PROOF We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated (Fig. 370). Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem we thus have

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz = 0, \tag{7}$$

the integral along C being taken *counterclockwise* and the other integrals *clockwise* (as in Figs. 351 and 352, Sec. 14.2). We take the integrals over $C_1, \cdots, C_k$ to the right and compensate the resulting minus sign by reversing the sense of integration. Thus,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz \tag{8}$$

where all the integrals are now taken counterclockwise. By (1) and (2),

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z), \qquad j = 1, \cdots, k,$$

so that (8) gives (6) and the residue theorem is proved. ■

This important theorem has various applications in connection with complex and real integrals. Let us first consider some complex integrals. (Real integrals follow in the next section.)

EXAMPLE 5 **Integration by the Residue Theorem. Several Contours**

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}\,dz$$

Solution. The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

[Confirm this by (4).] *Ans.* (a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0. ■

EXAMPLE 6 **Another Application of the Residue Theorem**

Integrate $(\tan z)/(z^2 - 1)$ counterclockwise around the circle C: $|z| = 3/2$.

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C. Because of the denominator $z^2 - 1 = (z-1)(z+1)$ the given function has simple poles at ± 1. We thus obtain from (4) and the residue theorem

$$\begin{aligned}\oint_C \frac{\tan z}{z^2-1}\,dz &= 2\pi i\left(\operatorname*{Res}_{z=1}\frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1}\frac{\tan z}{z^2-1}\right)\\ &= 2\pi i\left(\frac{\tan z}{2z}\bigg|_{z=1} + \frac{\tan z}{2z}\bigg|_{z=-1}\right)\\ &= 2\pi i \tan 1 = 9.7855i.\end{aligned}$$

■

EXAMPLE 7 **Poles and Essential Singularities**

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4 - 16} + ze^{\pi/z}\right) dz$$

Solution. Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4); note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4 - 16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16},$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4 - 16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2} \cdot \frac{1}{z} + \cdots \qquad (|z| > 0).$$

Ans. $2\pi i(-\frac{1}{16} - \frac{1}{16} + \frac{1}{2}\pi^2) = \pi(\pi^2 - \frac{1}{4})i = 30.221i$ by the residue theorem. ■

PROBLEM SET 16.3

1. Verify the calculations in Example 3 and find the other residues.
2. Verify the calculations in Example 4 and find the other residue.

3–12 RESIDUES

Find all the singular points and the corresponding residues. (Show the details of your work.)

3. $\dfrac{1}{4 + z^2}$
4. $\dfrac{\cos z}{z^6}$
5. $\dfrac{\sin z}{z^6}$
6. $\dfrac{z^2 + 1}{z^2 - z}$
7. $\cot z$
8. $\sec z$
9. $\dfrac{1}{(z^2 - 1)^2}$
10. $\dfrac{1/3}{z^4 - 1}$
11. $\tan z$
12. $\dfrac{z^2}{z^4 - 1}$

13. **CAS PROJECT. Residue at a Pole.** Write a program for calculating the residue at a pole of any order. Use it for solving Probs. 3–8.

14–25 RESIDUE INTEGRATION

Evaluate (counterclockwise). (Show the details.)

14. $\displaystyle\oint_C \frac{\sin \pi z}{z^4}\, dz, \quad C: |z - i| = 2$
15. $\displaystyle\oint_C e^{1/z}\, dz, \quad C: |z| = 1$
16. $\displaystyle\oint_C \frac{dz}{\sinh \frac{1}{2}\pi z}, \quad C: |z - 1| = 1.4$
17. $\displaystyle\oint_C \tan \pi z\, dz, \quad C: |z| = 1$
18. $\displaystyle\oint_C \tan \pi z\, dz, \quad C: |z| = 2$
19. $\displaystyle\oint_C \frac{e^z}{\cos z}\, dz, \quad C: |z| = 4.5$
20. $\displaystyle\oint_C \coth z\, dz, \quad C: |z| = 1$
21. $\displaystyle\oint_C \frac{e^z}{\cos \pi z}\, dz, \quad C: |z - i| = 1.5$
22. $\displaystyle\oint_C \frac{\cosh z}{z^2 - 3iz}\, dz, \quad C: |z| = 1$
23. $\displaystyle\oint_C \frac{\tan \pi z}{z^3}\, dz, \quad C: |z + \tfrac{1}{2}i| = 1$
24. $\displaystyle\oint_C \frac{1 - 4z + 6z^2}{(z^2 + \frac{1}{4})(2 - z)}\, dz, \quad C: |z| = 1$
25. $\displaystyle\oint_C \frac{30z^2 - 23z + 5}{(2z - 1)^2(3z - 1)}\, dz, \quad C: |z| = 1$

16.4 Residue Integration of Real Integrals

It is quite surprising that certain classes of complicated ***real*** integrals can be integrated by the residue theorem, as we shall see.

Integrals of Rational Functions of cos θ and sin θ

We first consider integrals of the type

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \tag{1}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite (does not become infinite) on the interval of integration. Setting $e^{i\theta} = z$, we obtain

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right). \end{aligned} \tag{2}$$

Since F is rational in $\cos\theta$ and $\sin\theta$, Eq. (2) shows that F is now a rational function of z, say, $f(z)$. Since $dz/d\theta = ie^{i\theta}$, we have $d\theta = dz/iz$ and the given integral takes the form

$$J = \oint_C f(z)\,\frac{dz}{iz} \tag{3}$$

and, as θ ranges from 0 to 2π in (1), the variable $z = e^{i\theta}$ ranges counterclockwise once around the unit circle $|z| = 1$. (Review Sec. 13.5 if necessary.)

EXAMPLE 1 An Integral of the Type (1)

Show by the present method that $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi.$

Solution. We use $\cos\theta = \frac{1}{2}(z + 1/z)$ and $d\theta = dz/iz$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

We see that the integrand has a simple pole at $z_1 = \sqrt{2} + 1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2 = \sqrt{2} - 1$ (where $z - \sqrt{2} + 1 = 0$) inside C with residue [by (3), Sec. 16.3]

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left[\frac{1}{z - \sqrt{2} - 1}\right]_{z=\sqrt{2}-1}$$

$$= -\frac{1}{2}.$$

Answer: $2\pi i(-2/i)(-1/2) = 2\pi$. (Here $-2/i$ is the factor in front of the last integral.) ■

As another large class, let us consider real integrals of the form

$$\int_{-\infty}^{\infty} f(x)\, dx. \tag{4}$$

Such an integral, whose interval of integration is not finite is called an **improper integral,** and it has the meaning

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{a\to-\infty} \int_{a}^{0} f(x)\, dx + \lim_{b\to\infty} \int_{0}^{b} f(x)\, dx. \tag{5'}$$

If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{5}$$

The limit in (5) is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx.$$

It may exist even if the limits in (5′) do not. *Example:*

$$\lim_{R\to\infty} \int_{-R}^{R} x\, dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{R^2}{2}\right) = 0, \qquad \text{but} \qquad \lim_{b\to\infty} \int_{0}^{b} x\, dx = \infty.$$

We assume that the function $f(x)$ in (4) is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of the numerator. Then the limits in (5′) exist, and we may start from (5). We consider the corresponding contour integral

$$\oint_C f(z)\, dz \tag{5*}$$

around a path C in Fig. 371. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose R large enough, then C encloses all these poles. By the residue theorem we then obtain

$$\oint_C f(z)\, dz = \int_S f(z)\, dz + \int_{-R}^{R} f(x)\, dx = 2\pi i \sum \text{Res}\, f(z)$$

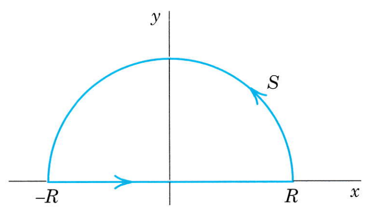


Fig. 371. Path C of the contour integral in (5*)

where the sum consists of all the residues of $f(z)$ at the points in the upper half-plane at which $f(z)$ has a pole. From this we have

$$\int_{-R}^{R} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)\, dz. \tag{6}$$

We prove that, if $R \to \infty$, the value of the integral over the semicircle S approaches zero. If we set $z = Re^{i\theta}$, then S is represented by $R = const$, and as z ranges along S, the variable θ ranges from 0 to π. Since, by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

for sufficiently large constants k and R_0. By the *ML*-inequality in Sec. 14.1,

$$\left|\int_S f(z)\, dz\right| < \frac{k}{R^2}\, \pi R = \frac{k\pi}{R} \qquad (R > R_0).$$

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z) \tag{7}$$

where we sum over all the residues of $f(z)$ at the poles of $f(z)$ in the upper half-plane.

EXAMPLE 2 An Improper Integral from 0 to ∞

Using (7), show that

$$\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}\,.$$

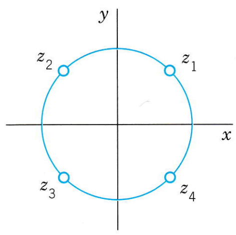


Fig. 372. Example 2

Solution. Indeed, $f(z) = 1/(1 + z^4)$ has four simple poles at the points (make a sketch)

$$z_1 = e^{\pi i/4}, \qquad z_2 = e^{3\pi i/4}, \qquad z_3 = e^{-3\pi i/4}, \qquad z_4 = e^{-\pi i/4}.$$

The first two of these poles lie in the upper half-plane (Fig. 372). From (4) in the last section we find the residues

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}\, e^{-3\pi i/4} = -\frac{1}{4}\, e^{\pi i/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}\, e^{-9\pi i/4} = \frac{1}{4}\, e^{-\pi i/4}.$$

(Here we used $e^{\pi i} = -1$ and $e^{-2\pi i} = 1$.) By (1) in Sec. 13.6 and (7) in this section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = -\frac{2\pi i}{4}\,(e^{\pi i/4} - e^{-\pi i/4}) = -\frac{2\pi i}{4} \cdot 2i \cdot \sin\frac{\pi}{4} = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}\,.$$

Since $1/(1+x^4)$ is an even function, we thus obtain, as asserted,

$$\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}\,.$$ ■

Fourier Integrals

The method of evaluating (4) by creating a closed contour (Fig. 371) and "blowing it up" extends to integrals

$$(8) \qquad \int_{-\infty}^{\infty} f(x) \cos sx \, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x) \sin sx \, dx \qquad (s \text{ real})$$

as they occur in connection with the Fourier integral (Sec. 11.7).

If $f(x)$ is a rational function satisfying the assumption on the degree as for (4), we may consider the corresponding integral

$$\oint_C f(z) e^{isz}\, dz \qquad (s \text{ real and positive})$$

over the contour C in Fig. 371 on p. 719. Instead of (7) we now get

$$(9) \qquad \int_{-\infty}^{\infty} f(x) e^{isx}\, dx = 2\pi i \sum \operatorname{Res}\,[f(z)e^{isz}] \qquad (s > 0)$$

where we sum the residues of $f(z)e^{isz}$ at its poles in the upper half-plane. Equating the real and the imaginary parts on both sides of (9), we have

$$\textbf{(10)} \qquad \begin{aligned} \int_{-\infty}^{\infty} f(x) \cos sx \, dx &= -2\pi \sum \operatorname{Im} \operatorname{Res}\,[f(z)e^{isz}], \\ \int_{-\infty}^{\infty} f(x) \sin sx \, dx &= 2\pi \sum \operatorname{Re} \operatorname{Res}\,[f(z)e^{isz}]. \end{aligned} \qquad (s > 0)$$

To establish (9), we must show [as for (4)] that the value of the integral over the semicircle S in Fig. 371 approaches 0 as $R \to \infty$. Now $s > 0$ and S lies in the upper half-plane $y \geqq 0$. Hence

$$|e^{isz}| = |e^{is(x+iy)}| = |e^{isx}|\,|e^{-sy}| = 1 \cdot e^{-sy} \leqq 1 \qquad (s > 0, \quad y \geqq 0).$$

From this we obtain the inequality $|f(z)e^{isz}| = |f(z)|\,|e^{isz}| \leqq |f(z)|$ $(s > 0,\ y \geqq 0)$. This reduces our present problem to that for (4). Continuing as before gives (9) and (10). ■

EXAMPLE 3 **An Application of (10)**

Show that
$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2}\, dx = \frac{\pi}{k} e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2}\, dx = 0 \qquad (s > 0, k > 0).$$

Solution. In fact, $e^{isz}/(k^2 + z^2)$ has only one pole in the upper half-plane, namely, a simple pole at $z = ik$, and from (4) in Sec. 16.3 we obtain
$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \frac{e^{-ks}}{2ik}.$$
Thus
$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2 + x^2}\, dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{-ks}.$$
Since $e^{isx} = \cos sx + i \sin sx$, this yields the above results [see also (15) in Sec. 11.7.] ■

Another Kind of Improper Integral

We consider an improper integral

$$\int_A^B f(x)\, dx \tag{11}$$

whose integrand becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty.$$

By definition, this integral (11) means

$$\int_A^B f(x)\, dx = \lim_{\epsilon \to 0} \int_A^{a-\epsilon} f(x)\, dx + \lim_{\eta \to 0} \int_{a+\eta}^{B} f(x)\, dx \tag{12}$$

where both ϵ and η approach zero independently and through positive values. It may happen that neither of these two limits exists if ϵ and η go to 0 independently, but the limit

$$\lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\, dx + \int_{a+\epsilon}^{B} f(x)\, dx\right] \tag{13}$$

exists. This is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_A^B f(x)\, dx.$$

For example,

$$\text{pr. v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\epsilon \to 0} \left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3}\right] = 0;$$

the principal value exists, although the integral itself has no meaning.

In the case of simple poles on the real axis we shall obtain a formula for the principal value of an integral from $-\infty$ to ∞. This formula will result from the following theorem.

THEOREM 1

Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then (Fig. 373)

$$\lim_{r \to 0} \int_{C_2} f(z)\, dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

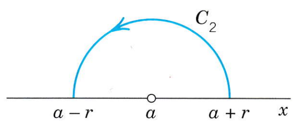


Fig. 373. Theorem 1

PROOF By the definition of a simple pole (Sec. 16.2) the integrand $f(z)$ has for $0 < |z - a| < R$ the Laurent series

$$f(z) = \frac{b_1}{z - a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z).$$

Here $g(z)$ is analytic on the semicircle of integration (Fig. 373)

$$C_2: \quad z = a + re^{i\theta}, \qquad 0 \leqq \theta \leqq \pi$$

and for all z between C_2 and the x-axis, and thus bounded on C_2, say, $|g(z)| \leqq M$. By integration,

$$\int_{C_2} f(z)\, dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}}\, ire^{i\theta}\, d\theta + \int_{C_2} g(z)\, dz = b_1 \pi i + \int_{C_2} g(z)\, dz.$$

The second integral on the right cannot exceed $M\pi r$ in absolute value, by the ML-inequality (Sec. 14.1), and $ML = M\pi r \to 0$ as $r \to 0$. ■

Figure 374 shows the idea of applying Theorem 1 to obtain the principal value of the integral of a rational function $f(x)$ from $-\infty$ to ∞. For sufficiently large R the integral over the entire contour in Fig. 374 has the value J given by $2\pi i$ times the sum of the residues of $f(z)$ at the singularities in the upper half-plane. We assume that $f(x)$ satisfies the degree

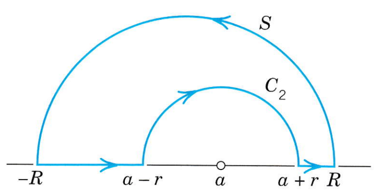


Fig. 374. Application of Theorem 1

condition imposed in connection with (4). Then the value of the integral over the large semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

by Theorem 1. Together this shows that the principal value P of the integral from $-\infty$ to ∞ plus K equals J; hence $P = J - K = J + \pi i \operatorname{Res}_{z=a} f(z)$. If $f(z)$ has several simple poles on the real axis, then K will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

$$\text{(14)} \qquad \text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

EXAMPLE 4 Poles on the Real Axis

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i \left(\frac{3-i}{20}\right) + \pi i \left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10}.$$

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.

PROBLEM SET 16.4

1–8 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals. (Show the details of your work.)

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{7 + 6\cos\theta}$

2. $\displaystyle\int_0^{\pi} \frac{d\theta}{2 + \cos\theta}$

3. $\displaystyle\int_0^{2\pi} \frac{d\theta}{37 - 12\cos\theta}$

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{8 - 2\sin\theta}$

5. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$

6. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$

7. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\, d\theta.$

Hint. $\cos 2\theta = \dfrac{1}{2}\left(z^2 + \dfrac{1}{z^2}\right)$

8. $\displaystyle\int_0^{2\pi} \frac{1 + 4\cos\theta}{17 - 8\cos\theta}\, d\theta$

9–22 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate (showing the details):

9. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^4 + 1}\, dx$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$

12. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4)^2}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 16}$

15. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3}{1 + x^8}\, dx$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 9)}$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2 - 2x + 2)^2}\, dx$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1}\, dx$

19. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^4 + 1}\, dx$

20. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1}\, dx$

21. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{x^4 + 1}\, dx$

22. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4 + 5x^2 + 4}\, dx$

23–27 IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

23. $\displaystyle\int_{-\infty}^{\infty} \frac{x + 2}{x^3 + x}\, dx$

24. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\, dx$

25. $\displaystyle\int_{-\infty}^{\infty} \frac{x + 5}{x^3 - x}\, dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$

27. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$

28. **TEAM PROJECT. Comments on Real Integrals.** **(a) Formula (10)** follows from (9). Give the details.

(b) Use of auxiliary results. Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a$, a, $a + ib$, $-a + ib$, letting $a \to \infty$, and using

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2},$$

show that

$$\int_0^{\infty} e^{-x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2}\, e^{-b^2}.$$

(This integral is needed in heat conduction in Sec. 12.6.)

(c) Inspection. Solve Probs. 15 and 21 without calculation.

29. **CAS EXPERIMENT. Check your CAS.** Find out to what extent your CAS can evaluate integrals of the form (1), (4), and (8) correctly. Do this by comparing the results of direct integration (which may come out false) with those of using residues.

30. **CAS EXPERIMENT. Simple Poles on the Real Axis.** Experiment with integrals $\int_{-\infty}^{\infty} f(x)\, dx$, $f(x) = [(x - a_1)(x - a_2) \cdots (x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.

CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS

1. Laurent series generalize Taylor series. Explain the details.

2. Can a function have several Laurent series with the same center? Explain. If your answer is yes, give examples.

3. What is the principal part of a Laurent series? Its significance?

4. What is a pole? An essential singularity? Give examples.

5. What is Picard's theorem? Why did it occur in this chapter?

6. What is the Riemann sphere? The extended complex plane? Its significance?

7. Is e^{1/z^2} analytic or singular at infinity? $\cosh z$? $(z-4)^3$? Explain.

8. What is the residue? Why is it important?

9. State formulas for residues from memory.

10. State some further methods for calculating residues.

11. What is residue integration? To what kind of complex integrals does it apply?

12. By what idea can we apply residue integration to real integrals from $-\infty$ to ∞? Give simple examples.

13. What is a zero of an analytic function? How are zeros classified?

14. What are improper integrals? Cauchy principal values? Give examples.

15. Can the residue at a singular point be 0? At a simple pole?

16. What is a meromorphic function? An entire function? Give examples.

17–28 COMPLEX INTEGRALS

Integrate counterclockwise around C. (Show the details.)

17. $\dfrac{\tan z}{z^4}$, C: $|z| = 1$

18. $\dfrac{\sin 2z}{z^4}$, C: $|z| = 1$

19. $\dfrac{10z}{2z + i}$, C: $|z - 2i| = 3$

20. $\dfrac{iz + 1}{z^2 - iz + 2}$, C: $|z - 1| = 3$

21. $\dfrac{\cosh 5z}{z^2 + 4}$, C: $|z - i| = 2$

22. $\dfrac{4z^3 + 7z}{\cos z}$, C: $|z + 1| = 1$

23. $\cot 8z$, C: $|z| = 0.2$

24. $\dfrac{z^2 \sin z}{4z^2 - 1}$, C: $|z - 1| = 2$

25. $\dfrac{\cos z}{z^n}$, $n = 1, 2, \cdots$, C: $|z| = 1$

26. $\dfrac{z^2 + 1}{z^2 - 2z}$, C: $\dfrac{1}{2}x^2 + y^2 = 1$

27. $\dfrac{15z + 9}{z^3 - 9z}$, C: $|z - 3| = 2$

28. $\dfrac{15z + 9}{z^3 - 9z}$, C: $|z| = 4$

29–35 REAL INTEGRALS

Evaluate by the methods of this chapter (showing the details):

29. $\displaystyle\int_0^{2\pi} \frac{d\theta}{25 - 24\cos\theta}$

30. $\displaystyle\int_0^{\pi} \frac{d\theta}{k + \cos\theta}$, $k > 1$

31. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - \frac{1}{2}\sin\theta}$

32. $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{3 + \cos\theta}\, d\theta$

33. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(1 + x^2)^2}\, dx$

34. $\displaystyle\int_0^{\infty} \frac{dx}{(1 + x^2)^2}$

35. $\displaystyle\int_0^{\infty} \frac{1 + 2x^2}{1 + 4x^4}\, dx$

36. Obtain the answer to Prob. 18 in Sec. 16.4 from the present Prob. 35.

SUMMARY OF CHAPTER 16
Laurent Series. Residue Integration

A **Laurent series** is a series of the form

$$(1)\qquad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \text{(Sec. 16.1)}$$

or, more briefly written [but this means the same as (1)!]

$$(1^*)\qquad f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*$$

where $n = 0, \pm 1, \pm 2, \cdots$. This series converges in an open annulus (ring) A with center z_0. In A the function $f(z)$ is analytic. At points not in A it may have singularities. The first series in (1) is a power series. In a given annulus, a Laurent series of $f(z)$ is unique, but $f(z)$ may have different Laurent series in different annuli with the same center.

