \quad z_0 = 1$

**21.** $\dfrac{4z - 1}{z^4 - 1}, \quad z_0 = 0$ **22.** $\dfrac{1}{z^2}, \quad z_0 = i$

**23.** $\dfrac{\sin z}{z + \frac{1}{2}\pi}, \quad z_0 = -\frac{1}{2}\pi$

**24. TEAM PROJECT. Laurent Series. (a) Uniqueness.** Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

**(b) Accumulation of singularities.** Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$? (Give a reason.)

**(c) Integrals.** Expand the following functions in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}\,dt, \qquad \frac{1}{z^3}\int_0^z \frac{\sin t}{t}\,dt.$$

**25. CAS PROJECT. Partial Fractions.** Write a program for obtaining Laurent series by the use of partial fractions. Using the program, verify the calculations in Example 5 of the text. Apply the program to two other functions of your choice.

# 16.2 Singularities and Zeros. Infinity

Roughly, a *singular point* of an analytic function $f(z)$ is a $z_0$ at which $f(z)$ ceases to be analytic, and a *zero* is a $z$ at which $f(z) = 0$. Precise definitions follow below. In this section we show that Laurent series can be used for classifying singularities and Taylor series for discussing zeros.

Singularities were defined in Sec. 15.4, as we shall now recall and extend. We also remember that, by definition, a function is a ***single-valued*** relation, as was emphasized in Sec. 13.3.

We say that a function $f(z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an **isolated singularity** of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$. *Example:* $\tan z$ has isolated singularities at $\pm\pi/2$, $\pm 3\pi/2$, etc.; $\tan(1/z)$ has a nonisolated singularity at 0. (Explain!)

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \text{(Sec. 16.1)} \tag{1}$$

valid ***in the immediate neighborhood*** of the singular point $z = z_0$, except at $z_0$ itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

The sum of the first series is analytic at $z = z_0$, as we know from the last section. The second series, containing the negative powers, is called the **principal part** of (1), as we remember from the last section. If it has only finitely many terms, it is of the form

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0). \tag{2}$$

Then the singularity of $f(z)$ at $z = z_0$ is called a **pole,** and $m$ is called its **order.** Poles of the first order are also known as **simple poles.**

If the principal part of (1) has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an **isolated essential singularity.**

We leave aside nonisolated singularities.

### EXAMPLE 1 Poles. Essential Singularities

The function

$$f(z) = \frac{1}{z(z - 2)^5} + \frac{3}{(z - 2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

and

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!z^{2n+1}} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots.$$

Section 16.1 provides further examples. For instance, Example 1 shows that $z^{-5} \sin z$ has a fourth-order pole at 0. Example 4 shows that $1/(z^3 - z^4)$ has a third-order pole at 0 and a Laurent series with infinitely many negative powers. This is no contradiction, since this series is valid for $|z| > 1$; it merely tells us that in classifying singularities it is quite important to consider the Laurent series valid ***in the immediate neighborhood*** of a singular point. In Example 4 this is the series (I), which has three negative powers. ■

The classification of singularities into poles and essential singularities is not merely a formal matter, because the behavior of an analytic function in a neighborhood of an essential singularity is entirely different from that in the neighborhood of a pole.

### EXAMPLE 2 Behavior Near a Pole

$f(z) = 1/z^2$ has a pole at $z = 0$, and $|f(z)| \to \infty$ as $z \to 0$ in any manner. This illustrates the following theorem. ■

**THEOREM 1**

**Poles**

*If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.*

The proof is left to the student (see Prob. 12).

**EXAMPLE 3** **Behavior Near an Essential Singularity**

The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. It has no limit for approach along the imaginary axis; it becomes infinite if $z \to 0$ through positive real values, but it approaches zero if $z \to 0$ through negative real values. It takes on any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small $\epsilon$-neighborhood of $z = 0$. To see the letter, we set $z = re^{i\theta}$, and then obtain the following complex equation for $r$ and $\theta$, which we must solve:

$$e^{1/z} = e^{(\cos\theta - i \sin\theta)/r} = c_0 e^{i\alpha}.$$

Equating the absolute values and the arguments, we have $e^{(\cos\theta)/r} = c_0$, that is

$$\cos\theta = r \ln c_0, \qquad \text{and} \qquad -\sin\theta = \alpha r$$

respectively. From these two equations and $\cos^2\theta + \sin^2\theta = r^2(\ln c_0)^2 + \alpha^2 r^2 = 1$ we obtain the formulas