Of particular importance is the Laurent series (1) that converges in a neighborhood of z_0 except at z_0 itself, say, for $0 < |z - z_0| < R$ ($R > 0$, suitable). The series (or finite sum) of the negative powers in this Laurent series is called the **principal part** of $f(z)$ at z_0. The coefficient b_1 of $1/(z - z_0)$ in this series is called the **residue** of $f(z)$ at z_0 and is given by [see (1) and (1*)]

$$(2)\quad b_1 = \operatorname*{Res}_{z \to z_0} f(z) = \frac{1}{2\pi i} \oint_C f(z^*)\, dz^*. \qquad \text{Thus} \qquad \oint_C f(z^*)\, dz^* = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

b_1 can be used for **integration** as shown in (2) because it can be found from

$$(3)\qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right), \qquad \text{(Sec. 16.3)},$$

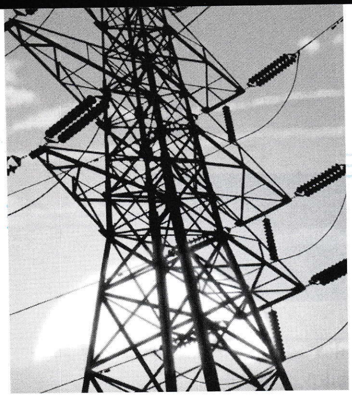
provided $f(z)$ has at z_0 a **pole of order *m*;** by definition this means that that principal part has $1/(z - z_0)^m$ as its highest negative power. Thus for a simple pole ($m = 1$),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z); \qquad \text{also,} \qquad \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

If the principal part is an infinite series, the singularity of $f(z)$ at z_0 is called an **essential singularity** (Sec. 16.2).

Section 16.2 also discusses the *extended complex plane,* that is, the complex plane with an improper point ∞ ("infinity") attached.

Residue integration may also be used to evaluate certain classes of complicated real integrals (Sec. 16.4).



CHAPTER 17

Conformal Mapping

If a complex function $w = f(z)$ is defined in a domain D of the z-plane, then to each point in D there corresponds a point in the w-plane. In this way we obtain a **mapping** of D onto the range of values of $f(z)$ in the w-plane. We shall see that if $f(z)$ is an *analytic* function, then the mapping given by $w = f(z)$ is **conformal** (angle-preserving), except at points where the derivative $f'(z)$ is zero.

Conformality appeared early in history in connection with constructing maps of the globe, which can be conformal (can give directions correctly) or "equiareal" (give areas correctly, except for a scale factor), but cannot have both properties, as can be proved (see [GR8] in App. 1).

Conformality is the most important geometric property of analytic functions and gives the possibility of a geometric approach to complex analysis. Indeed, just as in calculus we use curves of real functions $y = f(x)$ for studying "geometric" properties of functions, in complex analysis we can use conformal mappings for obtaining a deeper understanding of properties of functions, notably of those discussed in Chap. 13.

Indeed, we shall first define the concepts of conformal mapping and then consider mappings by those elementary analytic functions in Chap. 13.

This is one purpose of this chapter. A second purpose, more important to the engineer and physicist, is the use of conformal mapping in connection with potential problems. In fact, in this chapter and in the next one we shall see that conformal mapping yields a standard method for solving **boundary value problems** in (two-dimensional) potential theory by transforming a complicated region into a simpler one. Corresponding applications will concern problems from electrostatics, heat flow, and fluid flow.

In the last section (17.5) we explain the concept of a **Riemann surface,** which fits well into the present discussion of "geometric" ideas.

Prerequisite: Chap. 13.
Sections that may be omitted in a shorter course: 17.3 and 17.5
References and Answers to Problems: App. 1 Part D, App. 2.

17.1 Geometry of Analytic Functions: Conformal Mapping

A complex function

$$w = f(z) = u(x, y) + iv(x, y) \qquad (z = x + iy) \tag{1}$$

of a complex variable z gives a **mapping** of its domain of definition D in the complex z-plane ***into*** the complex w-plane or ***onto*** its range of values in that plane.[1] For any point z_0 in D the point $w_0 = f(z_0)$ is called the **image** of z_0 with respect to f. More generally, for the points of a curve C in D the image points form the **image** of C; similarly for other point sets in D. Also, instead of *the mapping by a function* $w = f(z)$ we shall say more briefly ***the mapping*** $\boldsymbol{w = f(z)}$.

EXAMPLE 1 Mapping $w = f(z) = z^2$

Using polar forms $z = re^{i\theta}$ and $w = Re^{i\phi}$, we have $w = z^2 = r^2e^{2i\theta}$. Comparing moduli and arguments gives $R = r^2$ and $\phi = 2\theta$. Hence circles $r = r_0$ are mapped onto circles $R = r_0^{\,2}$ and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$. Figure 375 shows this for the region $1 \leqq |z| \leqq 3/2$, $\pi/6 \leqq \theta \leqq \pi/3$, which is mapped onto the region $1 \leqq |w| \leqq 9/4$, $\pi/3 \leqq \theta \leqq 2\pi/3$.

In Cartesian coordinates we have $z = x + iy$ and

$$u = \operatorname{Re}(z^2) = x^2 - y^2, \qquad v = \operatorname{Im}(z^2) = 2xy.$$

Hence vertical lines $x = c = const$ are mapped onto $u = c^2 - y^2$, $v = 2cy$. From this we can eliminate y. We obtain $y^2 = c^2 - u$ and $v^2 = 4c^2y^2$. Together,

$$v^2 = 4c^2(c^2 - u) \qquad \text{(Fig. 376).}$$

These parabolas open to the left. Similarly, horizontal lines $y = k = const$ are mapped onto parabolas opening to the right,

$$v^2 = 4k^2(k^2 + u) \qquad \text{(Fig. 376).}$$ ■

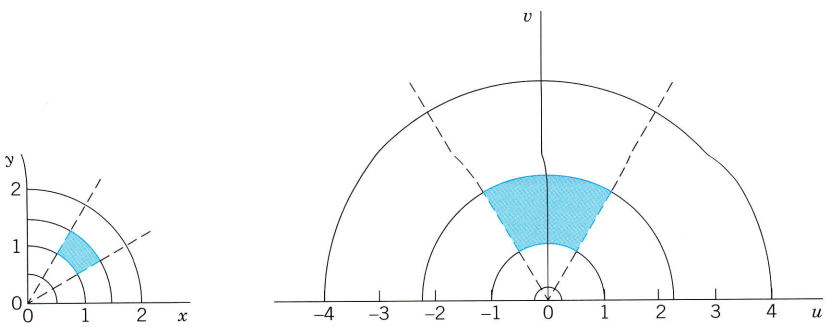


Fig. 375. Mapping $w = z^2$. Lines $|z| = const$, $\arg z = const$ and their images in the w-plane

[1]The general terminology is as follows. A mapping of a set A into a set B is called **surjective** or a mapping of A **onto** B if every element of B is the image of at least one element of A. It is called **injective** or **one-to-one** if different elements of A have different images in B. Finally, it is called **bijective** if it is both surjective and injective.

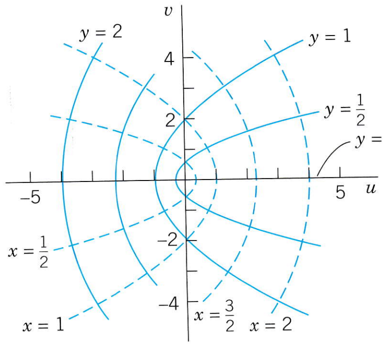


Fig. 376. Images of $x = const$, $y = const$ under $w = z^2$

Conformal Mapping

A mapping $w = f(z)$ is called **conformal** if it preserves angles between oriented curves in magnitude as well as in sense. Figure 377 shows what this means. The **angle** α $(0 \leqq \alpha \leqq \pi)$ between two intersecting curves C_1 and C_2 is defined to be the angle between their oriented tangents at the intersection point z_0. And *conformality* means that the images $C_1{}^*$ and $C_2{}^*$ of C_1 and C_2 make the same angle as the curves themselves in both magnitude and direction.

THEOREM 1

Conformality of Mapping by Analytic Functions

The mapping $w = f(z)$ by an analytic function f is conformal, except at **critical points,** *that is, points at which the derivative f' is zero.*

PROOF $w = z^2$ has a critical point at $z = 0$, where $f'(z) = 2z = 0$ and the angles are doubled (see Fig. 375), so that conformality fails.

The idea of proof is to consider a curve

$$C\colon\ z(t) = x(t) + iy(t) \tag{2}$$

in the domain of $f(z)$ and to show that $w = f(z)$ rotates all tangents at a point z_0 (where $f'(z_0) \neq 0$) through the same angle. Now $\dot{z}(t) = dz/dt = \dot{x}(t) + i\,\dot{y}(t)$ is tangent to C in (2) because this is the limit of $(z_1 - z_0)/\Delta t$ (which has the direction of the secant $z_1 - z_0$

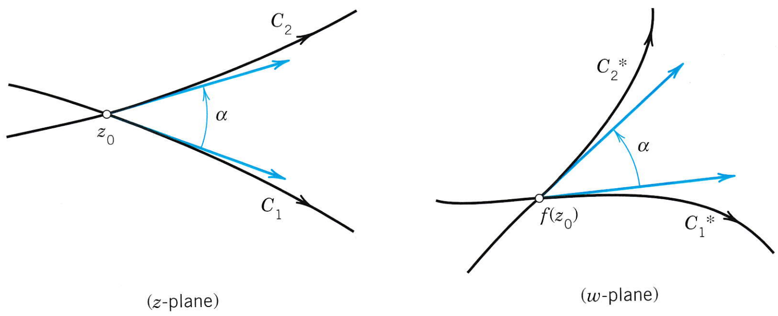


Fig. 377. Curves C_1 and C_2 and their respective images $C_1{}^*$ and $C_2{}^*$ under a conformal mapping $w = f(z)$

in Fig. 378) as z_1 approaches z_0 along C. The image C^* of C is $w = f(z(t))$. By the chain rule, $\dot{w} = f'(z(t))\dot{z}(t)$. Hence the tangent direction of C^* is given by the argument (use (9) in Sec. 13.2)

$$\arg \dot{w} = \arg f' + \arg \dot{z} \tag{3}$$

where $\arg \dot{z}$ gives the tangent direction of C. This shows that the mapping rotates *all* directions at a point z_0 in the domain of analyticity of f through the same angle $\arg f'(z_0)$, which exists as long as $f'(z_0) \neq 0$. But this means conformality, as Fig. 377 illustrates for an angle α between two curves, whose images $C_1{}^*$ and $C_2{}^*$ make the same angle (because of the rotation). ■

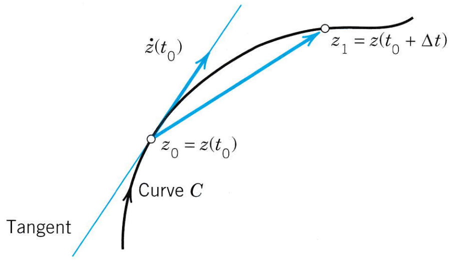


Fig. 378. Secant and tangent of the curve C

In the remainder of this section and in the next ones we shall consider various conformal mappings that are of practical interest, for instance, in modeling potential problems.

EXAMPLE 2 Conformality of $w = z^n$

The mapping $w = z^n$, $n = 2, 3, \cdots$, is conformal, except at $z = 0$, where $w' = nz^{n-1} = 0$. For $n = 2$ this is shown in Fig. 375; we see that at 0 the angles are doubled. For general n the angles at 0 are multiplied by a factor n under the mapping. Hence the sector $0 \leqq \theta \leqq \pi/n$ is mapped by z^n onto the upper half-plane $v \geqq 0$ (Fig. 379). ■

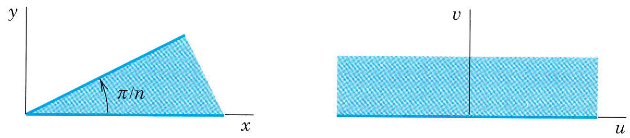


Fig. 379. Mapping by $w = z^n$

EXAMPLE 3 Mapping $w = z + 1/z$. Joukowski Airfoil

In terms of polar coordinates this mapping is

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta).$$

By separating the real and imaginary parts we thus obtain

$$u = a\cos\theta, \qquad v = b\sin\theta \qquad \text{where} \qquad a = r + \frac{1}{r}, \quad b = r - \frac{1}{r}.$$

Hence circles $|z| = r = \mathit{const} \neq 1$ are mapped onto ellipses $x^2/a^2 + y^2/b^2 = 1$. The circle $r = 1$ is mapped onto the segment $-2 \leqq u \leqq 2$ of the u-axis. See Fig. 380.

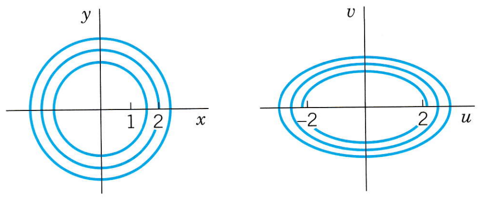


Fig. 380. Example 3

Now the derivative of w is

$$w' = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2}$$

which is 0 at $z = \pm 1$. These are the points at which the mapping is not conformal. The two circles in Fig. 381 pass through $z = -1$. The larger is mapped onto a *Joukowski airfoil.* The dashed circle passes through both -1 and 1 and is mapped onto a curved segment.

Another interesting application of $w = z + 1/z$ (the flow around a cylinder) will be considered in Sec. 18.4. ■

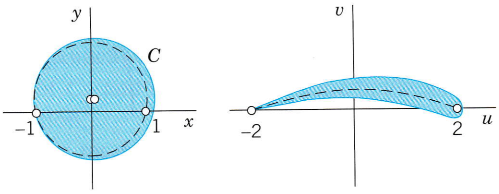


Fig. 381. Joukowski airfoil

EXAMPLE 4 **Conformality of $w = e^z$**

From (10) in Sec. 13.5 we have $|e^z| = e^x$ and Arg $z = y$. Hence e^z maps a vertical straight line $x = x_0 = const$ onto the circle $|w| = e^{x_0}$ and a horizontal straight line $y = y_0 = const$ onto the ray arg $w = y_0$. The rectangle in Fig. 382 is mapped onto a region bounded by circles and rays as shown.

The fundamental region $-\pi <$ Arg $z \leqq \pi$ of e^z in the z-plane is mapped bijectively and conformally onto the entire w-plane without the origin $w = 0$ (because $e^z = 0$ for no z). Figure 383 shows that the upper half $0 < y \leqq \pi$ of the fundamental region is mapped onto the upper half-plane $0 <$ arg $w \leqq \pi$, the left half being mapped inside the unit disk $|w| \leqq 1$ and the right half outside (why?). ■

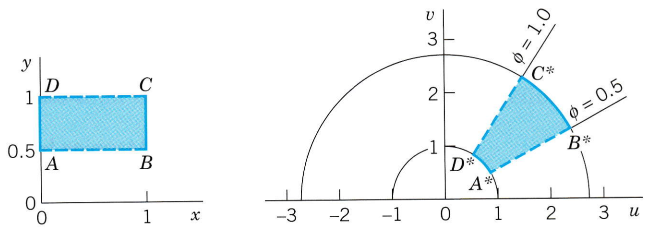


Fig. 382. Mapping by $w = e^z$

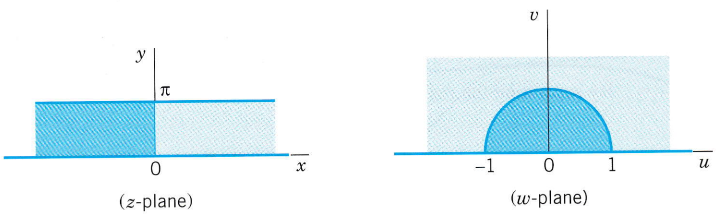


Fig. 383. Mapping by $w = e^z$

EXAMPLE 5 Principle of Inverse Mapping. Mapping $w = \mathrm{Ln}\, z$

Principle. *The mapping by the inverse $z = f^{-1}(w)$ of $w = f(z)$ is obtained by interchanging the roles of the z-plane and the w-plane in the mapping by $w = f(z)$.*

Now the principal value $w = f(z) = \mathrm{Ln}\, z$ of the natural logarithm has the inverse $z = f^{-1}(w) = e^w$. From Example 4 (with the notations z and w interchanged!) we know that $f^{-1}(w) = e^w$ maps the fundamental region of the exponential function onto the z-plane without $z = 0$ (because $e^w \neq 0$ for every w). Hence $w = f(z) = \mathrm{Ln}\, z$ maps the z-plane without the origin and cut along the negative real axis (where $\theta = \mathrm{Im}\,\mathrm{Ln}\, z$ jumps by 2π) conformally onto the horizontal strip $-\pi < v \leqq \pi$ of the w-plane, where $w = u + iv$.

Since the mapping $w = \mathrm{Ln}\, z + 2\pi i$ differs from $w = \mathrm{Ln}\, z$ by the translation $2\pi i$ (vertically upward), this function maps the z-plane (cut as before and 0 omitted) onto the strip $\pi < v \leqq 3\pi$. Similarly for each of the infinitely many mappings $w = \ln z = \mathrm{Ln}\, z \pm 2n\pi i$ ($n = 0, 1, 2, \cdots$). The corresponding horizontal strips of width 2π (images of the z-plane under these mappings) together cover the whole w-plane without overlapping. ■

Magnification Ratio. By the definition of the derivative we have

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|. \tag{4}$$

Therefore, the mapping $w = f(z)$ magnifies (or shortens) the lengths of short lines by approximately the factor $|f'(z_0)|$. The image of a small figure ***conforms*** to the original figure in the sense that it has approximately the same shape. However, since $f'(z)$ varies from point to point, a *large* figure may have an image whose shape is quite different from that of the original figure.

More on the Condition $f'(z) \neq 0$. From (4) in Sec. 13.4 and the Cauchy–Riemann equations we obtain

$$|f'(z)|^2 = \left| \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{5'}$$

that is,

$$|f'(z)|^2 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}. \tag{5}$$

This determinant is the so-called **Jacobian** (Sec. 10.3) of the transformation $w = f(z)$ written in real form $u = u(x, y)$, $v = v(x, y)$. Hence $f'(z_0) \neq 0$ implies that the Jacobian is not 0 at z_0. This condition is sufficient that the mapping $w = f(z)$ in a sufficiently small neighborhood of z_0 is one-to-one or injective (different points have different images). See Ref. [GR4] in App. 1.

PROBLEM SET 17.1

1. Verify all calculations in Example 1.
2. Why do the images of the curves $|z| = const$ and $\arg z = const$ under a mapping by an analytic function $f(z)$ intersect at right angles, except at points at which $f'(z) = 0$?
3. Does the mapping $w = \bar{z} = x - iy$ preserve angles in size as well as in sense?

4–6 MAPPING OF CURVES

Find and sketch or graph the image of the given curves under the given mapping.

4. $x = 1, 2, 3, 4$, $y = 1, 2, 3, 4$; $w = z^2$
5. Curves as in Prob. 4, $w = iz$ **(Rotation)**
6. $|z| = 1/3, 1/2, 1, 2, 3$; $\mathrm{Arg}\, z = 0, \pm\pi/4, \pm\pi/2, \pm 3\pi/2, \pm\pi$; $w = 1/z$

7–15 MAPPING OF REGIONS

Find and sketch or graph the image of the given region under the given mapping.

7. $-\pi/4 < \operatorname{Arg} z < \pi/4,\ |z| < 1/2,\ w = z^3$
8. $x \geqq 1,\ w = 1/z$
9. $|z| > 1,\ w = 3z$
10. $\operatorname{Im} z > 0,\ w = 1 - z$
11. $x \geqq 0,\ y \geqq 0,\ |z| \leqq 4;\ w = z^2$
12. $-1 \leqq x \leqq 1,\ -\pi < y < \pi;\ w = e^z$
13. $\ln 3 < x < \ln 5,\ w = e^z$
14. $-\pi < y \leqq 3\pi,\ w = e^z$
15. $2 \leqq |z| \leqq 3,\ \pi/4 \leqq \theta \leqq \pi/2;\ w = \operatorname{Ln} z$

16. **CAS EXPERIMENT. Orthogonal Nets.** Graph the orthogonal net of the two families of level curves $\operatorname{Re} f(z) = const$ and $\operatorname{Im} f(z) = const$, where (a) $f(z) = z^4$, (b) $f(z) = 1/z$, (c) $f(z) = 1/z^2$, (d) $f(z) = (z + i)/(1 + iz)$. Why do these curves generally intersect at right angles? In your work, experiment to get the best possible graphs. Also do the same for other functions of your own choice. Observe and record shortcomings of your CAS and means to overcome such deficiencies.

17–23 FAILURE OF CONFORMALITY

Find all points at which the following mappings are not conformal.

17. $z(z^4 - 5)$
18. $z^2 + 1/z^2$
19. $\cos \pi z$
20. $\cosh 2z$
21. $z^2 + az + b$
22. $\exp(z^5 - 80z)$
23. $(z - a)^3,\ (z^3 - a)^2$

24–28 MAGNIFICATION RATIO, JACOBIAN

Find the magnification ratio M. Describe what it tells you about the mapping. Where is M equal to 1? Find the Jacobian J.

24. $w = \frac{1}{2}z^2$
25. $w = e^z$
26. $w = z^3$
27. $w = \operatorname{Ln} z$
28. $w = 1/z$

29. **Magnification of Angles.** Let $f(z)$ be analytic at z_0. Suppose that $f'(z_0) = 0, \cdots, f^{(k-1)}(z_0) = 0$. Then the mapping $w = f(z)$ magnifies angles with vertex at z_0 by a factor k. Illustrate this with examples for $k = 2, 3, 4$.

30. Prove the statement in Prob. 29 for general $k = 1, 2, \cdots$. *Hint.* Use the Taylor series.

17.2 Linear Fractional Transformations

Conformal mappings can help in modeling and solving boundary value problems by first mapping regions conformally onto another. We shall explain this for standard regions (disks, half-planes, strips) in the next section. For this it is useful to know properties of special basic mappings. Accordingly, let us begin with the following very important class.

Linear fractional transformations (or **Möbius transformations**) are mappings

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

where a, b, c, d are complex or real numbers. Differentiation gives

$$w' = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2}. \tag{2}$$

This motivates our requirement $ad - bc \neq 0$. It implies conformality for all z and excludes the totally uninteresting case $w' \equiv 0$ once and for all. Special cases of (1) are

$$\begin{aligned} &w = z + b && \textit{(Translations)} \\ &w = az \quad \text{with } |a| = 1 && \textit{(Rotations)} \\ &w = az + b && \textit{(Linear transformations)} \\ &w = 1/z && \textit{(Inversion in the unit circle).} \end{aligned} \tag{3}$$

EXAMPLE 1 **Properties of the Inversion $w = 1/z$ (Fig. 384)**

In polar forms $z = re^{i\theta}$ and $w = Re^{i\phi}$ the inversion $w = 1/z$ is

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \qquad \text{and gives} \qquad R = \frac{1}{r}, \qquad \phi = -\theta.$$

Hence the unit circle $|z| = r = 1$ is mapped onto the unit circle $|w| = R = 1$; $w = e^{i\phi} = e^{-i\theta}$. For a general z the image $w = 1/z$ can be found geometrically by marking $|w| = R = 1/r$ on the segment from 0 to z and then reflecting the mark in the real axis. (Make a sketch.)

Figure 384 shows that $w = 1/z$ maps horizontal and vertical straight lines onto circles or straight lines. Even the following is true.

$w = 1/z$ maps every straight line or circle onto a circle or straight line.

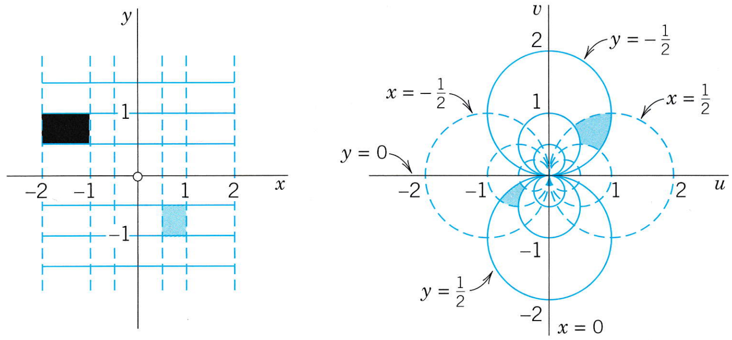


Fig. 384. Mapping (Inversion) $w = 1/z$

Proof. Every straight line or circle in the z-plane can be written

$$A(x^2 + y^2) + Bx + Cy + D = 0 \qquad (A, B\ C, D \text{ real}).$$

$A = 0$ gives a straight line and $A \neq 0$ a circle. In terms of z and $\bar{z}$ this equation becomes

$$Az\bar{z} + B\frac{z + \bar{z}}{2} + C\frac{z - \bar{z}}{2i} + D = 0.$$

Now $w = 1/z$. Substitution of $z = 1/w$ and multiplication by $w\bar{w}$ gives the equation

$$A + B\frac{\bar{w} + w}{2} + C\frac{\bar{w} - w}{2i} + Dw\bar{w} = 0$$

or, in terms of u and v,

$$A + Bu - Cv + D(u^2 + v^2) = 0.$$

This represents a circle (if $D \neq 0$) or a straight line (if $D = 0$) in the w-plane. ■

The proof in this example suggests the use of z and $\bar{z}$ instead of x and y, a ***general principle*** that is often quite useful in practice.

Surprisingly, *every* linear fractional transformation has the property just proved:

THEOREM 1 **Circles and Straight Lines**

Every linear fractional transformation (1) maps the totality of circles and straight lines in the z-plane onto the totality of circles and straight lines in the w-plane.

PROOF This is trivial for a translation or rotation, fairly obvious for a uniform expansion or contraction, and true for $w = 1/z$, as just proved. Hence it also holds for composites of these special mappings. Now comes the key idea of the proof: represent (1) in terms of these special mappings. When $c = 0$, this is easy. When $c \neq 0$, the representation is

$$w = K \frac{1}{cz + d} + \frac{a}{c} \qquad \text{where} \qquad K = -\frac{ad - bc}{c}.$$

This can be verified by substituting K, taking the common denominator and simplifying; this yields (1). We can now set

$$w_1 = cz, \qquad w_2 = w_1 + d, \qquad w_3 = \frac{1}{w_2}, \qquad w_4 = Kw_3,$$

and see from the previous formula that then $w = w_4 + a/c$. This tells us that (1) is indeed a composite of those special mappings and completes the proof. ■

Extended Complex Plane

The extended complex plane (the complex plane together with the point ∞ in Sec. 16.2) can now be motivated even more naturally by linear fractional transformations as follows.

To each z for which $cz + d \neq 0$ there corresponds a unique w in (1). Now let $c \neq 0$. Then for $z = -d/c$ we have $cz + d = 0$, so that no w corresponds to this z. This suggests that we let $w = \infty$ be the image of $z = -d/c$.

Also, the **inverse mapping** of (1) is obtained by solving (1) for z; this gives again a linear fractional transformation

(4) $$z = \frac{dw - b}{-cw + a}.$$

When $c \neq 0$, then $cw - a = 0$ for $w = a/c$, and we let a/c be the image of $z = \infty$. With these settings, the linear fractional transformation (1) is now a one-to-one mapping of the extended z-plane onto the extended w-plane. We also say that every linear fractional transformation maps "the extended complex plane in a one-to-one manner onto itself."

Our discussion suggests the following.

General Remark. If $z = \infty$, then the right side of (1) becomes the meaningless expression $(a \cdot \infty + b)/(c \cdot \infty + d)$. We assign to it the value $w = a/c$ if $c \neq 0$ and $w = \infty$ if $c = 0$.

Fixed Points

Fixed points of a mapping $w = f(z)$ are points that are mapped onto themselves, are "kept fixed" under the mapping. Thus they are obtained from

$$w = f(z) = z.$$

The **identity mapping** $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points, $w = 1/z$ has two, a rotation has one, and a translation none in the finite plane. (Find them in each case.) For (1), the fixed-point condition $w = z$ is

(5) $$z = \frac{az + b}{cz + d}, \qquad \text{thus} \qquad cz^2 - (a - d)z - b = 0.$$

This is a quadratic equation in z whose coefficients all vanish if and only if the mapping is the identity mapping $w = z$ (in this case, $a = d \neq 0$, $b = c = 0$). Hence we have

THEOREM 2

Fixed Points

A linear fractional transformation, not the identity, has at most two fixed points. If a linear fractional transformation is known to have three or more fixed points, it must be the identity mapping $w = z$.

To make our present general discussion of linear fractional transformations even more useful from a practical point of view, we extend it by further facts and typical examples, in the problem set as well as in the next section.

PROBLEM SET 17.2

1. Verify the calculations in the proof of Theorem 1.
2. **(Composition of LFTs)** Show that substituting a linear fractional transformation (LFT) into a LFT gives a LFT.
3. **(Matrices)** If you are familiar with 2×2 matrices, prove that the coefficient matrices of (1) and (4) are inverses of each other, provided $ad - bc = 1$, and that the composition of LFTs corresponds to the multiplication of the coefficient matrices.

4–7 INVERSE

Find the inverse $z = z(w)$. Check the result by solving $z(w)$ for w.

4. $w = \dfrac{4z + i}{-3iz + 1}$
5. $w = \dfrac{3z}{2z - i}$
6. $w = \dfrac{z + i}{z - i}$
7. $w = \dfrac{2z + 5i}{4z}$

8–14 FIXED POINTS

Find the fixed points.

8. $w = 81z^5$
9. $w = (4 + i)z$
10. $w = z + 4i$
11. $w = (z - i)^2$
12. $w = \dfrac{z - 1}{z + 1}$
13. $w = \dfrac{2iz - 1}{z + 2i}$
14. $w = \dfrac{3z + 2}{z - 1}$

15. Find a LFT whose (only) fixed points are -2 and 2.
16. Find a LFT (not $w = z$) with fixed points 0 and 1.
17. Find all LFTs with fixed points -1 and 1.
18. Find all LFTs whose only fixed point is 0.
19. Find all LFTs with fixed points 0 and ∞.
20. Find all LFTs without fixed points in the finite plane.

17.3 Special Linear Fractional Transformations

In this section we shall see how to determine linear fractional transformations

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

for mapping certain standard domains onto others and how to discuss properties of (1).

A mapping (1) is determined by a, b, c, d, actually by the ratios of three of these constants to the fourth because we can drop or introduce a common factor. This makes it plausible that three conditions determine a unique mapping (1):

THEOREM 1

Three Points and Their Images Given

Three given distinct points z_1, z_2, z_3 can always be mapped onto three prescribed distinct points w_1, w_2, w_3 by one, and only one, linear fractional transformation $w = f(z)$. This mapping is given implicitly by the equation

$$\text{(2)} \qquad \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

(If one of these points is the point ∞, the quotient of the two differences containing this point must be replaced by 1.)

PROOF Equation (2) is of the form $F(w) = G(z)$ with linear fractional F and G. Hence $w = F^{-1}(G(z)) = f(z)$, where F^{-1} is the inverse of F and is linear fractional (see (4) in Sec. 17.2) and so is the composite $F^{-1}(G(z))$ (by Prob. 21), that is, $w = f(z)$ is linear fractional. Now if in (2) we set $w = w_1, w_2, w_3$ on the left and $z = z_1, z_2, z_3$ on the right, we see that

$$F(w_1) = 0, \qquad F(w_2) = 1, \qquad F(w_3) = \infty$$
$$G(z_1) = 0, \qquad G(z_2) = 1, \qquad G(z_3) = \infty.$$

From the first column, $F(w_1) = G(z_1)$, thus $w_1 = F^{-1}(G(z_1)) = f(z_1)$. Similarly, $w_2 = f(z_2)$, $w_3 = f(z_3)$. This proves the existence of the desired linear fractional transformation.

To prove uniqueness, let $w = g(z)$ be a linear fractional transformation, which also maps z_j onto w_j, $j = 1, 2, 3$. Thus $w_j = g(z_j)$. Hence $g^{-1}(w_j) = z_j$, where $w_j = f(z_j)$. Together, $g^{-1}(f(z_j)) = z_j$, a mapping with the three fixed points z_1, z_2, z_3. By Theorem 2 in Sec. 17.2, this is the identity mapping, $g^{-1}(f(z)) = z$ for all z. Thus $f(z) = g(z)$ for all z, the uniqueness.

The last statement of Theorem 1 follows from the General Remark in Sec. 17.2. ■

Mapping of Standard Domains by Theorem 1

Using Theorem 1, we can now find linear fractional transformations according to the following

Principle. Prescribe three boundary points z_1, z_2, z_3 of the domain D in the z-plane. Choose their images w_1, w_2, w_3 on the boundary of the image D^* of D in the w-plane. Obtain the mapping from (2). Make sure that D is mapped onto D^*, not onto its complement. In the latter case, interchange two w-points. (Why does this help?)

EXAMPLE 1 **Mapping of a Half-Plane onto a Disk (Fig. 385)**

Find the linear fractional transformation (1) that maps $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$, respectively.

Solution. From (2) we obtain

$$\frac{w - (-1)}{w - 1} \cdot \frac{-i - 1}{-i - (-1)} = \frac{z - (-1)}{z - 1} \cdot \frac{0 - 1}{0 - (-1)},$$

thus

$$w = \frac{z - i}{-iz + 1}.$$

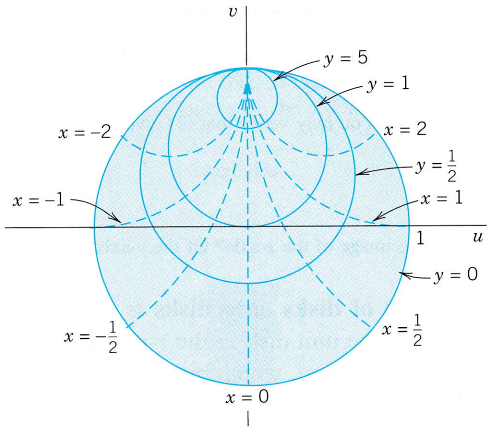


Fig. 385. Linear fractional transformation in Example 1

Let us show that we can determine the specific properties of such a mapping without much calculation. For $z = x$ we have $w = (x - i)/(-ix + 1)$, thus $|w| = 1$, so that the x-axis maps onto the unit circle. Since $z = i$ gives $w = 0$, the upper half-plane maps onto the interior of that circle and the lower half-plane onto the exterior. $z = 0, i, \infty$ go onto $w = -i, 0, i$, so that the positive imaginary axis maps onto the segment S: $u = 0$, $-1 \leqq v \leqq 1$. The vertical lines $x = const$ map onto circles (by Theorem 1, Sec. 17.2) through $w = i$ (the image of $z = \infty$) and perpendicular to $|w| = 1$ (by conformality; see Fig. 385). Similarly, the horizontal lines $y = const$ map onto circles through $w = i$ and perpendicular to S (by conformality). Figure 385 gives these circles for $y \geqq 0$, and for $y < 0$ they lie outside the unit disk shown. ■

EXAMPLE 2 **Occurrence of ∞**

Determine the linear fractional transformation that maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$, respectively.

Solution. From (2) we obtain the desired mapping

$$w = \frac{z - i}{z + i}.$$

This is sometimes called the *Cayley transformation.*[2] In this case, (2) gave at first the quotient $(1 - \infty)/(z - \infty)$, which we had to replace by 1. ■

EXAMPLE 3 **Mapping of a Disk onto a Half-Plane**

Find the linear fractional transformation that maps $z_1 = -1$, $z_2 = i$, $z_3 = 1$ onto $w_1 = 0$, $w_2 = i$, $w_3 = \infty$, respectively, such that the unit disk is mapped onto the right half-plane. (Sketch disk and half-plane.)

Solution. From (2) we obtain, after replacing $(i - \infty)/(w - \infty)$ by 1,

$$w = -\frac{z + 1}{z - 1}.$$ ■

Mapping half-planes onto half-planes is another task of practical interest. For instance, we may wish to map the upper half-plane $y \geqq 0$ onto the upper half-plane $v \geqq 0$. Then the x-axis must be mapped onto the u-axis.

[2]ARTHUR CAYLEY (1821–1895), English mathematician and professor at Cambridge, is known for his important work in algebra, matrix theory, and differential equations.

EXAMPLE 4 **Mapping of a Half-Plane onto a Half-Plane**

Find the linear fractional transformation that maps $z_1 = -2$, $z_2 = 0$, $z_3 = 2$ onto $w_1 = \infty$, $w_2 = 1/4$, $w_3 = 3/8$, respectively.

Solution. You may verify that (2) gives the mapping function

$$w = \frac{z + 1}{2z + 4}.$$

What is the image of the x-axis? Of the y-axis? ■

Mappings of disks onto disks is a third class of practical problems. We may readily verify that the unit disk in the z-plane is mapped onto the unit disk in the w-plane by the following function, which maps z_0 onto the center $w = 0$.

$$\text{(3)} \qquad w = \frac{z - z_0}{cz - 1}, \qquad c = \bar{z}_0, \qquad |z_0| < 1.$$

To see this, take $|z| = 1$, obtaining, with $c = \bar{z}_0$ as in (3),

$$\begin{aligned} |z - z_0| &= |\bar{z} - c| \\ &= |z|\,|\bar{z} - c| \\ &= |z\bar{z} - cz| = |1 - cz| = |cz - 1|. \end{aligned}$$

Hence

$$|w| = |z - z_0|/|cz - 1| = 1$$

from (3), so that $|z| = 1$ maps onto $|w| = 1$, as claimed, with z_0 going onto 0, as the numerator in (3) shows.

Formula (3) is illustrated by the following example. Another interesting case will be given in Prob. 10 of Sec. 18.2.

EXAMPLE 5 **Mapping of the Unit Disk onto the Unit Disk**

Taking $z_0 = \frac{1}{2}$ in (3), we obtain (verify!)

$$w = \frac{2z - 1}{z - 2} \qquad \text{(Fig. 386).}$$ ■

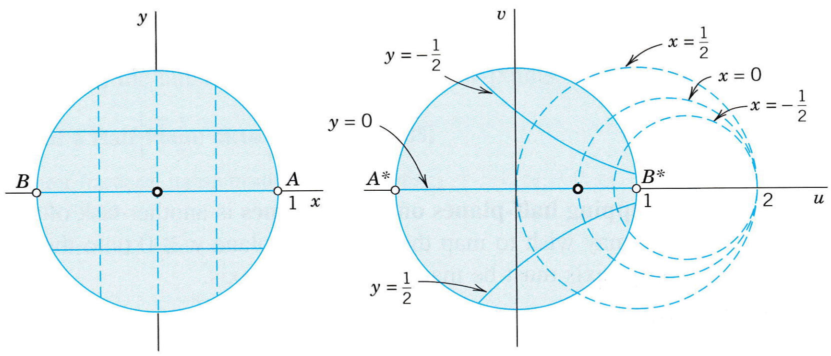


Fig. 386. Mapping in Example 5

EXAMPLE 6 Mapping of an Angular Region onto the Unit Disk

Certain mapping problems can be solved by combining linear fractional transformations with others. For instance, to map the angular region D: $-\pi/6 \leqq \arg z \leqq \pi/6$ (Fig. 387) onto the unit disk $|w| \leqq 1$, we may map D by $Z = z^3$ onto the right Z-half-plane and then the latter onto the disk $|w| \leqq 1$ by

$$w = i\,\frac{Z-1}{Z+1}, \qquad \text{combined} \qquad w = i\,\frac{z^3-1}{z^3+1}.$$

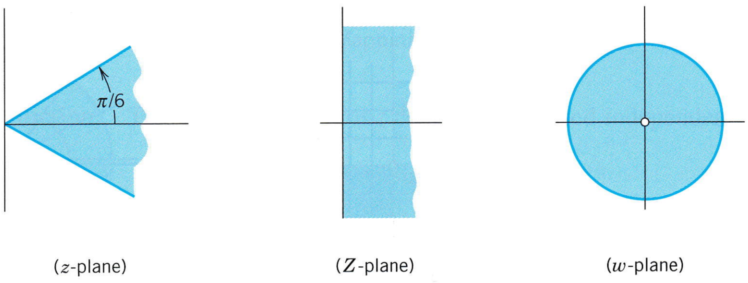


Fig. 387. Mapping in Example 6

This is the end of our discussion of linear fractional transformations. In the next section we turn to conformal mappings by other analytic functions (sine, cosine, etc.).

PROBLEM SET 17.3

1. Derive the mapping in Example 2 from (2).
2. **(Inverse)** Find the inverse of the mapping in Example 1. Show that under that inverse the lines $x = const$ are the images of circles in the w-plane with centers on the line $v = 1$.
3. Verify the formula (3) for disks.
4. Derive the mapping in Example 4 from (2). Find its inverse and prove by calculation that it has the same fixed points as the mapping itself. Is this surprising?
5. **(Inverse)** If $w = f(z)$ is any transformation that has an inverse, prove the (trivial!) fact that f and its inverse have the same fixed points.
6. **CAS EXPERIMENT. Linear Fractional Transformations (LFTs). (a)** Graph typical regions (squares, disks, etc.) and their images under the LFTs in Examples 1–5.
(b) Make an experimental study of the continuous dependence of LFTs on their coefficients. For instance, change the LFT in Example 4 continuously and graph the changing image of a fixed region (applying animation if available).

7–15 LFTs FROM THREE POINTS AND THEIR IMAGES

Find the LFT that maps the given three points onto the three given points in the respective order.

7. $-1, 0, 1$ onto $-0.6 - 0.8i, -1, -0.6, + 0.8i$
8. $0, 1, 2$ onto $1, \frac{1}{2}, \frac{1}{3}$
9. $2i, -2i, 4$ onto $-4 + 2i, -4 - 2i, 0$
10. $i, -1, 1$ onto $-1, -i, i$
11. $0, 1, \infty$ onto $\infty, 1, 0$
12. $0, -i, i$ onto $-1, 0, \infty$
13. $2i, i, 0$ onto $\frac{5}{2}i, 2i, \infty$
14. $0, 2i, -2i$ onto $-1, 0, \infty$
15. $-1, 0, 1$ onto $0, 1, -1$
16. Find all LFTs $w(z)$ that map the x-axis onto the u-axis.
17. Find a LFT that maps $|z| \leqq 1$ onto $|w| \leqq 1$ so that $z = i/2$ is mapped onto $w = 0$. Sketch the images of the lines $x = const$ and $y = const$.
18. Find an analytic function that maps the second quadrant of the z-plane onto the interior of the unit circle in the w-plane.
19. Find an analytic function $w = f(z)$ that maps the region $0 \leqq \arg z \leqq \pi/4$ onto the unit disk $|w| \leqq 1$.
20. **(Composite)** Show that the composite of two LFTs is a LFT.

17.4 Conformal Mapping by Other Functions

So far we have discussed the mapping by z^n, e^z (Sec. 17.1) and linear fractional transformations (Secs. 17.2, 17.3), and we shall now turn to the mapping by trigonometric and hyperbolic analytic functions.