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2} \qquad \text{and} \qquad \tan\theta = -\frac{\alpha}{\ln c_0}.$$

Hence $r$ can be made arbitrarily small by adding multiples of $2\pi$ to $\alpha$, leaving $c$ unaltered. This illustrates the very famous *Picard's theorem* (with $z = 0$ as the exceptional value). For the rather complicated proof, see Ref. [D4], vol. 2, p. 258. For Picard, see Sec. 1.7. ■

**THEOREM 2**

**Picard's Theorem**

*If $f(z)$ is analytic and has an isolated essential singularity at a point $z_0$, it takes on every value, with at most one exceptional value, in an arbitrarily small $\epsilon$-neighborhood of $z_0$.*

**Removable Singularities.** We say that a function $f(z)$ has a *removable singularity* at $z = z_0$ if $f(z)$ is not analytic at $z = z_0$, but can be made analytic there by assigning a suitable value $f(z_0)$. Such singularities are of no interest since they can be removed as just indicated. *Example:* $f(z) = (\sin z)/z$ becomes analytic at $z = 0$ if we define $f(0) = 1$.

## Zeros of Analytic Functions

A **zero** of an analytic function $f(z)$ in a domain $D$ is a $z = z_0$ in $D$ such that $f(z_0) = 0$. A zero has **order** $n$ if not only $f$ but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$ but $f^{(n)}(z_0) \neq 0$. A first-order zero is also called a **simple zero.** For a second-order zero, $f(z_0) = f'(z_0) = 0$ but $f''(z_0) \neq 0$. And so on.

**EXAMPLE 4** **Zeros**

The function $1 + z^2$ has simple zeros at $\pm i$. The function $(1 - z^4)^2$ has second-order zeros at $\pm 1$ and $\pm i$. The function $(z - a)^3$ has a third-order zero at $z = a$. The function $e^z$ has no zeros (see Sec. 13.5). The function $\sin z$ has simple zeros at $0, \pm\pi, \pm 2\pi, \cdots$, and $\sin^2 z$ has second-order zeros at these points. The function $1 - \cos z$ has second-order zeros at $0, \pm 2\pi, \pm 4\pi, \cdots$, and the function $(1 - \cos z)^2$ has fourth-order zeros at these points. ■

**Taylor Series at a Zero.** At an $n$th-order zero $z = z_0$ of $f(z)$, the derivatives $f'(z_0), \cdots, f^{(n-1)}(z_0)$ are zero, by definition. Hence the first few coefficients $a_0, \cdots, a_{n-1}$ of the Taylor series (1), Sec. 15.4, are zero, too, whereas $a_n \neq 0$, so that this series takes the form

$$
\begin{aligned}
f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots \\
&= (z - z_0)^n [a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots] \qquad (a_n \neq 0).
\end{aligned} \tag{3}
$$

This is characteristic of such a zero, because if $f(z)$ has such a Taylor series, it has an $n$th-order zero at $z = z_0$, as follows by differentiation.

Whereas nonisolated singularities may occur, for zeros we have

**THEOREM 3**

**Zeros**

*The zeros of an analytic function $f(z)$ ($\not\equiv 0$) are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.*

**PROOF** The factor $(z - z_0)^n$ in (3) is zero only at $z = z_0$. The power series in the brackets $[\cdots]$ represents an analytic function (by Theorem 5 in Sec. 15.3), call it $g(z)$. Now $g(z_0) = a_n \neq 0$, since an analytic function is continuous, and because of this continuity, also $g(z) \neq 0$ in some neighborhood of $z = z_0$. Hence the same holds of $f(z)$. ■

This theorem is illustrated by the functions in Example 4.

Poles are often caused by zeros in the denominator. (*Example:* $\tan z$ has poles where $\cos z$ is zero.) This is a major reason for the importance of zeros. The key to the connection is the following theorem, whose proof follows from (3) (see Team Project 24).