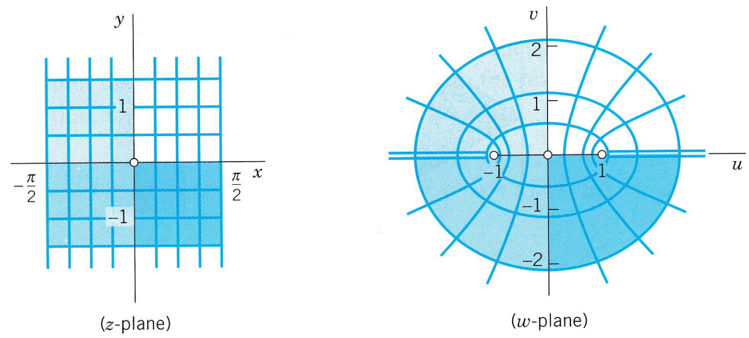


Fig. 388. Mapping $w = u + iv = \sin z$

Sine Function. Figure 388 shows the mapping by

$$w = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y \qquad \text{(Sec. 13.6).} \tag{1}$$

Hence

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y. \tag{2}$$

Since $\sin z$ is periodic with period 2π, the mapping is certainly not one-to-one if we consider it in the full z-plane. We restrict z to the vertical strip S: $-\frac{1}{2}\pi \leqq x \leqq \frac{1}{2}\pi$ in Fig. 388. Since $f'(z) = \cos z = 0$ at $z = \pm\frac{1}{2}\pi$, the mapping is not conformal at these two critical points. We claim that the rectangular net of straight lines $x = const$ and $y = const$ in Fig. 388 is mapped onto a net in the w-plane consisting of hyperbolas (the images of the vertical lines $x = const$) and ellipses (the images of the horizontal lines $y = const$) intersecting the hyperbolas at right angles (conformality!). Corresponding calculations are simple. From (2) and the relations $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$ we obtain

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1 \qquad \text{(Hyperbolas)}$$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = \sin^2 x + \cos^2 x = 1 \qquad \text{(Ellipses).}$$

Exceptions are the vertical lines $x = \pm\frac{1}{2}\pi$, which are "folded" onto $u \leqq -1$ and $u \geqq 1$ $(v = 0)$, respectively.

Figure 389 illustrates this further. The upper and lower sides of the rectangle are mapped onto semi-ellipses and the vertical sides onto $-\cosh 1 \leqq u \leqq -1$ and $1 \leqq u \leqq \cosh 1$ $(v = 0)$, respectively. An application to a potential problem will be given in Prob. 5 of Sec. 18.2.

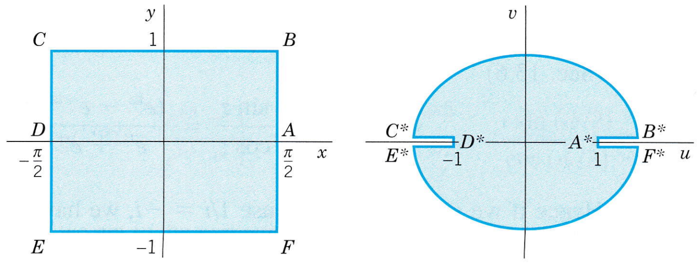


Fig. 389. Mapping by $w = \sin z$

Cosine Function. The mapping $w = \cos z$ could be discussed independently, but since

$$w = \cos z = \sin\left(z + \tfrac{1}{2}\pi\right), \tag{3}$$

we see at once that this is the same mapping as $\sin z$ preceded by a translation to the right through $\frac{1}{2}\pi$ units.

Hyperbolic Sine. Since

$$w = \sinh z = -i \sin(iz), \tag{4}$$

the mapping is a counterclockwise rotation $Z = iz$ through $\frac{1}{2}\pi$ (i.e., 90°), followed by the sine mapping $Z^* = \sin Z$, followed by a clockwise 90°-rotation $w = -iZ^*$.

Hyperbolic Cosine. This function

$$w = \cosh z = \cos(iz) \tag{5}$$

defines a mapping that is a rotation $Z = iz$ followed by the mapping $w = \cos Z$.

Figure 390 shows the mapping of a semi-infinite strip onto a half-plane by $w = \cosh z$. Since $\cosh 0 = 1$, the point $z = 0$ is mapped onto $w = 1$. For real $z = x \geqq 0$, $\cosh z$ is real and increases with increasing x in a monotone fashion, starting from 1. Hence the positive x-axis is mapped onto the portion $u \geqq 1$ of the u-axis.

For pure imaginary $z = iy$ we have $\cosh iy = \cos y$. Hence the left boundary of the strip is mapped onto the segment $1 \geqq u \geqq -1$ of the u-axis, the point $z = \pi i$ corresponding to

$$w = \cosh i\pi = \cos \pi = -1.$$

On the upper boundary of the strip, $y = \pi$, and since $\sin \pi = 0$ and $\cos \pi = -1$, it follows that this part of the boundary is mapped onto the portion $u \leqq -1$ of the u-axis. Hence the boundary of the strip is mapped onto the u-axis. It is not difficult to see that the interior of the strip is mapped onto the upper half of the w-plane, and the mapping is one-to-one.

This mapping in Fig. 390 has applications in potential theory, as we shall see in Prob. 12 of Sec. 18.3.

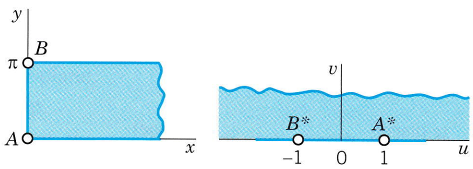


Fig. 390. Mapping by $w = \cosh z$

Tangent Function. Figure 391 shows the mapping of a vertical infinite strip onto the unit circle by $w = \tan z$, accomplished in three steps as suggested by the representation (Sec. 13.6)

$$w = \tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz} - e^{-iz})/i}{e^{iz} + e^{-iz}} = \frac{(e^{2iz} - 1)/i}{e^{2iz} + 1}.$$

Hence if we set $Z = e^{2iz}$ and use $1/i = -i$, we have

$$w = \tan z = -iW, \qquad W = \frac{Z - 1}{Z + 1}, \qquad Z = e^{2iz}. \tag{6}$$

We now see that $w = \tan z$ is a linear fractional transformation preceded by an exponential mapping (see Sec. 17.1) and followed by a clockwise rotation through an angle $\frac{1}{2}\pi$ (90°).

The strip is S: $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$, and we show that it is mapped onto the unit disk in the w-plane. Since $Z = e^{2iz} = e^{-2y+2ix}$, we see from (10) in Sec. 13.5 that $|Z| = e^{-2y}$, $\operatorname{Arg} Z = 2x$. Hence the vertical lines $x = -\pi/4, 0, \pi/4$ are mapped onto the rays $\operatorname{Arg} Z = -\pi/2, 0, \pi/2$, respectively. Hence S is mapped onto the right Z-half-plane. Also $|Z| = e^{-2y} < 1$ if $y > 0$ and $|Z| > 1$ if $y < 0$. Hence the upper half of S is mapped inside the unit circle $|Z| = 1$ and the lower half of S outside $|Z| = 1$, as shown in Fig. 391.

Now comes the linear fractional transformation in (6), which we denote by $g(Z)$:

$$W = g(Z) = \frac{Z - 1}{Z + 1}. \tag{7}$$

For real Z this is real. Hence the real Z-axis is mapped onto the real W-axis. Furthermore, the imaginary Z-axis is mapped onto the unit circle $|W| = 1$ because for pure imaginary $Z = iY$ we get from (7)

$$|W| = |g(iY)| = \left|\frac{iY - 1}{iY + 1}\right| = 1.$$

The right Z-half-plane is mapped inside this unit circle $|W| = 1$, not outside, because $Z = 1$ has its image $g(1) = 0$ inside that circle. Finally, the unit circle $|Z| = 1$ is mapped

Fig. 391. Mapping by $w = \tan z$

onto the imaginary W-axis, because this circle is $Z = e^{i\phi}$, so that (7) gives a pure imaginary expression, namely,

$$g(e^{i\phi}) = \frac{e^{i\phi} - 1}{e^{i\phi} + 1} = \frac{e^{i\phi/2} - e^{-i\phi/2}}{e^{i\phi/2} + e^{-i\phi/2}} = \frac{i \sin(\phi/2)}{\cos(\phi/2)} .$$

From the W-plane we get to the w-plane simply by a clockwise rotation through $\pi/2$; see (6).

Together we have shown that $w = \tan z$ maps S: $-\pi/4 < \text{Re } z < \pi/4$ onto the unit disk $|w| = 1$, with the four quarters of S mapped as indicated in Fig. 391. This mapping is conformal and one-to-one.

PROBLEM SET 17.4

1–7 CONFORMAL MAPPING $w = e^z$

Find and sketch the image of the given region under $w = e^z$.

1. $0 \leqq x \leqq 2,\ -\pi \leqq y \leqq \pi$
2. $-1 \leqq x \leqq 0,\ 0 \leqq y \leqq \pi/2$
3. $-0.5 < x < 0.5,\ 3\pi/4 < y < 5\pi/4$
4. $-3 < x < 3,\ \pi/4 < y < 3\pi/4$
5. $0 < x < 1,\ 0 < y < \pi$
6. $x < 0,\ -\pi/2 < y < \pi/2$
7. x arbitrary, $0 \leqq y \leqq 2\pi$

8. **CAS EXPERIMENT. Conformal Mapping.** If your CAS can do conformal mapping, use it to solve Prob. 5. Then increase y beyond π, say, to 50π or 100π. State what you expected. See what you get as the image. Explain.

9–12 CONFORMAL MAPPING $w = \sin z$

Find and sketch or graph the image of the given region under $w = \sin z$.

9. $0 \leqq x \leqq \pi,\ 0 \leqq y \leqq 1$
10. $0 < x < \pi/6$, y arbitrary
11. $0 < x < 2\pi,\ 1 < y < 5$
12. $-\pi/4 < x < \pi/4,\ 0 < y < 3$

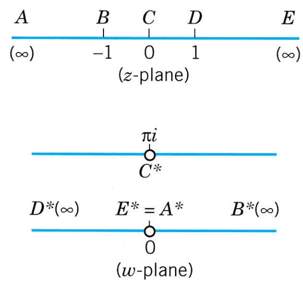
13. Determine all points at which $w = \sin z$ is not conformal.
14. Find and sketch or graph the images of the lines $x = 0$, $\pm\pi/6$, $\pm\pi/3$, $\pm\pi/2$ under the mapping $w = \sin z$.
15. Find an analytic function that maps the region R bounded by the positive x- and y-axes and the hyperbola $xy = \pi/2$ in the first quadrant onto the upper half-plane. *Hint.* First map the region onto a horizontal strip.
16. Describe the mapping $w = \cosh z$ in terms of the mapping $w = \sin z$ and rotations and translations.
17. Find all points at which the mapping $w = \cosh \pi z$ is not conformal.

18–22 CONFORMAL MAPPING $w = \cos z$

Find and sketch or graph the image of the given region under $w = \cos z$.

18. $0 < x < \pi/2,\ 0 < y < 2$
19. $0 < x < \pi,\ 0 < y < 1$
20. $-1 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$
21. $\pi < x < 2\pi,\ y < 0$
22. $0 < x < 2\pi,\ 1/2 < y < 1$

23. Find the images of the lines $y = c = const$ under the mapping $w = \cos z$.
24. Show that $w = \text{Ln}\,\dfrac{z-1}{z+1}$ maps the upper half-plane onto the horizontal strip $0 \leqq \text{Im } w \leqq \pi$ as shown in the figure.



Problem 24

25. Find and sketch the image of R: $2 \leqq |z| \leqq 3$, $\pi/4 \leqq \theta \leqq \pi/2$ under the mapping $w = \text{Ln } z$.

17.5 Riemann Surfaces. *Optional*

Riemann surfaces are surfaces on which multivalued relations, such as $w = \sqrt{z}$ or $w = \ln z$, become single-valued, that is, functions in the usual sense. We explain the idea, which is simple—but ingenious, one of the greatest in complex analysis.

The mapping given by

$$w = u + iv = z^2 \qquad \text{(1)} \qquad \text{(Sec. 17.1)}$$

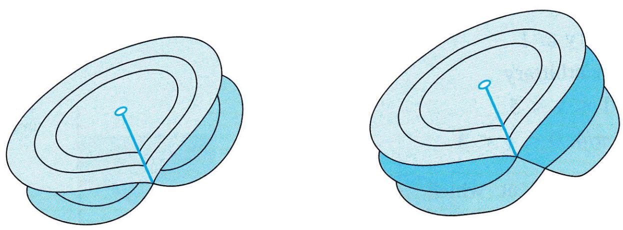
is conformal, except at $z = 0$, where $w' = 2z = 0$. At $z = 0$, angles are doubled under the mapping. Thus the right z-half-plane (including the positive y-axis) is mapped onto the full w-plane, cut along the negative half of the u-axis; this mapping is one-to-one. Similarly for the left z-half-plane (including the negative y-axis). Hence the image of the full z-plane under $w = z^2$ "covers the w-plane twice" in the sense that every $w \neq 0$ is the image of two z-points; if z_1 is one, the other is $-z_1$. For example, $z = i$ and $-i$ are both mapped onto $w = -1$.

Now comes the crucial idea. We place those two copies of the cut w-plane upon each other so that the upper sheet is the image of the right half z-plane R and the lower sheet is the image of the left half z-plane L. We join the two sheets crosswise along the cuts (along the negative u-axis) so that if z moves from R to L, its image can move from the upper to the lower sheet. The two origins are fastened together because $w = 0$ is the image of just one z-point, $z = 0$. The surface obtained is called a **Riemann surface** (Fig. 392a). $w = 0$ is called a "winding point" or **branch point.** $w = z^2$ maps the full z-plane onto this surface in a one-to-one manner.

By interchanging the roles of the variables z and w it follows that the double-valued relation

$$w = \sqrt{z} \qquad \text{(2)} \qquad \text{(Sec. 13.2)}$$

becomes single-valued on the Riemann surface in Fig. 392a, that is, a function in the usual sense. We can let the upper sheet correspond to the principal value of $\sqrt{z}$. Its image is the right w-half-plane. The other sheet is then mapped onto the left w-half-plane.



(a) Riemann surface of $\sqrt{z}$ (b) Riemann surface of $\sqrt[3]{z}$

Fig. 392. Riemann surfaces

Similarly, the triple-valued relation $w = \sqrt[3]{z}$ becomes single-valued on the three-sheeted Riemann surface in Fig. 392b, which also has a branch point at $z = 0$.

The infinitely many-valued natural logarithm (Sec. 13.7)

$$w = \ln z = \text{Ln}\, z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

becomes single-valued on a Riemann surface consisting of infinitely many sheets. $w = \text{Ln}\, z$ corresponds to one of them. This sheet is cut along the negative x-axis and the upper edge of the slit is joined to the lower edge of the next sheet, which corresponds to the argument $\pi < \theta \leqq 3\pi$, that is, to

$$w = \text{Ln}\, z + 2\pi i.$$

The principal value $\text{Ln}\, z$ maps its sheet onto the horizontal strip $-\pi < v \leqq \pi$. The function $w = \text{Ln}\, z + 2\pi i$ maps its sheet onto the neighboring strip $\pi < v \leqq 3\pi$, and so on. The mapping of the points $z \neq 0$ of the Riemann surface onto the points of the w-plane is one-to-one. See also Example 5 in Sec. 17.1.

PROBLEM SET 17.5

1. Consider $w = \sqrt{z}$. Find the path of the image point w of a point z that moves twice around the unit circle, starting from the initial position $z = 1$.
2. Show that the Riemann surface of $w = \sqrt[n]{z}$ consists of n sheets and has a branch point at $z = 0$.
3. Make a sketch, similar to Fig. 392, of the Riemann surface of $\sqrt[4]{z}$.
4. Show that the Riemann surface of $w = \sqrt{(z-1)(z-2)}$ has branch points at $z = 1$ and $z = 2$ and consists of two sheets that may be cut along the line segment from 1 to 2 and joined crosswise. *Hint.* Introduce polar coordinates $z - 1 = r_1 e^{i\theta_1}$, $z - 2 = r_2 e^{i\theta_2}$.

5–10 RIEMANN SURFACES

Find the branch points and the number of sheets of the Riemann surface.

5. $\sqrt{3z + 5}$
6. $\sqrt{(1 - z^2)(4 - z^2)}$
7. $5 + \sqrt[3]{2z + i}$
8. $\ln(3z - 4i)$
9. $e^{\sqrt{z}}$
10. $\sqrt{e^z}$

CHAPTER 17 REVIEW QUESTIONS AND PROBLEMS

1. How did we define the angle of intersection of two oriented curves, and what does it mean to say that a mapping is conformal?
2. At what points is a mapping $w = f(z)$ by an analytic function not conformal? Give examples.
3. What happens to angles at z_0 under a mapping $w = f(z)$ if $f'(z_0) = 0$, $f''(z_0) = 0$, $f'''(z_0) \neq 0$?
4. What do "surjective," "injective," and "bijective" mean?
5. What mapping gave the Joukowski airfoil?
6. What are linear fractional transformations (LFTs)? Why are they important in connection with the extended complex plane?
7. Why did we require that $ad - bc \neq 0$ for a LFT?
8. What are fixed points of a mapping? Give examples.
9. Can you remember mapping properties of $w = \sin z$? $\cos z$? e^z?
10. What is a Riemann surface? Why was it introduced? Explain the simplest example.

11–16 MAPPING $w = z^2$

Find and sketch the image of the given curve or region under $w = z^2$.

11. $y = -1$, $y = 1$
12. $xy = -4$
13. $|z| = 4.5$, $|\arg z| < \pi/4$
14. $0 < y < 2$
15. $\frac{1}{2} < x < 1$
16. $\text{Im}\, z > 0$

17–22 MAPPING $w = 1/z$

Find and sketch the image of the given curve or region under $w = 1/z$.

17. $x = -1$
18. $y = 1$

19. $|z - \frac{1}{2}| = \frac{1}{2}$

20. $|z| < \frac{1}{2}, y < 0$

21. $|\arg z| < \pi/4$

22. $|z| < 1, x < 0, y > 0$

23–28 FAILURE OF CONFORMALITY

Where is the mapping by the given function not conformal? (Give reason.)

23. $5z^7 + 7z^5$

24. $\cosh 2z$

25. $\sin 2z + \cos 2z$

26. $\cos \pi z^2$

27. $\exp(z^4 + z^2)$

28. $z + 1/z \ (z \neq 0)$

29–34 LINEAR FRACTIONAL TRANSFORMATIONS (LFTs)

Find the LFT that maps

29. 0, 1, 2 onto 0, i, $2i$, respectively

30. -1, 1, 2 onto 0, 2, 3/2, respectively

31. 1, -1, $-i$ onto 1, -1, i, respectively

32. -1, $-i$, i onto $1 - i$, 2, 0, respectively

33. 0, ∞, -2 onto 0, 1, ∞, respectively

34. 0, i, $2i$ onto 0, ∞, $2i$

35–40 Fixed Points.

Find all fixed points of

35. $w = \dfrac{z+2}{z+1}$

36. $w = \dfrac{2iz - 1}{z + 2i}$

37. $w = \dfrac{3z+2}{z-1}$

38. $w = \dfrac{iz+5}{5z+i}$

39. $w = \dfrac{(2+i)z+1}{z-i}$

40. $w = z^4 + z - 81$

41–45 GIVEN REGIONS

Find an analytic function $w = f(z)$ that maps:

41. The infinite strip $0 < y < \pi/3$ onto the upper half-plane $v > 0$.

42. The interior of the unit circle $|z| = 1$ onto the exterior of the circle $|w + 1| = 5$.

43. The region $x > 0$, $y > 0$, $xy < k$ onto the strip $0 < v < 1$.

44. The semi-disk $|z| < 1$, $x > 0$ onto the exterior of the unit circle $|w| = 1$.

45. The sector $0 < \arg z < \pi/3$ onto the region $u < 1$.

SUMMARY OF CHAPTER 17
Conformal Mapping

A complex function $w = f(z)$ gives a **mapping** of its domain of definition in the complex z-plane onto its range of values in the complex w-plane. If $f(z)$ is *analytic,* this mapping is **conformal,** that is, angle-preserving: the images of any two intersecting curves make the same angle of intersection, in both magnitude and sense, as the curves themselves (Sec. 17.1). Exceptions are the points at which $f'(z) = 0$ (**"critical points,"** e.g. $z = 0$ for $w = z^2$).

For mapping properties of e^z, $\cos z$, $\sin z$, etc. see Secs. 17.1 and 17.4.

Linear fractional transformations, also called *Möbius transformations*

$$w = \frac{az + b}{cz + d} \qquad (1) \qquad \text{(Secs. 17.2, 17.3)}$$

($ad - bc \neq 0$) map the extended complex plane (Sec. 17.2) onto itself. They solve the problems of mapping half-planes onto half-planes or disks, and disks onto disks or half-planes. Prescribing the images of three points determines (1) uniquely.

Riemann surfaces (Sec. 17.5) consist of several sheets connected at certain points called *branch points.* On them, multivalued relations become single-valued, that is, functions in the usual sense. *Examples.* For $w = \sqrt{z}$ we need two sheets (with branch point 0) since this relation is doubly-valued. For $w = \ln z$ we need infinitely many sheets since this relation is infinitely many-valued (see Sec. 13.7).

CHAPTER 18

Complex Analysis and Potential Theory

Laplaces's equation $\nabla^2\Phi = 0$ is one of the most important PDEs in engineering mathematics, because it occurs in gravitation (Secs. 9.7, 12.10), electrostatics (Sec. 9.7), steady-state heat conduction (Sec. 12.5), incompressible fluid flow, etc. The theory of solutions of this equation is called **potential theory** (although "potential" is also used in a more general sense in connection with gradients; see Sec. 9.7).

In the "two-dimensional case" when Φ depends only on two Cartesian coordinates x and y, Laplace's equation becomes

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} = 0.$$

From Sec. 13.4 we know that then its solutions Φ are closely related to complex analytic functions $\Phi + i\Psi$. This relation is the main reason for the importance of complex analysis in physics and engineering. (We use the notation $\Phi + i\Psi$ since $u + iv$ will be needed in conformal mapping.)

In this chapter we shall consider this connection and its consequences in detail and illustrate it by modeling typical examples from electrostatics (Secs. 18.1, 18.2), heat conduction (Sec. 18.3), and hydrodynamics (Sec. 18.4). This will lead to **boundary value problems,** some of which involving functions whose mapping properties we have studied in Chap. 17. Further relating to that chapter, in Sec. 18.2 we explain conformal mapping as a method in potential theory.

In Sec. 18.5 we derive the important Poisson formula for potentials in a circular disk.

Finally, in Sec. 18.6 we show that results on analytic functions can be used to characterize *general* properties of **harmonic functions** (solutions of Laplace's equation whose second partial derivatives are continuous).

Prerequisite: Chaps. 13, 14, 17.
References and Answers to Problems: App. 1 Part D, App. 2.

18.1 Electrostatic Fields

The electrical force of attraction or repulsion between charged particles is governed by Coulomb's law. This force is the gradient of a function Φ, called the **electrostatic potential.** At any points free of charges, Φ is a solution of Laplace's equation

$$\nabla^2\Phi = 0.$$

The surfaces $\Phi = const$ are called **equipotential surfaces.** At each point P at which the gradient of Φ is not the zero vector, it is perpendicular to the surface $\Phi = const$ through P; that is, the electrical force has the direction perpendicular to the equipotential surface. (See also Secs. 9.7 and 12.10.)

The problems we shall discuss in this entire chapter are **two-dimensional** (for the reason just given in the chapter opening), that is, they model physical systems that lie in three-dimensional space (of course!), but are such that the potential Φ is independent of one of the space coordinates, so that Φ depends only on two coordinates, which we call x and y. Then **Laplace's equation** becomes

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0. \tag{1}$$

Equipotential surfaces now appear as **equipotential lines** (curves) in the xy-plane.

Let us illustrate these ideas by a few simple basic examples.

EXAMPLE 1 **Potential Between Parallel Plates**

Find the potential Φ of the field between two parallel conducting plates extending to infinity (Fig. 393), which are kept at potentials Φ_1 and Φ_2, respectively.

Solution. From the shape of the plates it follows that Φ depends only on x, and Laplace's equation becomes $\Phi'' = 0$. By integrating twice we obtain $\Phi = ax + b$, where the constants a and b are determined by the given boundary values of Φ on the plates. For example, if the plates correspond to $x = -1$ and $x = 1$, the solution is

$$\Phi(x) = \tfrac{1}{2}(\Phi_2 - \Phi_1)x + \tfrac{1}{2}(\Phi_2 + \Phi_1).$$

The equipotential surfaces are parallel planes. ■

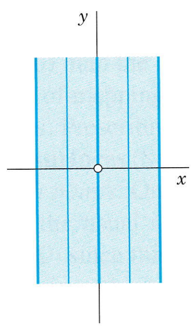


Fig. 393. Potential in Example 1

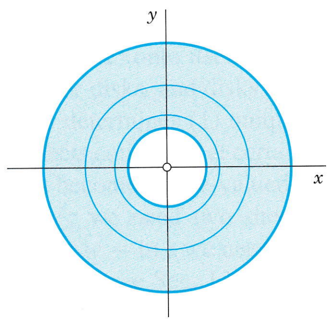


Fig. 394. Potential in Example 2

EXAMPLE 2 **Potential Between Coaxial Cylinders**

Find the potential Φ between two coaxial conducting cylinders extending to infinity on both ends (Fig. 394) and kept at potentials Φ_1 and Φ_2, respectively.

Solution. Here Φ depends only on $r = \sqrt{x^2 + y^2}$, for reasons of symmetry, and Laplace's equation $r^2u_{rr} + ru_r + u_{\theta\theta} = 0$ [(5), Sec. 12.9] with $u_{\theta\theta} = 0$ and $u = \Phi$ becomes $r\Phi'' + \Phi' = 0$. By separating variables and integrating we obtain

$$\frac{\Phi''}{\Phi'} = -\frac{1}{r}, \qquad \ln \Phi' = -\ln r + \tilde{a}, \qquad \Phi' = \frac{a}{r}, \qquad \Phi = a \ln r + b$$

and a and b are determined by the given values of Φ on the cylinders. Although no infinitely extended conductors exist, the field in our idealized conductor will approximate the field in a long finite conductor in that part which is far away from the two ends of the cylinders. ■

EXAMPLE 3 **Potential in an Angular Region**

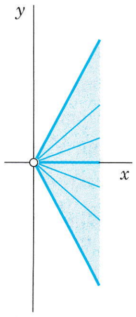


Fig. 395. Potential in Example 3

Find the potential Φ between the conducting plates in Fig. 395, which are kept at potentials Φ_1 (the lower plate) and Φ_2, and make an angle α, where $0 < \alpha \leqq \pi$. (In the figure we have $\alpha = 120° = 2\pi/3$.)

Solution. $\theta = \operatorname{Arg} z$ $(z = x + iy \neq 0)$ is constant on rays $\theta = const$. It is harmonic since it is the imaginary part of an analytic function, $\operatorname{Ln} z$ (Sec. 13.7). Hence the solution is

$$\Phi(x, y) = a + b \operatorname{Arg} z$$

with a and b determined from the two boundary conditions (given values on the plates)

$$a + b(-\tfrac{1}{2}\alpha) = \Phi_1, \qquad a + b(\tfrac{1}{2}\alpha) = \Phi_2.$$

Thus $a = (\Phi_2 + \Phi_1)/2$, $b = (\Phi_2 - \Phi_1)/\alpha$. The answer is

$$\Phi(x, y) = \frac{1}{2}(\Phi_2 + \Phi_1) + \frac{1}{\alpha}(\Phi_2 - \Phi_1)\theta, \qquad \theta = \arctan \frac{y}{x}.$$ ■

Complex Potential

Let $\Phi(x, y)$ be harmonic in some domain D and $\Psi(x, y)$ a harmonic conjugate of Φ in D. (See Sec. 13.4, where we wrote u and v, now needed in conformal mapping from the next section on; hence the change to Φ and Ψ.) Then

$$F(z) = \Phi(x, y) + i\Psi(x, y) \tag{2}$$

is an analytic function of $z = x + iy$. This function F is called the **complex potential** corresponding to the real potential Φ. Recall from Sec. 13.4 that for given Φ, a conjugate Ψ is uniquely determined except for an additive real constant. Hence we may say ***the*** complex potential, without causing misunderstandings.

The use of F has two advantages, a technical one and a physical one. Technically, F is easier to handle than real or imaginary parts, in connection with methods of complex analysis. Physically, Ψ has a meaning. By conformality, the curves $\Psi = const$ intersect the equipotential lines $\Phi = const$ in the xy-plane at right angles [except where $F'(z) = 0$]. Hence they have the direction of the electrical force and, therefore, are called **lines of force.** They are the paths of moving charged particles (electrons in an electron microscope, etc.).

EXAMPLE 4 **Complex Potential**

In Example 1, a conjugate is $\Psi = ay$. It follows that the complex potential is

$$F(z) = az + b = ax + b + iay,$$

and the lines of force are horizontal straight lines $y = const$ parallel to the x-axis. ■

EXAMPLE 5 **Complex Potential**

In Example 2 we have $\Phi = a \ln r + b = a \ln |z| + b$. A conjugate is $\Psi = a \operatorname{Arg} z$. Hence the complex potential is

$$F(z) = a \operatorname{Ln} z + b$$

and the lines of force are straight lines through the origin. $F(z)$ may also be interpreted as the complex potential of a source line (a wire perpendicular to the xy-plane) whose trace in the xy-plane is the origin. ■

EXAMPLE 6 **Complex Potential**

In Example 3 we get $F(z)$ by noting that $i \operatorname{Ln} z = i \ln |z| - \operatorname{Arg} z$, multiplying this by $-b$, and adding a:

$$F(z) = a - ib \operatorname{Ln} z = a + b \operatorname{Arg} z - ib \ln |z|.$$

We see from this that the lines of force are concentric circles $|z| = const$. Can you sketch them? ■

Superposition

More complicated potentials can often be obtained by superposition.

EXAMPLE 7 **Potential of a Pair of Source Lines (a Pair of Charged Wires)**

Determine the potential of a pair of oppositely charged source lines of the same strength at the points $z = c$ and $z = -c$ on the real axis.

Solution. From Examples 2 and 5 it follows that the potential of each of the source lines is

$$\Phi_1 = K \ln |z - c| \qquad \text{and} \qquad \Phi_2 = -K \ln |z + c|,$$

respectively. Here the real constant K measures the strength (amount of charge). These are the real parts of the complex potentials

$$F_1(z) = K \operatorname{Ln}(z - c) \qquad \text{and} \qquad F_2(z) = -K \operatorname{Ln}(z + c).$$

Hence the complex potential of the combination of the two source lines is

$$F(z) = F_1(z) + F_2(z) = K[\operatorname{Ln}(z - c) - \operatorname{Ln}(z + c)]. \tag{3}$$

The **equipotential lines** are the curves

$$\Phi = \operatorname{Re} F(z) = K \ln \left| \frac{z - c}{z + c} \right| = const, \qquad \text{thus} \qquad \left| \frac{z - c}{z + c} \right| = const.$$

These are circles, as you may show by direct calculation. The **lines of force** are

$$\Psi = \operatorname{Im} F(z) = K[\operatorname{Arg}(z - c) - \operatorname{Arg}(z + c)] = const.$$

We write this briefly (Fig. 396)

$$\Psi = K(\theta_1 - \theta_2) = const.$$

Now $\theta_1 - \theta_2$ is the angle between the line segments from z to c and $-c$ (Fig. 396). Hence the lines of force are the curves along each of which the line segment S: $-c \leqq x \leqq c$ appears under a constant angle. These curves are the totality of circular arcs over S, as is (or should be) known from elementary geometry. Hence the lines of force are circles. Figure 397 shows some of them together with some equipotential lines.

In addition to the interpretation as the potential of two source lines, this potential could also be thought of as the potential between two circular cylinders whose axes are parallel but do not coincide, or as the potential between two equal cylinders that lie outside each other, or as the potential between a cylinder and a plane wall. Explain this, using Fig. 397. ■

The idea of the complex potential as just explained is the key to a close relation of potential theory to complex analysis and will recur in heat flow and fluid flow.

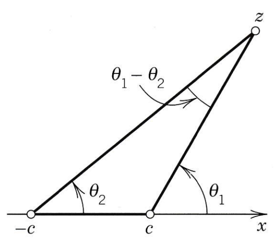


Fig. 396. Arguments in Example 7

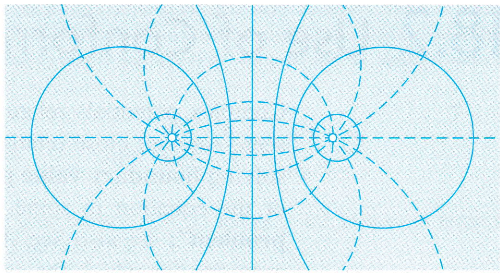


Fig. 397. Equipotential lines and lines of force (dashed) in Example 7

PROBLEM SET 18.1

1–4 POTENTIAL

Find and sketch the potential. Find the complex potential:

1. Between parallel plates at $x = -3$ and 3, potentials 140 V and 260 V, respectively
2. Between parallel plates at $x = -4$ and 10, potentials 4.4 kV and 10 kV, respectively
3. Between the axes (potential 110 V) and the hyperbola $xy = 1$ (potential 60 V)
4. Between parallel plates at $y = x$ and $x + k$, potentials 0 and 100 V, respectively

5–8 COAXIAL CYLINDERS

Find the potential between two infinite coaxial cylinders of radii r_1 and r_2 having potentials U_1 and U_2, respectively. Find the complex potential.

5. $r_1 = 0.5$, $r_2 = 2.0$, $U_1 = -110$ V, $U_2 = 110$ V
6. $r_1 = 1$, $r_2 = 10$, $U_1 = 100$ V, $U_2 = 1$ kV
7. $r_1 = 1$, $r_2 = 4$, $U_1 = 200$ V, $U_2 = 0$
8. $r_1 = 0.1$, $r_2 = 10$, $U_1 = 150$ V, $U_2 = 50$ V

9. Show that $\Phi = \theta/\pi = (1/\pi) \arctan(y/x)$ is harmonic in the upper half-plane and satisfies the boundary condition $\Phi(x, 0) = 1$ if $x < 0$ and 0 if $x > 0$, and the corresponding complex potential is $F(z) = -(i/\pi) \operatorname{Ln} z$.
10. Map the upper half z-plane onto the unit disk $|w| \leqq 1$ so that $0, \infty, -1$ are mapped onto $1, i, -i$, respectively. What are the boundary conditions on $|w| = 1$ resulting from the potential in Prob. 9? What is the potential at $w = 0$?
11. Verify by calculation that the equipotential lines in Example 7 are circles.
12. **CAS EXPERIMENT. Complex Potentials.** Graph the equipotential lines and lines of force in (a)–(d) (four graphs, Re $F(z)$ and Im $F(z)$ on the same axes). Then explore further complex potentials of your choice with the purpose of discovering configurations that might be of practical interest.

    (a) $F(z) = z^2$ **(b)** $F(z) = iz^2$
    (c) $F(z) = 1/z$ **(d)** $F(z) = i/z$

13–15 POTENTIALS FOR OTHER CONFIGURATIONS

13. Show that $F(z) = \arccos z$ (defined in Problem Set 13.7) gives the potential in Figs. 398 and 399.

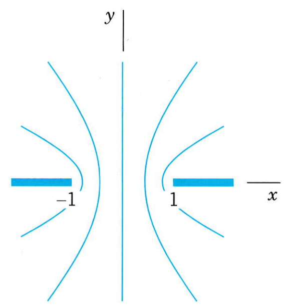


Fig. 398. Slit

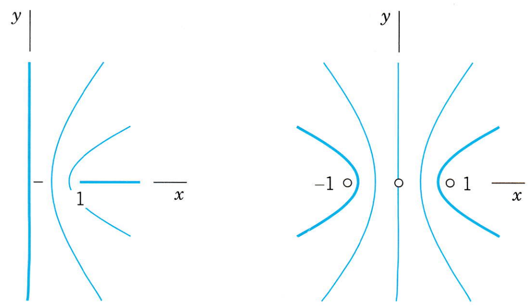


Fig. 399. Other apertures

14. Find the real and complex potentials in the sector $-\pi/6 \leqq \theta \leqq \pi/6$ between the boundary $\theta = \pm\pi/6$ (kept at 0) and the curve $x^3 - 3xy^2 = 1$, kept at 110 V.

15. Find the potential in the first quadrant of the xy-plane between the axes (having potential 220 V) and the hyperbola $xy = 1$ (having potential 110 V).

18.2 Use of Conformal Mapping. Modeling

Complex potentials relate potential theory closely to complex analysis, as we have just seen. Another close relation results from the use of conformal mapping in modeling and solving **boundary value problems** for the Laplace equation, that is, in finding a solution of the equation in some domain assuming given values on the boundary (**"Dirichlet problem"**; see also Sec. 12.5). Then conformal mapping is used to map a given domain onto one for which the solution is known or can be found more easily. This solution is then mapped back to the given domain. This is the idea. That it works is due to the fact that harmonic functions remain harmonic under conformal mapping:

THEOREM 1

Harmonic Functions Under Conformal Mapping

Let Φ^* be harmonic in a domain D^* in the w-plane. Suppose that $w = u + iv = f(z)$ is analytic in a domain D in the z-plane and maps D conformally onto D^*. Then the function

$$\Phi(x, y) = \Phi^*(u(x, y), v(x, y)) \tag{1}$$

is harmonic in D.

PROOF The composite of analytic functions is analytic, as follows from the chain rule. Hence, taking a harmonic conjugate $\Psi^*(u, v)$ of Φ^*, as defined in Sec. 13.4, and forming the analytic function $F^*(w) = \Phi^*(u, v) + i\Psi^*(u, v)$, we conclude that $F(z) = F^*(f(z))$ is analytic in D. Hence its real part $\Phi(x, y) = \operatorname{Re} F(z)$ is harmonic in D. This completes the proof.

We mention without proof that if D^* is simply connected (Sec. 14.2), then a harmonic conjugate of Φ^* exists. Another proof of Theorem 1 without the use of a harmonic conjugate is given in App. 4. ■

EXAMPLE 1 **Potential Between Noncoaxial Cylinders**

Model the electrostatic potential between the cylinders C_1: $|z| = 1$ and C_2: $|z - 2/5| = 2/5$ in Fig. 400. Then give the solution for the case that C_1 is grounded, $U_1 = 0$ V, and C_2 has the potential $U_2 = 110$ V.

Solution. We map the unit disk $|z| = 1$ onto the unit disk $|w| = 1$ in such a way that C_2 is mapped onto some cylinder C_2^*: $|w| = r_0$. By (3), Sec. 17.3, a linear fractional transformation mapping the unit disk onto the unit disk is

$$w = \frac{z - b}{bz - 1} \tag{2}$$

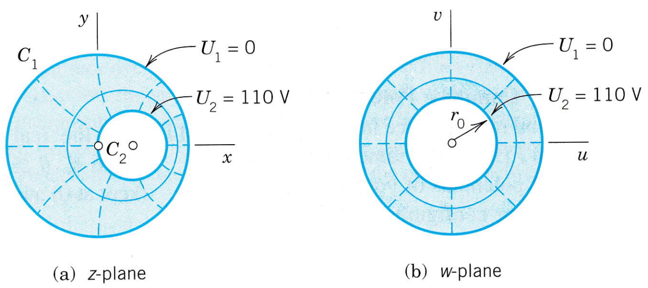


Fig. 400. Example 1

where we have chosen $b = z_0$ real without restriction. z_0 is of no immediate help here because centers of circles do not map onto centers of the images, in general. However, we now have two free constants b and r_0 and shall succeed by imposing two reasonable conditions, namely, that 0 and 4/5 (Fig. 400) should be mapped onto r_0 and $-r_0$, respectively. This gives by (2)

$$r_0 = \frac{0-b}{0-1} = b, \qquad \text{and with this,} \qquad -r_0 = \frac{4/5 - b}{4b/5 - 1} = \frac{4/5 - r_0}{4r_0/5 - 1},$$

a quadratic equation in r_0 with solutions $r_0 = 2$ (no good because $r_0 < 1$) and $r_0 = 1/2$. Hence our mapping function (2) with $b = 1/2$ becomes that in Example 5 of Sec. 17.3,

$$w = f(z) = \frac{2z-1}{z-2}. \tag{3}$$

From Example 5 in Sec. 18.1, writing w for z we have as the complex potential in the w-plane the function $F^*(w) = a \operatorname{Ln} w + k$ and from this the real potential

$$\Phi^*(u, v) = \operatorname{Re} F^*(w) = a \ln |w| + k.$$

This is our model. We now determine a and k from the boundary conditions. If $|w| = 1$, then $\Phi^* = a \ln 1 + k = 0$, hence $k = 0$. If $|w| = r_0 = 1/2$, then $\Phi^* = a \ln (1/2) = 110$, hence $a = 110/\ln (1/2) = -158.7$. Substitution of (3) now gives the desired solution in the given domain in the z-plane

$$F(z) = F^*(f(z)) = a \operatorname{Ln} \frac{2z-1}{z-2}.$$

The real potential is

$$\Phi(x, y) = \operatorname{Re} F(z) = a \ln \left| \frac{2z-1}{z-2} \right|, \qquad a = -158.7.$$

Can we "see" this result? Well, $\Phi(x, y) = \mathit{const}$ if and only if $|(2z-1)/(z-2)| = \mathit{const}$, that is, $|w| = \mathit{const}$ by (2) with $b = 1/2$. These circles are images of circles in the z-plane because the inverse of a linear fractional transformation is linear fractional (see (4), Sec. 17.2), and any such mapping maps circles onto circles (or straight lines), by Theorem 1 in Sec. 17.2. Similarly for the rays $\arg w = \mathit{const}$. Hence the equipotential lines $\Phi(x, y) = \mathit{const}$ are circles, and the lines of force are circular arcs (dashed in Fig. 400). These two families of curves intersect orthogonally, that is, at right angles, as shown in Fig. 400. ■

EXAMPLE 2 **Potential Between Two Semicircular Plates**

Model the potential between two semicircular plates P_1 and P_2 in Fig. 401a having potentials -3000 V and 3000 V, respectively. Use Example 3 in Sec. 18.1 and conformal mapping.