**THEOREM 4**

**Poles and Zeros**

*Let $f(z)$ be analytic at $z = z_0$ and have a zero of $n$th order at $z = z_0$. Then $1/f(z)$ has a pole of $n$th order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.*

## Riemann Sphere. Point at Infinity

When we want to study complex functions for large $|z|$, the complex plane will generally become rather inconvenient. Then it may be better to use a representation of complex numbers on the so-called **Riemann sphere.** This is a sphere $S$ of diameter 1 touching the complex $z$-plane at $z = 0$ (Fig. 369), and we let the image of a point $P$ (a number $z$ in the plane) be the intersection $P^*$ of the segment $PN$ with $S$, where $N$ is the "North Pole" diametrically opposite to the origin in the plane. Then to each $z$ there corresponds a point on $S$.

Conversely, each point on $S$ represents a complex number $z$, except for $N$, which does not correspond to any point in the complex plane. This suggests that we introduce an additional point, called the **point at infinity** and denoted $\infty$ ("infinity") and let its image be $N$. The complex plane together with $\infty$ is called the **extended complex plane.** The complex plane is often called the *finite complex plane,* for distinction, or simply the

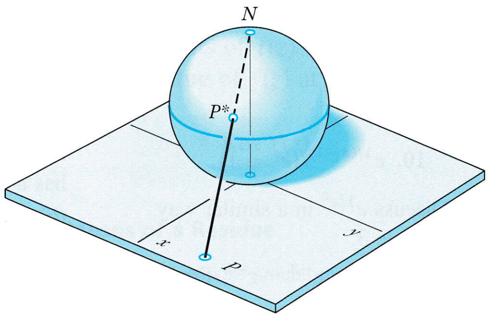



Fig. 369. Riemann sphere

*complex plane* as before. The sphere $S$ is called the **Riemann sphere.** The mapping of the extended complex plane onto the sphere is known as a **stereographic projection.** (What is the image of the Northern Hemisphere? Of the Western Hemisphere? Of a straight line through the origin?)

## Analytic or Singular at Infinity

If we want to investigate a function $f(z)$ for large $|z|$, we may now set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighborhood of $w = 0$. We define $f(z)$ to be **analytic** or **singular at infinity** if $g(w)$ is analytic or singular, respectively, at $w = 0$. We also define

$$g(0) = \lim_{w \to 0} g(w) \tag{4}$$

if this limit exists.

Furthermore, we say that $f(z)$ has an *nth-order zero at infinity* if $f(1/w)$ has such a zero at $w = 0$. Similarly for poles and essential singularities.

**EXAMPLE 5 Functions Analytic or Singular at Infinity. Entire and Meromorphic Functions**

The function $f(z) = 1/z^2$ is analytic at $\infty$ since $g(w) = f(1/w) = w^2$ is analytic at $w = 0$, and $f(z)$ has a second-order zero at $\infty$. The function $f(z) = z^3$ is singular at $\infty$ and has a third-order pole there since the function $g(w) = f(1/w) = 1/w^3$ has such a pole at $w = 0$. The function $e^z$ has an essential singularity at $\infty$ since $e^{1/w}$ has such a singularity at $w = 0$. Similarly, $\cos z$ and $\sin z$ have an essential singularity at $\infty$.

Recall that an **entire function** is one that is analytic everywhere in the (finite) complex plane. Liouville's theorem (Sec. 14.4) tells us that the only *bounded* entire functions are the constants, hence any nonconstant entire function must be unbounded. Hence it has a singularity at $\infty$, a pole if it is a polynomial or an essential singularity if it is not. The functions just considered are typical in this respect.

An analytic function whose only singularities in the finite plane are poles is called a **meromorphic function.** Examples are rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$. ■

In this section we used Laurent series for investigating singularities. In the next section we shall use these series for an elegant integration method.

## PROBLEM SET 16.2

**1–10 SINGULARITIES**

Determine the location and kind of the singularities of the following functions in the finite plane and at infinity. In the case of poles also state the order.