Solution. ***Step 1.*** We map the unit disk in Fig. 401a onto the right half of the w-plane (Fig. 401b) by using the linear fractional transformation in Example 3, Sec. 17.3:

$$w = f(z) = \frac{1+z}{1-z}.$$

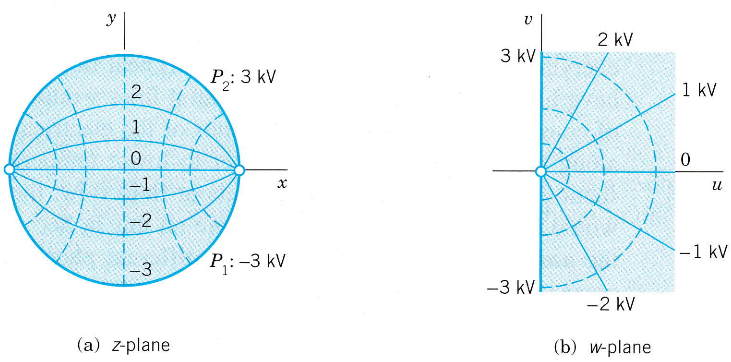


Fig. 401. Example 2

The boundary $|z| = 1$ is mapped onto the boundary $u = 0$ (the v-axis), with $z = -1, i, 1$ going onto $w = 0, i, \infty$, respectively, and $z = -i$ onto $w = -i$. Hence the upper semicircle of $|z| = 1$ is mapped onto the upper half, and the lower semicircle onto the lower half of the v-axis, so that the boundary conditions in the w-plane are as indicated in Fig. 401b.

Step 2. We determine the potential $\Phi^*(u, v)$ in the right half-plane of the w-plane. Example 3 in Sec. 18.1 with $\alpha = \pi$, $U_1 = -3000$, and $U_2 = 3000$ [with $\Phi^*(u, v)$ instead of $\Phi(x, y)$] yields

$$\Phi^*(u, v) = \frac{6000}{\pi} \varphi, \qquad \varphi = \arctan \frac{v}{u}.$$

On the positive half of the imaginary axis ($\varphi = \pi/2$), this equals 3000 and on the negative half -3000, as it should be. Φ^* is the real part of the complex potential

$$F^*(w) = -\frac{6000\, i}{\pi} \operatorname{Ln} w.$$

Step 3. We substitute the mapping function into F^* to get the complex potential $F(z)$ in Fig. 401a in the form

$$F(z) = F^*(f(z)) = -\frac{6000\, i}{\pi} \operatorname{Ln} \frac{1 + z}{1 - z}.$$

The real part of this is the potential we wanted to determine:

$$\Phi(x, y) = \operatorname{Re} F(z) = \frac{6000}{\pi} \operatorname{Im} \operatorname{Ln} \frac{1 + z}{1 - z} = \frac{6000}{\pi} \operatorname{Arg} \frac{1 + z}{1 - z}.$$

As in Example 1 we conclude that the equipotential lines $\Phi(x, y) = const$ are circular arcs because they correspond to Arg $[(1 + z)/(1 - z)] = const$, hence to Arg $w = const$. Also, Arg $w = const$ are rays from 0 to ∞, the images of $z = -1$ and $z = 1$, respectively. Hence the equipotential lines all have -1 and 1 (the points where the boundary potential jumps) as their endpoints (Fig. 401a). The lines of force are circular arcs, too, and since they must be orthogonal to the equipotential lines, their centers can be obtained as intersections of tangents to the unit circle with the x-axis, (Explain!) ■

Further examples can easily be constructed. Just take any mapping $w = f(z)$ in Chap. 17, a domain D in the z-plane, its image D^* in the w-plane, and a potential Φ^* in D^*. Then (1) gives a potential in D. Make up some examples of your own, involving, for instance, linear fractional transformations.

Basic Comment on Modeling

We formulated the examples in this section as models on the electrostatic potential. It is quite important to realize that this is accidental. We could equally well have phrased everything in terms of (time-independent) heat flow; then instead of voltages we would have had temperatures, the equipotential lines would have become isotherms (= lines of constant temperature), and the lines of the electrical force would have become lines along which heat flows from higher to lower temperatures (more on this in the next section). Or we could have talked about fluid flow; then the electrostatic lines of force would have become streamlines (more on this in Sec. 18.4). What we again see here is the ***unifying power of mathematics:*** different phenomena and systems from different areas in physics having the same types of model can be treated by the same mathematical methods. What differs from area to area is just the kinds of problems that are of practical interest.

PROBLEM SET 18.2

1. Verify Theorem 1 for $\Phi^*(u, v) = u^2 - v^2$, $w = f(z) = e^z$ and any domain D.
2. Verify Theorem 1 for $\Phi^*(u, v) = uv$, $w = f(z) = e^z$, and D: $x \leqq 0$, $0 \leqq y \leqq \pi$. Sketch D and D^*.
3. Carry out all steps of the second proof of Theorem 1 (given in App. 4) in detail.
4. Derive (3) from (2).
5. Let D^* be the image of the rectangle D: $0 \leqq x \leqq \frac{1}{2}\pi$, $0 \leqq y \leqq 1$ under $w = \sin z$, and $\Phi^*(u, v) = u^2 - v^2$. Find the corresponding potential Φ in D and its boundary values.
6. What happens in Prob. 5 if you replace the potential by the conjugate $\Phi^* = 2uv$? Sketch or graph some of the equipotential lines $\Phi = const.$
7. **CAS PROJECT. Graphing Potential Fields.** **(a)** Graph equipotential lines in Probs. 1 and 2.
   (b) Graph equipotential lines if the complex potential is $F(z) = iz^2$, $F(z) = e^z$, $F(z) = ie^z$, $F(z) = e^{iz}$.
   (c) Graph equipotential surfaces corresponding to $F(z) = \ln z$ as cylinders in space.
8. **TEAM PROJECT. Noncoaxial Cylinders.** Find the potential between the cylinders C_1: $|z| = 1$ (potential $U_1 = 0$) and C_2: $|z - c| = c$ ($U_2 = 110$ V), where $0 < c < \frac{1}{2}$. Sketch or graph the equipotential curves and their orthogonal trajectories for $c = 0.1, 0.2, 0.3, 0.4$. Try to think of the further extension C_1: $|z| = 1$, C_2: $|z - c| = \rho \neq c$.
9. Find the potential Φ in the region R in the first quadrant of the z-plane bounded by the axes (having potential U_1) and the hyperbola $y = 1/x$ (having potential 0) in two ways, (i) directly, (ii) by mapping R onto a suitable infinite strip.
10. **(Extension of Example 2)** Find the linear fractional transformation $z = g(Z)$ that maps $|Z| \leqq 1$ onto $|z| \leqq 1$ with $Z = i/2$ being mapped onto $z = 0$. Show that $Z_1 = 0.6 + 0.8i$ is mapped onto $z = -1$ and $Z_2 = -0.6 + 0.8i$ onto $z = 1$, so that the equipotential lines of Example 2 look in $|Z| \leqq 1$ as shown in Fig. 402.

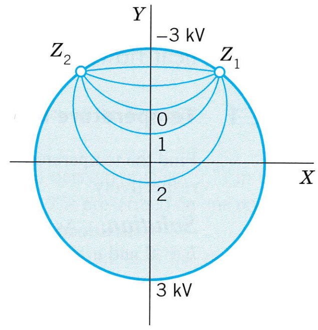


Fig. 402. Problem 10

11. The equipotential lines in Prob. 10 are circles. Why?
12. Show that in Example 2 the y-axis is mapped onto the unit circle in the w-plane.
13. Find the complex and real potentials in the upper half-plane with boundary values 0 if $x < 4$ and 10 kV if $x > 4$ on the x-axis.
14. **(Angular region)** Applying a suitable conformal mapping, obtain from Fig. 401b the potential Φ in the angular region $-\frac{1}{4}\pi < \text{Arg } z < \frac{1}{4}\pi$ such that $\Phi = -3$ kV if $\text{Arg } z = -\frac{1}{4}\pi$ and $\Phi = 3$ kV if $\text{Arg } z = \frac{1}{4}\pi$.
15. At $z = \pm 1$ in Fig. 401a the tangents to the equipotential lines shown make equal angles ($\pi/6$). Why?

18.3 Heat Problems

Laplace's equation also governs heat flow problems that are **steady,** that is, time-independent. Indeed, heat conduction in a body of homogeneous material is modeled by the **heat equation**

$$T_t = c^2 \nabla^2 T$$

where the function T is temperature, $T_t = \partial T/\partial t$, t is time, and c^2 is a positive constant (depending on the material of the body; see Sec. 12.5). Hence if a problem is steady, so that $T_t = 0$, and two-dimensional, then the heat equation reduces to the two-dimensional Laplace equation

$$\nabla^2 T = T_{xx} + T_{yy} = 0, \tag{1}$$

so that the problem can be treated by our present methods.

$T(x, y)$ is called the **heat potential.** It is the real part of the **complex heat potential**

$$F(z) = T(x, y) + i\Psi(x, y).$$

The curves $T(x, y) = const$ are called **isotherms** (= lines of constant temperature) and the curves $\Psi(x, y) = const$ **heat flow lines,** because along them, heat flows from higher to lower temperatures.

It follows that all the examples considered so far (Secs. 18.1, 18.2) can now be reinterpreted as problems on heat flow. The electrostatic equipotential lines $\Phi(x, y) = const$ now become isotherms $T(x, y) = const$, and the lines of electrical force become lines of heat flow, as in the following two problems.

EXAMPLE 1 Temperature Between Parallel Plates

Find the temperature between two parallel plates $x = 0$ and $x = d$ in Fig. 403 having temperatures 0 and 100°C, respectively.

Solution. As in Example 1 of Sec. 18.1 we conclude that $T(x, y) = ax + b$. From the boundary conditions, $b = 0$ and $a = 100/d$. The answer is

$$T(x, y) = \frac{100}{d}\, x \text{ [°C]}.$$

The corresponding complex potential is $F(z) = (100/d)z$. Heat flows horizontally in the negative x-direction along the lines $y = const$. ■

EXAMPLE 2 Temperature Distribution Between a Wire and a Cylinder

Find the temperature field around a long thin wire of radius $r_1 = 1$ mm that is electrically heated to $T_1 = 500$°F and is surrounded by a circular cylinder of radius $r_2 = 100$ mm, which is kept at temperature $T_2 = 60$°F by cooling it with air. See Fig. 404. (The wire is at the origin of the coordinate system.)

Solution. T depends only on r, for reasons of symmetry. Hence, as in Sec. 18.1 (Example 2),

$$T(x, y) = a \ln r + b.$$

The boundary conditions are

$$T_1 = 500 = a \ln 1 + b, \qquad T_2 = 60 = a \ln 100 + b.$$

Hence $b = 500$ (since $\ln 1 = 0$) and $a = (60 - b)/\ln 100 = -95.54$. The answer is

$$T(x, y) = 500 - 95.54 \ln r \text{ [°F]}.$$

The isotherms are concentric circles. Heat flows from the wire radially outward to the cylinder. Sketch T as a function of r. Does it look physically reasonable? ■

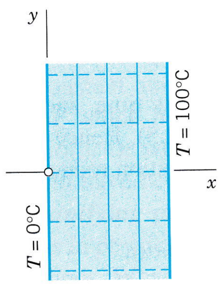


Fig. 403. Example 1

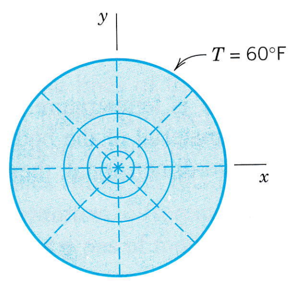


Fig. 404. Example 2

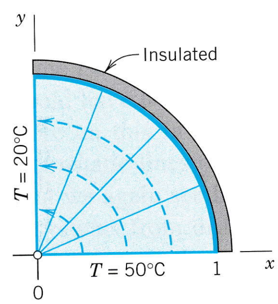


Fig. 405. Example 3

Mathematically the calculations remain the same in the transition to another field of application. Physically, new problems may arise, with boundary conditions that would make no sense physically or would be of no practical interest. This is illustrated by the next two examples.

EXAMPLE 3 A Mixed Boundary Value Problem

Find the temperature distribution in the region in Fig. 405 (cross section of a solid quarter-cylinder), whose vertical portion of the boundary is at 20°C, the horizontal portion at 50°C, and the circular portion is insulated.

Solution. The insulated portion of the boundary must be a heat flow line, since by the insulation, heat is prevented from crossing such a curve, hence heat must flow along the curve. Thus the isotherms must meet such a curve at right angles. Since T is constant along an isotherm, this means that

$$\frac{\partial T}{\partial n} = 0 \qquad \text{along an insulated portion of the boundary.} \tag{2}$$

Here $\partial T/\partial n$ is the **normal derivative** of T, that is, the directional derivative (Sec. 9.7) in the direction normal (perpendicular) to the insulated boundary. Such a problem in which T is prescribed on one portion of the boundary and $\partial T/\partial n$ on the other portion is called a **mixed boundary value problem.**

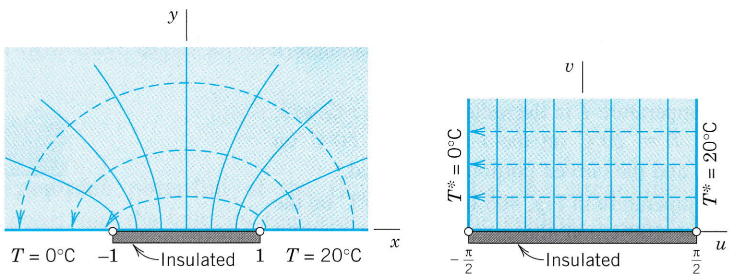
In our case, the normal direction to the insulated circular boundary curve is the radial direction toward the origin. Hence (2) becomes $\partial T/\partial r = 0$, meaning that along this curve the solution must not depend on r. Now $\operatorname{Arg} z = \theta$ satisfies (1), as well as this condition, and is constant (0 and $\pi/2$) on the straight portions of the boundary. Hence the solution is of the form

$$T(x, y) = a\theta + b.$$

The boundary conditions yield $a \cdot \pi/2 + b = 20$ and $a \cdot 0 + b = 50$. This gives

$$T(x, y) = 50 - \frac{60}{\pi}\,\theta, \qquad \theta = \arctan\frac{y}{x}.$$

The isotherms are portions of rays $\theta = const.$ Heat flows from the x-axis along circles $r = const$ (dashed in Fig. 405) to the y-axis. ■



(a) z-plane (b) w-plane

Fig. 406. Example 4

EXAMPLE 4 Another Mixed Boundary Value Problem in Heat Conduction

Find the temperature field in the upper half-plane when the x-axis is kept at $T = 0$°C for $x < -1$, is insulated for $-1 < x < 1$, and is kept at $T = 20$°C for $x > 1$ (Fig. 406a).

Solution. We map the half-plane in Fig. 406a onto the vertical strip in Fig. 406b, find the temperature $T^*(u, v)$ there, and map it back to get the temperature $T(x, y)$ in the half-plane.

The idea of using that strip is suggested by Fig. 388 in Sec. 17.4 with the roles of $z = x + iy$ and $w = u + iv$ interchanged. The figure shows that $z = \sin w$ maps our present strip onto our half-plane in Fig. 406a. Hence the inverse function

$$w = f(z) = \arcsin z$$

maps that half-plane onto the strip in the w-plane. This is the mapping function that we need according to Theorem 1 in Sec. 18.2.

The insulated segment $-1 < x < 1$ on the x-axis maps onto the segment $-\pi/2 < u < \pi/2$ on the u-axis. The rest of the x-axis maps onto the two vertical boundary portions $u = -\pi/2$ and $\pi/2$, $v > 0$, of the strip. This gives the transformed boundary conditions in Fig. 406b for $T^*(u, v)$, where on the insulated horizontal boundary, $\partial T^*/\partial n = \partial T^*/\partial v = 0$ because v is a coordinate normal to that segment.

Similarly to Example 1 we obtain

$$T^*(u, v) = 10 + \frac{20}{\pi} u$$

which satisfies all the boundary conditions. This is the real part of the complex potential $F^*(w) = 10 + (20/\pi)w$. Hence the complex potential in the z-plane is

$$F(z) = F^*(f(z)) = 10 + \frac{20}{\pi} \arcsin z$$

and $T(x, y) = \operatorname{Re} F(z)$ is the solution. The isotherms are $u = const$ in the strip and the hyperbolas in the z-plane, perpendicular to which heat flows along the dashed ellipses from the 20°-portion to the cooler 0°-portion of the boundary, a physically very reasonable result. ■

This section and the last one show the usefulness of conformal mappings and complex potentials. The latter will also play a role in the next section on fluid flow.

PROBLEM SET 18.3

1. **CAS PROJECT. Isotherms.** Graph isotherms and lines of heat flow in Examples 2–4. Can you see from the graphs where the heat flow is very rapid?
2. Find the temperature and the complex potential in an infinite plate with edges $y = x - 2$ and $y = x + 2$ kept at -10°C and 20°C, respectively.
3. Find the temperature between two parallel plates $y = 0$ and $y = d$ kept at temperatures 0°C and 100°C, respectively. (i) Proceed directly. (ii) Use Example 1 and a suitable mapping.
4. Find the temperature T in the sector $0 \leqq \operatorname{Arg} z \leqq \pi/3$, $|z| \leqq 1$ if $T = 20$°C on the x-axis, $T = 50$°C on $y = \sqrt{3}\, x$, and the curved portion is insulated.
5. Find the temperature in Fig. 405 if $T = -20$°C on the y-axis, $T = 100$°C on the x-axis, and the circular portion of the boundary is insulated as before.
6. Interpret Prob. 10 in Sec. 18.2 as a heat flow problem (with boundary temperatures, say, 20°C and 300°C). Along what curves does the heat flow?
7. Find the temperature and the complex potential in the first quadrant of the z-plane if the y-axis is kept at 100°C, the segment $0 < x < 1$ of the x-axis is insulated and the portion $x > 1$ of the x-axis is kept at 200°C. *Hint.* Use Example 4.
8. **TEAM PROJECT. Piecewise Constant Boundary Temperatures. (a) A basic building block** is shown in Fig. 407. Find the corresponding temperature and complex potential in the upper half-plane.

   (b) Conformal mapping. What temperature in the first quadrant of the z-plane is obtained from (a) by the mapping $w = a + z^2$ and what are the transformed boundary conditions?

   (c) Superposition. Find the temperature T^* and the complex potential F^* in the upper half-plane satisfying the boundary condition in Fig. 408.

   (d) Semi-infinite strip. Applying $w = \cosh z$ to (c), obtain the solution of the boundary value problem in Fig. 409.

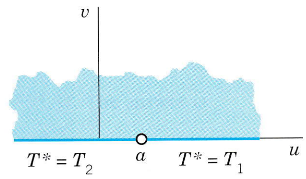


Fig. 407. Team Project 8(a)

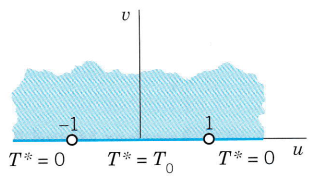


Fig. 408. Team Project 8(c)

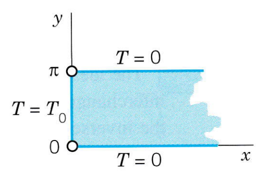
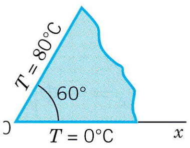


Fig. 409. Team Project 8(d)

9–14 TEMPERATURE DISTRIBUTIONS IN PLATES

Find the temperature $T(x, y)$ in the given thin metal plate whose faces are insulated and whose edges are kept at the indicated temperatures or are insulated as shown.

9.



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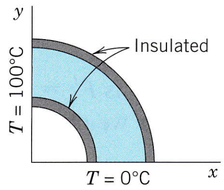

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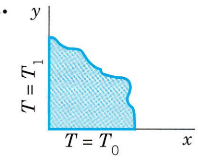

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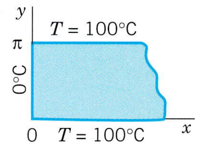

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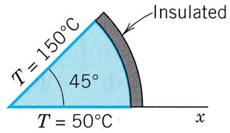

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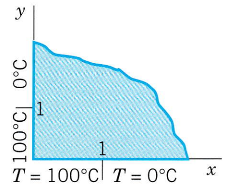

18.4 Fluid Flow

Laplace's equation also plays a basic role in hydrodynamics, in steady nonviscous fluid flow under physical conditions discussed later in this section. In order that methods of complex analysis can be applied, our problems will be ***two-dimensional,*** so that the **velocity vector** V by which the motion of the fluid can be given depends only on two space variables x and y and the motion is the same in all planes parallel to the xy-plane.

Then we can use for the velocity vector V a complex function

$$V = V_1 + iV_2 \tag{1}$$

giving the magnitude $|V|$ and direction Arg V of the velocity at each point $z = x + iy$. Here V_1 and V_2 are the components of the velocity in the x and y directions. V is tangential to the path of the moving particles, called a **streamline** of the motion (Fig. 410).

We show that under suitable assumptions (explained in detail following the examples), for a given flow there exists an analytic function

$$F(z) = \Phi(x, y) + i\Psi(x, y), \tag{2}$$

called the **complex potential** of the flow, such that the streamlines are given by $\Psi(x, y) = const,$ and the velocity vector or, briefly, the **velocity** is given by

$$V = V_1 + iV_2 = \overline{F'(z)} \tag{3}$$

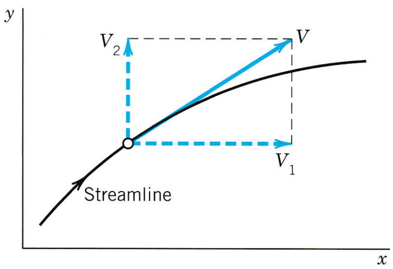

Fig. 410. Velocity

where the bar denotes the complex conjugate. Ψ is called the **stream function.** The function Φ is called the **velocity potential.** The curves $\Phi(x, y) = const$ are called **equipotential lines.** The velocity vector V is the **gradient** of Φ; by definition, this means that

$$V_1 = \frac{\partial \Phi}{\partial x}, \qquad V_2 = \frac{\partial \Phi}{\partial y}. \tag{4}$$

Indeed, for $F = \Phi + i\Psi$, Eq. (4) in Sec. 13.4 is $F' = \Phi_x + i\Psi_x$ with $\Psi_x = -\Phi_y$ by the second Cauchy–Riemann equation. Together we obtain (3):

$$\overline{F'(z)} = \Phi_x - i\Psi_x = \Phi_x + i\Phi_y = V_1 + iV_2 = V.$$

Furthermore, since $F(z)$ is analytic, Φ and Ψ satisfy Laplace's equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \qquad \nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0. \tag{5}$$

Whereas in electrostatics the boundaries (conducting plates) are equipotential lines, in fluid flow the boundaries across which fluid cannot flow must be streamlines. Hence in fluid flow the stream function is of particular importance.