1. $\tan^2 \pi z$
2. $z + \dfrac{2}{z} - \dfrac{3}{z^2}$
3. $\cot z^2$
4. $z^3 e^{1/(z-1)}$
5. $\cos z - \sin z$
6. $1/(\cos z - \sin z)$

**7.** $\dfrac{\sin 3z}{(z^4-1)^4}$

**8.** $\dfrac{4}{z-1} + \dfrac{2}{(z-1)^2} - \dfrac{8}{(z-1)^3}$

**9.** $\cosh\,[1/(z^2+1)]$ **10.** $e^{1/(z-1)}/(e^z-1)$

**11. (Essential singularity)** Discuss $e^{1/z^2}$ in a similar way as $e^{1/z}$ is discussed in Example 3.

**12. (Poles)** Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.

### 13–22 ZEROS

Determine the location and order of the zeros.

**13.** $(z+16i)^4$ **14.** $(z^4-16)^4$

**15.** $z^{-3}\sin^3 \pi z$ **16.** $\cosh^2 z$

**17.** $(3z^2+1)e^{-z}$ **18.** $(z^2-1)^2(e^{z^2}-1)$

**19.** $(z^2+4)(e^z-1)^2$ **20.** $(\sin z - 1)^3$

**21.** $(1-\cos z)^2$ **22.** $e^z - e^{2z}$

**23. (Zeros)** If $f(z)$ is analytic and has a zero of order $n$ at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$.

**24. TEAM PROJECT. Zeros. (a) Derivative.** Show that if $f(z)$ has a zero of order $n > 1$ at $z = z_0$, then $f'(z)$ has a zero of order $n - 1$ at $z_0$.

**(b) Poles and zeros.** Prove Theorem 4.

**(c) Isolated *k*-points.** Show that the points at which a nonconstant analytic function $f(z)$ has a given value $k$ are isolated.

**(d) Identical functions.** If $f_1(z)$ are analytic in a domain $D$ and equal at a sequence of points $z_n$ in $D$ that converges in $D$, show that $f_1(z) \equiv f_2(z)$ in $D$.

**25. (Riemann sphere)** Assuming that we let the image of the $x$-axis be meridians 0° and 180°, describe and sketch (or graph) the images of the following regions on the Riemann sphere: (a) $|z| > 100$, (b) the lower half-plane, (c) $\frac{1}{2} \leqq |z| \leqq 2$.

# 16.3 Residue Integration Method

The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\,dz$$

taken around a simple close path $C$. The idea is as follows.

If $f(z)$ is analytic everywhere on $C$ and inside $C$, such an integral is zero by Cauchy's integral theorem (Sec. 14.2), and we are done.

If $f(z)$ has a singularity at a point $z = z_0$ inside $C$, but is otherwise analytic on $C$ and inside $C$, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$ (sometimes called a **deleted neighborhood,** an old-fashioned term that we shall not use). Now comes the key idea. The coefficient $b_1$ of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula (2) in Sec. 16.1 with $n = 1$, namely,

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz.$$

Now, since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients (see the examples in Sec. 16.1), we can find $b_1$ by one of those methods and then use the formula for $b_1$ for evaluating the integral, that is,

$$\oint_C f(z)\,dz = 2\pi i b_1. \tag{1}$$

Here we integrate conunterclockwise around a simple closed path $C$ that contains $z = z_0$ in its interior (but no other singular points of $f(z)$ on or inside $C$!).

The coefficient $b_1$ is called the **residue** of $f(z)$ at $z = z_0$ and we denote it by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z). \tag{2}$$

**EXAMPLE 1** **Evaluation of an Integral by Means of a Residue**

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle $C$.

***Solution.*** From (14) in Sec. 15.4 we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots$$

which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -1/3!$. From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4}\, dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

**EXAMPLE 2** **CAUTION! Use the Right Laurent Series!**

Integrate $f(z) = 1/(z^3 - z^4)$ clockwise around the circle $C$: $|z| = 1/2$.

***Solution.*** $z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$. Now $z = 1$ lies outside $C$. Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$. This is series (I) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1).$$

We see from it that this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

*CAUTION!* Had we used the wrong series (II) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots \qquad (|z| > 1),$$

we would have obtained the wrong answer, 0, because this series has no power $1/z$.

## Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

**Simple Poles.** Two formulas for the residue of $f(z)$ at a simple pole at $z_0$ are

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z) \tag{3}$$

and, assuming that $f(z) = p(z)/q(z)$, $p(z_0) \neq 0$, and $q(z)$ has a simple zero at $z_0$ (so that $f(z)$ has at $z_0$ a simple pole, by Theorem 4 in Sec. 16.2),

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{4}$$

**PROOF** For a simple pole at $z = z_0$ the Laurent series (1), Sec. 16.1, is

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R).$$

Here $b_1 \neq 0$. (Why?) Multiplying both sides by $z - z_0$ and then letting $z \to z_0$, we obtain the formula (3):

$$\lim_{z \to z_0} (z - z_0)f(z) = b_1 + \lim_{z \to z_0} (z - z_0)[a_0 + a_1(z - z_0) + \cdots] = b_1$$

where the last equality follows from continuity (Theorem 1, Sec. 15.3).

We prove (4). The Taylor series of $q(z)$ at a simple zero $z_0$ is

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots .$$

Substituting this into $f = p/q$ and then $f$ into (3) gives

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q''(z_0)/2 + \cdots]} .$$

$z - z_0$ cancels. By continuity, the limit of the denominator is $q'(z_0)$ and (4) follows. ■

**EXAMPLE 3** **Residue at a Simple Pole**

$f(z) = (9z + i)/(z^3 + z)$ has a simple pole at $i$ because $z^2 + 1 = (z + i)(z - i)$, and (3) gives the residue

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z \to i} (z - i) \frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

By (4) with $p(i) = 9i + i$ and $q'(z) = 3z^2 + 1$ we confirm the result,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1}\right]_{z=i} = \frac{10i}{-2} = -5i.$$ ■

**Poles of Any Order.** The residue of $f(z)$ at an $m$th-order pole at $z_0$ is

**(5)** $$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m - 1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[ (z - z_0)^m f(z) \right] \right\}.$$

In particular, for a second-order pole ($m = 2$),

**(5*)** $$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \{[(z - z_0)^2 f(z)]'\}.$$

**PROOF** The Laurent series of $f(z)$ converging near $z_0$ (except at $z_0$ itself) is (Sec. 16.2)

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

where $b_m \neq 0$. The residue wanted is $b_1$. Multiplying both sides by $(z - z_0)^m$ gives

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_1(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots .$$

We see that $b_1$ is now the coefficient of the power $(z - z_0)^{m-1}$ of the power series of $g(z) = (z - z_0)^m f(z)$. Hence Taylor's theorem (Sec. 15.4) gives (5):

$$\begin{aligned} b_1 &= \frac{1}{(m-1)!}\, g^{(m-1)}(z_0) \\ &= \frac{1}{(m-1)!}\, \frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right]. \end{aligned}$$

**EXAMPLE 4 Residue at a Pole of Higher Order**

$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$ has a pole of second order at $z = 1$ because the denominator equals $(z + 4)(z - 1)^2$ (verify!). From (5*) we obtain the residue

$$\begin{aligned} \operatorname*{Res}_{z=1} f(z) &= \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right] \\ &= \lim_{z \to 1} \frac{d}{dz}\left(\frac{50z}{z+4}\right) \\ &= \frac{200}{5^2} = 8. \end{aligned}$$

## Several Singularities Inside the Contour. Residue Theorem

Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour $C$. This is the purpose of the residue theorem. The extension is surprisingly simple.

**THEOREM 1**

**Residue Theorem**

*Let $f(z)$ be analytic inside a simple closed path $C$ and on $C$, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside $C$. Then the integral of $f(z)$ taken counterclockwise around $C$ equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_k$:*

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{6}$$

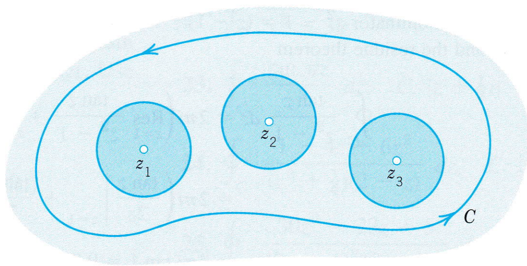



**Fig. 370.** Residue theorem