Before discussing the conditions for the validity of the statements involving (2)–(5), let us consider two flows of practical interest, so that we first see what is going on from a practical point of view. Further flows are included in the problem set.

EXAMPLE 1 Flow Around a Corner

The complex potential $F(z) = z^2 = x^2 - y^2 + 2ixy$ models a flow with

Equipotential lines	$\Phi = x^2 - y^2 = const$	(Hyperbolas)
Streamlines	$\Psi = 2xy = const$	(Hyperbolas).

From (3) we obtain the velocity vector

$$V = 2\bar{z} = 2(x - iy), \qquad \text{that is,} \qquad V_1 = 2x, \qquad V_2 = -2y.$$

The speed (magnitude of the velocity) is

$$|V| = \sqrt{V_1^2 + V_2^2} = 2\sqrt{x^2 + y^2}.$$

The flow may be interpreted as the flow in a channel bounded by the positive coordinates axes and a hyperbola, say, $xy = 1$ (Fig. 411). We note that the speed along a streamline S has a minimum at the point P where the cross section of the channel is large. ■

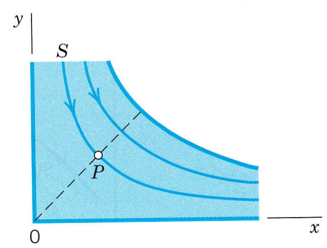

Fig. 411. Flow around a corner (Example 1)

EXAMPLE 2 Flow Around a Cylinder

Consider the complex potential

$$F(z) = \Phi(x, y) + i\Psi(x, y) = z + \frac{1}{z}.$$

Using the polar form $z = re^{i\theta}$, we obtain

$$F(z) = re^{i\theta} + \frac{1}{r}e^{-i\theta} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Hence the streamlines are

$$\Psi(x, y) = \left(r - \frac{1}{r}\right)\sin\theta = const.$$

In particular, $\Psi(x, y) = 0$ gives $r - 1/r = 0$ or $\sin\theta = 0$. Hence this streamline consists of the unit circle ($r = 1/r$ gives $r = 1$) and the x-axis ($\theta = 0$ and $\theta = \pi$). For large $|z|$ the term $1/z$ in $F(z)$ is small in absolute value, so that for these z the flow is nearly uniform and parallel to the x-axis. Hence we can interpret this as a flow around a long circular cylinder of unit radius that is perpendicular to the z-plane and intersects it in the unit circle $|z| = 1$ and whose axis corresponds to $z = 0$.

The flow has two **stagnation points** (that is, points at which the velocity V is zero), at $z = \pm 1$. This follows from (3) and

$$F'(z) = 1 - \frac{1}{z^2}, \qquad \text{hence} \qquad z^2 - 1 = 0. \qquad \text{(See Fig. 412.)}$$ ■

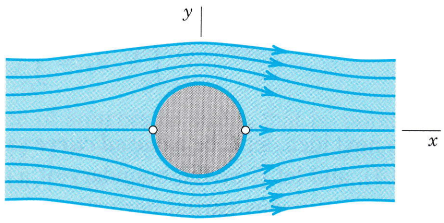

Fig. 412. Flow around a cylinder (Example 2)

Assumptions and Theory Underlying (2)–(5)

THEOREM 1 Complex Potential of a Flow

If the domain of flow is simply connected and the flow is irrotational and incompressible, then the statements involving (2)–(5) *hold. In particular, then the flow has a complex potential $F(z)$, which is an analytic function.* (Explanation of terms below.)

PROOF We prove this theorem, along with a discussion of basic concepts related to fluid flow.

(a) ***First Assumption: Irrotational.*** Let C be any smooth curve in the z-plane given by $z(s) = x(s) + iy(s)$, where s is the arc length of C. Let the real variable V_t be the component of the velocity V tangent to C (Fig. 413). Then the value of the real line integral

$$\int_C V_t \, ds \tag{6}$$

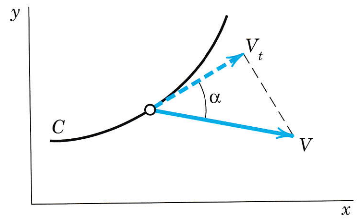

Fig. 413. Tangential component of the velocity with respect to a curve C

taken along C in the sense of increasing s is called the **circulation** of the fluid along C, a name that will be motivated as we proceed in this proof. Dividing the circulation by the length of C, we obtain the *mean velocity*[1] of the flow along the curve C. Now

$$V_t = |V| \cos \alpha \qquad \text{(Fig. 413).}$$

Hence V_t is the dot product (Sec. 9.2) of V and the tangent vector dz/ds of C (Sec. 17.1); thus in (6),

$$V_t\, ds = \left(V_1 \frac{dx}{ds} + V_2 \frac{dy}{ds}\right) ds = V_1\, dx + V_2\, dy.$$

The circulation (6) along C now becomes

$$\int_C V_t\, ds = \int_C (V_1\, dx + V_2\, dy). \tag{7}$$

As the next idea, let C be a ***closed curve*** satisfying the assumption as in Green's theorem (Sec. 10.4), and let C be the boundary of a simply connected domain D. Suppose further that V has continuous partial derivatives in a domain containing D and C. Then we can use Green's theorem to represent the circulation around C by a double integral,

$$\oint_C (V_1\, dx + V_2\, dy) = \iint_D \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) dx\, dy. \tag{8}$$

The integrand of this double integral is called the **vorticity** of the flow. The vorticity divided by 2 is called the **rotation**

$$\omega(x, y) = \frac{1}{2}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right). \tag{9}$$

---

[1]*Definitions:* $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\, dx =$ **mean value** of f on the interval $a \leqq x \leqq b$,

$\dfrac{1}{L}\displaystyle\int_C f(s)\, ds =$ **mean value** of f on C $\quad$ ($L =$ length of C),

$\dfrac{1}{A}\displaystyle\iint_D f(x, y)\, dx\, dy =$ **mean value** of f on D $\quad$ ($A =$ area of D).

We assume the flow to be **irrotational,** that is, $\omega(x, y) \equiv 0$ throughout the flow; thus,

$$\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} = 0. \tag{10}$$

To understand the physical meaning of vorticity and rotation, take for C in (8) a circle. Let r be the radius of C. Then the circulation divided by the length $2\pi r$ of C is the mean velocity of the fluid along C. Hence by dividing this by r we obtain the mean **angular velocity** ω_0 of the fluid about the center of the circle:

$$\omega_0 = \frac{1}{2\pi r^2} \iint_D \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) dx\, dy = \frac{1}{\pi r^2} \iint_D \omega(x, y)\, dx\, dy.$$

If we now let $r \to 0$, the limit of ω_0 is the value of ω at the center of C. Hence $\omega(x, y)$ is the limiting angular velocity of a circular element of the fluid as the circle shrinks to the point (x, y). Roughly speaking, *if a spherical element of the fluid were suddenly solidified and the surrounding fluid simultaneously annihilated, the element would rotate with the angular velocity* ω.

(b) ***Second Assumption: Incompressible.*** Our second assumption is that the fluid is incompressible. (Fluids include liquids, which are incompressible, and gases, such as air, which are compressible.) Then

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0 \tag{11}$$

in every region that is free of **sources** or **sinks,** that is, points at which fluid is produced or disappears, respectively. The expression in (11) is called the **divergence** of V and is denoted by div V. (See also (7) in Sec. 9.8.)

(c) ***Complex Velocity Potential.*** If the domain D of the flow is simply connected (Sec. 14.2) and the flow is irrotational, then (10) implies that the line integral (7) is independent of path in D (by Theorem 3 in Sec. 10.2, where $F_1 = V_1$, $F_2 = V_2$, $F_3 = 0$, and z is the third coordinate in space and has nothing to do with our present z). Hence if we integrate from a fixed point (a, b) in D to a variable point (x, y) in D, the integral becomes a function of the point (x, y), say, $\Phi(x, y)$:

$$\Phi(x, y) = \int_{(a, b)}^{(x, y)} (V_1\, dx + V_2\, dy). \tag{12}$$

We claim that the flow has a velocity potential Φ, which is given by (12). To prove this, all we have to do is to show that (4) holds. Now since the integral (7) is independent of path, $V_1\, dx + V_2\, dy$ is exact (Sec. 10.2), namely, the differential of Φ, that is,

$$V_1\, dx + V_2\, dy = \frac{\partial \Phi}{\partial x}\, dx + \frac{\partial \Phi}{\partial y}\, dy.$$

From this we see that $V_1 = \partial\Phi/\partial x$ and $V_2 = \partial\Phi/\partial y$, which gives (4).

That Φ is harmonic follows at once by substituting (4) into (11), which gives the first Laplace equation in (5).

We finally take a harmonic conjugate Ψ of Φ. Then the other equation in (5) holds. Also, since the second partial derivatives of Φ and Ψ are continuous, we see that the complex function

$$F(z) = \Phi(x, y) + i\Psi(x, y)$$

is analytic in D. Since the curves $\Psi(x, y) = const$ are perpendicular to the equipotential curves $\Phi(x, y) = const$ (except where $F'(z) = 0$), we conclude that $\Psi(x, y) = const$ are the streamlines. Hence Ψ is the stream function and $F(z)$ is the complex potential of the flow. This completes the proof of Theorem 1 as well as our discussion of the important role of complex analysis in compressible fluid flow. ■

PROBLEM SET 18.4

1–15 FLOW PATTERNS: STREAMLINES, COMPLEX POTENTIAL

These problems should encourage you to experiment with various functions $F(z)$, many of which model interesting flow patterns.

1. **(Parallel flow)** Show that $F(z) = -iKz$ (K positive real) describes a uniform flow upward, which can be interpreted as a uniform flow between two parallel lines (parallel planes in three-dimensional space). See Fig. 414. Find the velocity vector, the streamlines, and the equipotential lines.

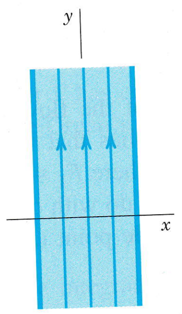

Fig. 414. Parallel flow in Problem 1

2. **(Conformal mapping)** Obtain the flow in Example 1 from that in Prob. 1 by a suitable conformal mapping.
3. Find the complex potential of a uniform flow parallel to the x-axis in the positive x-direction.
4. What happens to the flow in Prob. 1 if you replace z by $ze^{-i\alpha}$ with constant α, e.g., $\alpha = \pi/4$?
5. What is the complex potential of an upward parallel flow in the direction of $y = 2x$?
6. **(Extension of Example 1)** Sketch or graph the flow in Example 1 on the whole upper half-plane. Show that you can interpret it as as flow against a horizontal wall (the x-axis).
7. What $F(z)$ would be suitable in Example 1 if the angle of the corner were $\pi/3$?
8. Sketch or graph the streamlines and equipotential lines of $F(z) = iz^3$. Find V. Find all points at which V is parallel to the x-axis.
9. Find and graph the streamlines of $F(z) = z^2 + 2z$. Interpret the flow.
10. Show that $F(z) = iz^2$ models a flow around a corner. Sketch the streamlines and equipotential lines. Find V.
11. **(Potential $F(z) = 1/z$)** Show that the streamlines of $F(z) = 1/z$ are circles through the origin.
12. **(Cylinder)** What happens in Example 2 if you replace z by z^2? Sketch and interpret the resulting flow in the first quadrant.
13. Change $F(z)$ in Example 2 slightly to obtain a flow around a cylinder of radius r_0 that gives the flow in Example 2 if $r_0 \to 1$.
14. **(Aperture)** Show that $F(z) = \text{arccosh}\, z$ gives confocal hyperbolas as streamlines, with foci at $z = \pm 1$, and the flow may be interpreted as a flow through an aperture (Fig. 415).
15. **(Elliptical cylinder)** Show that $F(z) = \arccos z$ gives confocal ellipses as streamlines, with foci at $z = \pm 1$, and that the flow circulates around an elliptic cylinder or a plate (the segment from -1 to 1 in Fig. 416).

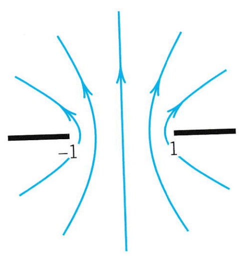

Fig. 415. Flow through an aperture in Problem 14

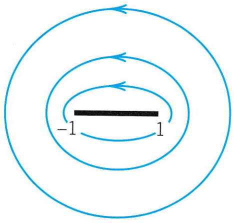

Fig. 416. Flow around a plate in Problem 15

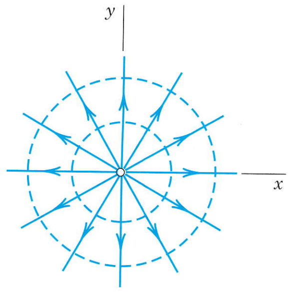

Fig. 417. Point source

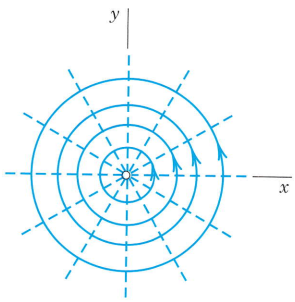

Fig. 418. Vortex flow

16. TEAM PROJECT. Role of the Natural Logarithm in Modeling Flows. (a) Basic flows: Source and sink. Show that $F(z) = (c/2\pi) \ln z$ with constant positive real c gives a flow directed radially outward (Fig. 417), so that F models a **point source** at $z = 0$ (that is, a **source line** $x = 0$, $y = 0$ in space) at which fluid is produced. c is called the **strength** or **discharge** of the source. If c is negative real, show that the flow is directed radially inward, so that F models a **sink** at $z = 0$, a point at which fluid disappears. Note that $z = 0$ is the singular point of $F(z)$.

(b) Basic flows: Vortex. Show that $F(z) = -(Ki/2\pi) \ln z$ with positive real K gives a flow circulating counterclockwise around $z = 0$ (Fig. 418). $z = 0$ is called a **vortex.** Note that each time we travel around the vortex, the potential increases by K.

(c) Addition of flows. Show that addition of the velocity vectors of two flows gives a flow whose complex potential is obtained by adding the complex potentials of those flows.

(d) Source and sink combined. Find the complex potentials of a flow with a source of strength 1 at $z = -a$ and of a flow with a sink of strength 1 at $z = a$. Add both and sketch or graph the streamlines. Show that for small $|a|$ these lines look similar to those in Prob. 11.

(e) Flow with circulation around a cylinder. Add the potential in (b) to that in Example 2. Show that this gives a flow for which the cylinder wall $|z| = 1$ is a streamline. Find the speed and show that the stagnation points are

$$z = \frac{iK}{4\pi} \pm \sqrt{\frac{-K^2}{16\pi^2} + 1}\,;$$

if $K = 0$ they are at ± 1; as K increases they move up on the unit circle until they unite at $z = i$ ($K = 4\pi$, see Fig. 419), and if $K > 4\pi$ they lie on the imaginary axis (one lies in the field of flow and the other one lies inside the cylinder and has no physical meaning).

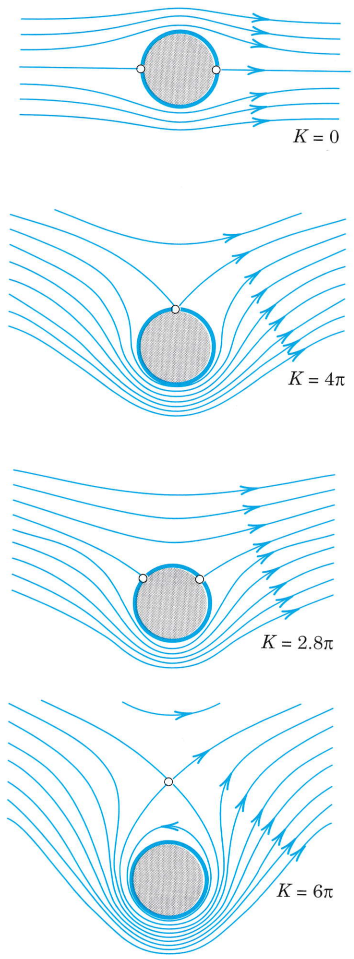

Fig. 419. Flow around a cylinder without circulation ($K = 0$) and with circulation

18.5 Poisson's Integral Formula for Potentials

So far in this chapter we have seen that complex analysis offers powerful methods for modeling and solving two-dimensional potential problems based on conformal mappings and complex potentials. A further method results from complex integration. As a most important result it yields Poisson's integral formula (5) for potentials in a standard domain (a circular disk) and from (5) a useful series (7) for these potentials. Hence we can solve problems for disks and then map solutions conformally onto other domains.

Poisson's formula will follow from Cauchy's integral formula (Sec. 14.3)

$$F(z) = \frac{1}{2\pi i} \oint_C \frac{F(z^*)}{z^* - z}\, dz^*. \tag{1}$$

Here C is the circle $z^* = Re^{i\alpha}$ (counterclockwise, $0 \leqq \alpha \leqq 2\pi$), and we assume that $F(z^*)$ is analytic in a domain containing C and its full interior. Since $dz^* = iRe^{i\alpha}\, d\alpha = iz^*\, d\alpha$, we obtain from (1)

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} F(z^*) \frac{z^*}{z^* - z}\, d\alpha \qquad (z^* = Re^{i\alpha},\ z = re^{i\theta}). \tag{2}$$

Now comes a little trick. If instead of z inside C we take a Z outside C, the integrals (1) and (2) are zero by Cauchy's integral theorem (Sec. 14.2). We choose $Z = z^*\bar{z}^*/\bar{z} = R^2/\bar{z}$, which is outside C because $|Z| = R^2/|z| = R^2/r > R$. From (2) we thus have

$$0 = \frac{1}{2\pi} \int_0^{2\pi} F(z^*) \frac{z^*}{z^* - Z}\, d\alpha = \frac{1}{2\pi} \int_0^{2\pi} F(z^*) \frac{z^*}{z^* - \dfrac{z^*\bar{z}^*}{\bar{z}}}\, d\alpha$$

and by straightforward simplification of the last expression on the right,

$$0 = \frac{1}{2\pi} \int_0^{2\pi} F(z^*) \frac{\bar{z}}{\bar{z} - \bar{z}^*}\, d\alpha.$$

We subtract this from (2) and use the following formula that you can verify by direct calculation ($\bar{z}z^*$ cancels):

$$\frac{z^*}{z^* - z} - \frac{\bar{z}}{\bar{z} - \bar{z}^*} = \frac{z^*\bar{z}^* - z\bar{z}}{(z^* - z)(\bar{z}^* - \bar{z})}. \tag{3}$$

We then have

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} F(z^*) \frac{z^*\bar{z}^* - z\bar{z}}{(z^* - z)(\bar{z}^* - \bar{z})}\, d\alpha. \tag{4}$$

From the polar representations of z and z^* we see that the quotient in the integrand is real and equal to

$$\frac{R^2 - r^2}{(Re^{i\alpha} - re^{i\theta})(Re^{-i\alpha} - re^{-i\theta})} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \alpha) + r^2}.$$

We now write $F(z) = \Phi(r, \theta) + i\Psi(r, \theta)$ and take the real part on both sides of (4). Then we obtain **Poisson's integral formula**[2]

$$\textbf{(5)} \qquad \Phi(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(R, \alpha)\, \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \alpha) + r^2}\, d\alpha.$$

This formula represents the harmonic function Φ in the disk $|z| \leqq R$ in terms of its values $\Phi(R, \alpha)$ on the boundary (the circle) $|z| = R$.

Formula (5) is still valid if the boundary function $\Phi(R, \alpha)$ is merely piecewise continuous (as is practically often the case; see Fig. 401 in Sec. 18.2 for an example). Then (5) gives a function harmonic in the open disk, and on the circle $|z| = R$ equal to the given boundary function, except at points where the latter is discontinuous. A proof can be found in Ref. [D1] in App. 1.

Series for Potentials in Disks

From (5) we may obtain an important series development of Φ in terms of simple harmonic functions. We remember that the quotient in the integrand of (5) was derived from (3). We claim that the right side of (3) is the real part of

$$\frac{z^* + z}{z^* - z} = \frac{(z^* + z)(\bar{z}^* - \bar{z})}{(z^* - z)(\bar{z}^* - \bar{z})} = \frac{z^*\bar{z}^* - z\bar{z} - z^*\bar{z} + z\bar{z}^*}{|z^* - z|^2}.$$

Indeed, the last denominator is real and so is $z^*\bar{z}^* - z\bar{z}$ in the numerator, whereas $-z^*\bar{z} + z\bar{z}^* = 2i\,\mathrm{Im}\,(z\bar{z}^*)$ in the numerator is pure imaginary. This verifies our claim. Now by the use of the geometric series we obtain (develop the denominator)

$$(6) \qquad \frac{z^* + z}{z^* - z} = \frac{1 + (z/z^*)}{1 - (z/z^*)} = \left(1 + \frac{z}{z^*}\right)\sum_{n=0}^{\infty}\left(\frac{z}{z^*}\right)^n = 1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{z^*}\right)^n.$$

Since $z = re^{i\theta}$ and $z^* = Re^{i\alpha}$, we have

$$\mathrm{Re}\left[\left(\frac{z}{z^*}\right)^n\right] = \mathrm{Re}\left[\frac{r^n}{R^n}\, e^{in\theta}e^{-in\alpha}\right] = \left(\frac{r}{R}\right)^n \cos(n\theta - n\alpha).$$

On the right, $\cos(n\theta - n\alpha) = \cos n\theta \cos n\alpha + \sin n\theta \sin n\alpha$. Hence from (6) we obtain

$$(6^*) \qquad \begin{aligned} \mathrm{Re}\,\frac{z^* + z}{z^* - z} &= 1 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left(\frac{z}{z^*}\right)^n \\ &= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (\cos n\theta \cos n\alpha + \sin n\theta \sin n\alpha). \end{aligned}$$

[2]SIMÉON DENIS POISSON (1781–1840), French mathematician and physicist, professor in Paris from 1809. His work includes potential theory, partial differential equations (Poisson equation, Sec. 12.1), and probability (Sec. 24.7).

This expression is equal to the quotient in (5), as we have mentioned before, and by inserting it into (5) and integrating term by term with respect to α from 0 to 2π we obtain

$$\Phi(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \tag{7}$$

where the coefficients are [the 2 in (6*) cancels the 2 in $1/(2\pi)$ in (5)]

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} \Phi(R, \alpha)\, d\alpha, \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(R, \alpha) \cos n\alpha\, d\alpha, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} \Phi(R, \alpha) \sin n\alpha\, d\alpha, \end{aligned} \qquad n = 1, 2, \cdots, \tag{8}$$

the Fourier coefficients of $\Phi(R, \alpha)$; see Sec. 11.1. Now for $r = R$ the series (7) becomes the Fourier series of $\Phi(R, \alpha)$. Hence the representation (7) will be valid whenever the given $\Phi(R, \alpha)$ on the boundary can be represented by a Fourier series.

EXAMPLE 1 Dirichlet Problem for the Unit Disk

Find the electrostatic potential $\Phi(r, \theta)$ in the unit disk $r < 1$ having the boundary values

$$\Phi(1, \alpha) = \begin{cases} -\alpha/\pi & \text{if } -\pi < \alpha < 0 \\ \alpha/\pi & \text{if } 0 < \alpha < \pi \end{cases} \qquad \text{(Fig. 420).}$$

Solution. Since $\Phi(1, \alpha)$ is even, $b_n = 0$, and from (8) we obtain $a_0 = \frac{1}{2}$ and

$$a_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \frac{\alpha}{\pi} \cos n\alpha\, d\alpha + \int_0^{\pi} \frac{\alpha}{\pi} \cos n\alpha\, d\alpha\right] = \frac{2}{n^2\pi^2}(\cos n\pi - 1).$$

Hence, $a_n = -4/(n^2\pi^2)$ if n is odd, $a_n = 0$ if $n = 2, 4, \cdots$, and the potential is

$$\Phi(r, \theta) = \frac{1}{2} - \frac{4}{\pi^2}\left[r \cos \theta + \frac{r^3}{3^2} \cos 3\theta + \frac{r^5}{5^2} \cos 5\theta + \cdots\right].$$

Figure 421 shows the unit disk and some of the equipotential lines (curves $\Phi = const$). ■

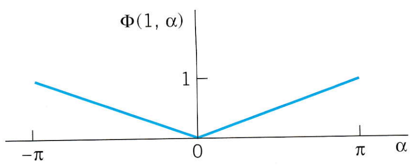

Fig. 420. Boundary values in Example 1

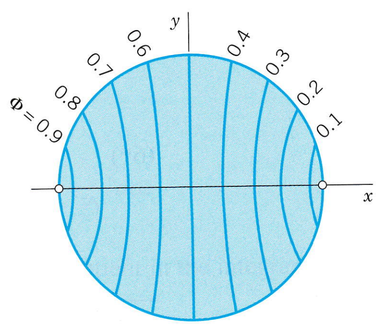

Fig. 421. Potential in Example 1

PROBLEM SET 18.5

1. Verify (3).

2. Show that every term in (7) is a harmonic function in the disk $r < R$.

3. Give the details of the derivation of the series (7) from the Poisson formula (5).

4–13 HARMONIC FUNCTIONS IN A DISK

Using (7), find the potential $\Phi(r, \theta)$ in the unit disk $r < 1$ having the given boundary values $\Phi(1, \theta)$. Using the sum of the first few terms of the series, compute some values of Φ and sketch a figure of the equipotential lines.

4. $\Phi(1, \theta) = \sin 2\theta$

5. $\Phi(1, \theta) = 2 \sin^2 \theta$

6. $\Phi(1, \theta) = \cos^2 5\theta$

7. $\Phi(1, \theta) = \theta$ if $-\pi < \theta < \pi$

8. $\Phi(1, \theta) = \theta$ if $0 < \theta < 2\pi$

9. $\Phi(1, \theta) = \sin^3 2\theta$

10. $\Phi(1, \theta) = \cos^4 \theta$

11. $\Phi(1, \theta) = \theta^2$ if $-\pi < \theta < \pi$

12. $\Phi(1, \theta) = 1$ if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,
$\Phi(1, \theta) = 0$ if $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$

13. $\Phi(1, \theta) = \theta$ if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,
$\Phi(1, \theta) = \pi - \theta$ if $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$

14. TEAM PROJECT. Potential in a Disk. (a) Mean value property. Show that the value of a harmonic function Φ at the center of a circle C equals the mean of the value of Φ on C (see Sec. 18.4, footnote 1, for definitions of mean values).

(b) Separation of variables. Show that the terms of (7) appear as solutions in separating the Laplace equation in polar coordinates.

(c) Harmonic conjugate. Find a series for a harmonic conjugate Ψ of Φ from (7).

(d) Power series. Find a series for $F(z) = \Phi + i\Psi$.

15. CAS EXPERIMENT. Series (7). Write a program for series developments (7). Experiment on accuracy by computing values from partial sums and comparing them with values that you obtain from your CAS graph. Do this **(a)** for Example 1 and Fig. 421, **(b)** for Φ in Prob. 8 (which is discontinuous on the boundary!), **(c)** for a Φ of your choice with continuous boundary values, **(d)** for Φ with discontinuous boundary values.

18.6 General Properties of Harmonic Functions

General properties of harmonic functions can often be obtained from properties of analytic functions in a simple fashion. Specifically, important mean value properties of harmonic functions follow readily from those of analytic functions. The details are as follows.

THEOREM 1

Mean Value Property of Analytic Functions

Let $f(z)$ be analytic in a simply connected domain D. Then the value of $F(z)$ at a point z_0 in D is equal to the mean value of $F(z)$ on any circle in D with center at z_0.

PROOF In Cauchy's integral formula (Sec. 14.3)

$$F(z_0) = \frac{1}{2\pi i} \oint_C \frac{F(z)}{z - z_0}\, dz \tag{1}$$

we choose for C the circle $z = z_0 + re^{i\alpha}$ in D. Then $z - z_0 = re^{i\alpha}$, $dz = ire^{i\alpha}\, d\alpha$, and (1) becomes

$$F(z_0) = \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\alpha})\, d\alpha. \tag{2}$$

The right side is the mean value of F on the circle (= value of the integral divided by the length 2π of the interval of integration). This proves the theorem. ■

For harmonic functions, Theorem 1 implies

THEOREM 2

Two Mean Value Properties of Harmonic Functions

Let $\Phi(x, y)$ be harmonic in a simply connected domain D. Then the value of $\Phi(x, y)$ at a point (x_0, y_0) in D is equal to the mean value of $\Phi(x, y)$ on any circle in D with center at (x_0, y_0). This value is also equal to the mean value of $\Phi(x, y)$ on any circular disk in D with center (x_0, y_0). [See footnote 1 in Sec. 18.4.]

PROOF The first part of the theorem follows from (2) by taking the real parts on both sides,

$$\Phi(x_0, y_0) = \operatorname{Re} F(x_0 + iy_0) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(x_0 + r \cos \alpha, y_0 + r \sin \alpha)\, d\alpha.$$

The second part of the theorem follows by integrating this formula over r from 0 to r_0 (the radius of the disk) and dividing by $r_0^2/2$,

$$\Phi(x_0, y_0) = \frac{1}{\pi r_0^2} \int_0^{r_0} \int_0^{2\pi} \Phi(x_0 + r \cos \alpha, y_0 + r \sin \alpha) r \, d\alpha \, dr. \tag{3}$$

The right side is the indicated mean value (integral divided by the area of the region of integration). ■

Returning to analytic functions, we state and prove another famous consequence of Cauchy's integral formula. The proof is indirect and shows quite a nice idea of applying the ML-inequality. (A *bounded region* is a region that lies entirely in some circle about the origin.)

THEOREM 3

Maximum Modulus Theorem for Analytic Functions

Let $F(z)$ be analytic and nonconstant in a domain containing a bounded region R and its boundary. Then the absolute value $|F(z)|$ cannot have a maximum at an interior point of R. Consequently, the maximum of $|F(z)|$ is taken on the boundary of R. If $F(z) \neq 0$ in R, the same is true with respect to the minimum of $|F(z)|$.

PROOF We assume that $|F(z)|$ has a maximum at an interior point z_0 of R and show that this leads to a contradiction. Let $|F(z_0)| = M$ be this maximum. Since $F(z)$ is not constant, $|F(z)|$ is not constant, as follows from Example 3 in Sec. 13.4. Consequently, we can find a circle C of radius r with center at z_0 such that the interior of C is in R and $|F(z)|$ is smaller than M at some point P of C. Since $|F(z)|$ is continuous, it will be smaller than M on an arc C_1 of C that contains P (see Fig. 422), say,

$$|F(z)| \leqq M - k \quad (k > 0) \qquad \text{for all } z \text{ on } C_1.$$

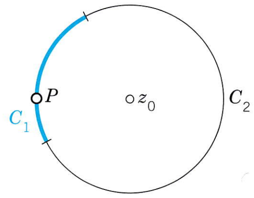

Fig. 422. Proof of Theorem 3

Let C_1 have the length L_1. Then the complementary arc C_2 of C has the length $2\pi r - L_1$. We now apply the ML-inequality (Sec. 14.1) to (1) and note that $|z - z_0| = r$. We then obtain (using straightforward calculation in the second line of the formula)

$$M = |F(z_0)| \leqq \frac{1}{2\pi}\left|\int_{C_1} \frac{F(z)}{z - z_0}\, dz\right| + \frac{1}{2\pi}\left|\int_{C_2} \frac{F(z)}{z - z_0}\, dz\right|$$

$$\leqq \frac{1}{2\pi}\left(\frac{M - k}{r}\right) L_1 + \frac{1}{2\pi}\left(\frac{M}{r}\right)(2\pi r - L_1) = M - \frac{kL_1}{2\pi r} < M$$

that is, $M < M$, which is impossible. Hence our assumption is false and the first statement is proved.

Next we prove the second statement. If $F(z) \neq 0$ in R, then $1/F(z)$ is analytic in R. From the statement already proved it follows that the maximum of $1/|F(z)|$ lies on the boundary of R. But this maximum corresponds to the minimum of $|F(z)|$. This completes the proof. ■

This theorem has several fundamental consequences for harmonic functions, as follows.

THEOREM 4

Harmonic Functions

Let $\Phi(x, y)$ be harmonic in a domain containing a simply connected bounded region R and its boundary curve C. Then:

(I) (Maximum principle) *If $\Phi(x, y)$ is not constant, it has neither a maximum nor a minimum in R. Consequently, the maximum and the minimum are taken on the boundary of R.*

(II) *If $\Phi(x, y)$ is constant on C, then $\Phi(x, y)$ is a constant.*

(III) *If $h(x, y)$ is harmonic in R and on C and if $h(x, y) = \Phi(x, y)$ on C, then $h(x, y) = \Phi(x, y)$ everywhere in R.*

PROOF **(I)** Let $\Psi(x, y)$ be a conjugate harmonic function of $\Phi(x, y)$ in R. Then the complex function $F(z) = \Phi(x, y) + i\Psi(x, y)$ is analytic in R, and so is $G(z) = e^{F(z)}$. Its absolute value is

$$|G(z)| = e^{\text{Re } F(z)} = e^{\Phi(x, y)}.$$

From Theorem 3 it follows that $|G(z)|$ cannot have a maximum at an interior point of R. Since e^{Φ} is a monotone increasing function of the real variable Φ, the statement about the maximum of Φ follows. From this, the statement about the minimum follows by replacing Φ by $-\Phi$.

(II) By (I) the function $\Phi(x, y)$ takes its maximum and its minimum on C. Thus, if $\Phi(x, y)$ is constant on C, its minimum must equal its maximum, so that $\Phi(x, y)$ must be a constant.

(III) If h and Φ are harmonic in R and on C, then $h - \Phi$ is also harmonic in R and on C, and by assumption, $h - \Phi = 0$ everywhere on C. By (II) we thus have $h - \Phi = 0$ everywhere in R, and (III) is proved. ■

The last statement of Theorem 4 is very important. It means that a *harmonic function is uniquely determined in R by its values on the boundary of R*. Usually, $\Phi(x, y)$ is required to be harmonic in R and continuous on the boundary of R, that is,

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \Phi(x, y) = \Phi(x_0, y_0), \text{ where } (x_0, y_0) \text{ is on the boundary and } (x, y) \text{ is in } R.$$

Under these assumptions the maximum principle (I) is still applicable. The problem of determining $\Phi(x, y)$ when the boundary values are given is called the **Dirichlet problem** for the Laplace equation in two variables, as we know. From (III) we thus have, as a highlight of our discussion,

THEOREM 5

Uniqueness Theorem for the Dirichlet Problem

If for a given region and given boundary values the Dirichlet problem for the Laplace equation in two variables has a solution, the solution is unique.

PROBLEM SET 18.6

1. Integrate $|z|^2$ around the unit circle. Does your result contradict Theorem 1?

2–4 VERIFY THEOREM 1 for the given $F(z)$, z_0, and circle of radius 1.

2. $(z + 1)^3$, $z_0 = 2$

3. $(z - 2)^2$, $z_0 = \frac{1}{3}$

4. $10z^4$, $z_0 = 0$

5–7 VERIFY THEOREM 2 for the given $\Phi(x, y)$, (x_0, y_0) and circle of radius 1.

5. $(x - 2)(y - 2)$, $(4, -4)$

6. $x^2 - y^2$, $(3, 8)$

7. $x^3 - 3xy^2$, $(1, 1)$

8. Derive Theorem 2 from Poisson's integral formula.

9. CAS EXPERIMENT. Graphing Potentials. Graph the potentials in Probs. 5 and 7 and for three other functions of your choice as surfaces over a rectangle or a disk in the xy-plane. Find the locations of maxima and minima by inspecting these graphs.

10. TEAM PROJECT. Maximum Modulus of Analytic Functions. (a) Verify Theorem 3 for (i) $F(z) = z^2$ and the square $4 \leqq x \leqq 6$, $2 \leqq y \leqq 4$, (ii) $F(z) = e^{3z}$ and any bounded domain, (iii) $F(z) = \sin z$ and the unit disk.

(b) $F(x) = \cos x$ (x real) has a maximum 1 at 0. How does it follow that this cannot be a maximum of $|F(z)| = |\cos z|$ in a domain containing $z = 0$?

(c) $F(z) = 1 + |z|^2$ is not zero in the disk $|z| \leqq 4$ and has a minimum at an interior point. Does this contradict Theorem 3?

(d) If $F(z)$ is analytic and not constant in the closed unit disk D: $|z| \leqq 1$ and $|F(z)| = c = \mathit{const}$ on the unit circle, show that $F(z)$ must have a zero in D. Can you extend this to an arbitrary simple closed curve?

11–13 MAXIMUM MODULUS

Find the location and size of the maximum of $|F(z)|$ in the unit disk $|z| \leqq 1$.

11. $F(z) = z^2 - 1$

12. $F(z) = az + b$ (a, b complex)

13. $F(z) = \cos 2z$

14. Verify the maximum principle for $\Phi(x, y) = e^x \cos y$ and the rectangle $a \leqq x \leqq b$, $0 \leqq y \leqq 2\pi$.

15. (Conjugate) Do Φ and a harmonic conjugate Ψ of Φ in a region R have their maximum at the same point of R?

16. (Conformal mapping) Find the location (u_1, v_1) of the maximum of $\Phi^* = e^u \cos v$ in R^*: $|w| \leqq 1$, $v \geqq 0$, where $w = u + iv$. Find the region R that is mapped onto R^* by $w = f(z) = z^2$. Find the potential in R resulting from Φ^* and the location (x_1, y_1) of the maximum. Is (u_1, v_1) the image of (x_1, y_1)? If so, is this just by chance?

CHAPTER 18 REVIEW QUESTIONS AND PROBLEMS

1. Why can potential problems be modeled and solved by complex analysis? For what dimensions?

2. What is a harmonic function? A harmonic conjugate?

3. Give a few examples of potential problems considered in this chapter.

4. What is a complex potential? What does it give physically?

5. How can conformal mapping be used in connection with the Dirichlet problem?

6. What heat problems reduce to potential problems? Give a few examples.

7. Write a short essay on potential theory in fluid flow from memory.

8. What is a mixed boundary value problem? Where did it occur?

9. State Poisson's formula and its derivation from Cauchy's formula.

10. State the maximum modulus theorem and mean value theorems for harmonic functions.

11. Find the potential and complex potential between the plates $y = x$ and $y = x + 10$ kept at 10 V and 110 V, respectively.

12. Find the potential between the cylinders $|z| = 1$ cm having potential 0 and $|z| = 10$ cm having potential 20 kV.

13. Find the complex potential in Prob. 12.

14. Find the equipotential line $U = 0$ V between the cylinders $|z| = 0.25$ cm and $|z| = 4$ cm kept at -220 V and 220 V, respectively. (Guess first.)

15. Find the potential between the cylinders $|z| = 10$ cm and $|z| = 100$ cm kept at the potentials 10 kV and 0, respectively.

16. Find the potential in the angular region between the plates $\text{Arg } z = \pi/6$, kept at 8 kV, and $\text{Arg } z = \pi/3$, kept at 6 kV.

17. Find the equipotential lines of $F(z) = i \text{ Ln } z$.

18. Find and sketch the equipotential lines of $F(z) = (1 + i)/z$.

19. What is the complex potential in the upper half-plane if the negative half of the x-axis has potential 1 kV and the positive half is grounded?

20. Find the potential on the ray $y = x$, $x > 0$, and on the positive half of the x-axis if the positive half of the y-axis is at 1200 V and the negative half is grounded.

21. Interpret Prob. 20 as a problem in heat conduction.

22. Find the temperature in the upper half-plane if the portion $x > 2$ of the x-axis is kept at 50°C and the other portion at 0°C.

23. Show that the isotherms of $F(z) = -iz^2 + z$ are hyperbolas.

24. If the region between two concentric cylinders of radii 2 cm and 10 cm contains water and the outer cylinder is kept at 20°C, to what temperature must we heat the inner cylinder in order to have 30°C at distance 5 cm from the axis?

25. What are the streamlines of $F(z) = i/z$?

26. What is the complex potential of a flow around a cylinder of radius 4 without circulation?

27. Find the complex potential of a source at $z = 5$. What are the streamlines?

28. Find the temperature in the unit disk $|z| \leqq 1$ in the form of an infinite series if the left semicircle of $|z| = 1$ has the temperature of 50°C and the right semicircle has the temperature 0°C.

29. Same task as in Prob. 28 if the upper semicircle is at 40°C and the lower at 0°C.

30. Find a series for the potential in the unit disk with boundary values $\Phi(1, \theta) = \theta^2$ $(-\pi < \theta < \pi)$.

SUMMARY OF CHAPTER 18

Complex Analysis and Potential Theory

Potential theory is the theory of solutions of **Laplace's equation**

$$\nabla^2 \Phi = 0. \tag{1}$$

Solutions whose second partial derivatives are ***continuous*** are called **harmonic functions.** Equation (1) is the most important PDE in physics, where it is of interest in two and three dimensions. It appears in electrostatics (Sec. 18.1), steady-state heat problems (Sec. 18.3), fluid flow (Sec. 18.4), gravity, etc. Whereas the three-dimensional case requires other methods (see Chap. 12), two-dimensional potential theory can be handled by complex analysis, since the real and imaginary parts of an analytic function are harmonic (Sec. 13.4). They remain harmonic under conformal mapping (Sec. 18.2), so that **conformal mapping** becomes a powerful tool in solving boundary value problems for (1), as is illustrated in this chapter. With a real potential Φ in (1) we can associate a **complex potential**

$$F(z) = \Phi + i\Psi \qquad \text{(Sec. 18.1).} \tag{2}$$

Then both families of curves $\Phi = const$ and $\Psi = const$ have a physical meaning. In electrostatics, they are equipotential lines and lines of electrical force (Sec. 18.1). In heat problems, they are isotherms (curves of constant temperature) and lines of heat flow (Sec. 18.3). In fluid flow, they are equipotential lines of the velocity potential and streamlines (Sec. 18.4).

For the disk, the solution of the Dirichlet problem is given by the **Poisson formula** (Sec. 18.5) or by a series that on the boundary circle becomes the Fourier series of the given boundary values (Sec. 18.5).

Harmonic functions, like analytic functions, have a number of general properties; particularly important are the **mean value property** and the **maximum modulus property** (Sec. 18.6), which implies the uniqueness of the solution of the Dirichlet problem (Theorem 5 in Sec. 18.6).