SPECIFIC NEEDS IN MATHEMATICS
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INTRODUCTION

The presented text is devoted to specific learning disabilities in mathematics and is written in two languages. It is intended for foreign students studying at Masaryk University, and for students of the elementary school teaching, students of the follow-up master study programme of teaching mathematics for primary schools and for students of Special Pedagogy.

The percentage of children with educational problems at primary schools has been constantly increasing. These can be children whom some of the specific learning disabilities have been diagnosed, while these children can be of average or above-average talent in certain areas. They require the professional special pedagogical help, understanding of the teacher of the respective subject and of the parents as well. In the text we will focus on specific learning disabilities in mathematics, in particular dyscalculia, but also on the influence of other specific learning disabilities on the pupil's success in mathematics. Monitoring of the basic functional deficits of their mathematical abilities is not of less significance.

The professional literature contains plenty of opinions on the possibilities how to provide the care to children with the diagnosed dyscalculia. These opinions can arise from medical, psychotherapeutic, specially pedagogical, psychological or pedagogical approaches. As a rule, individual approaches cannot be applied separately, but in most cases the child and its problems are considered in the more complex way. The suggestions for re-education procedures stated in the text are based on pedagogical approaches; however, they respect the construction of the mathematical notions in their connections. Although the text contains minimum of the mathematical theory, it should lead the reader to reflect on the mathematical correctness of the used procedures, so that they do not contradict the mathematical theory and thus do not cause the children problems in their further study.

1 PREREQUISITES

Before we focus on the specific manifestations of dyscalculia and its re-education, let us mention basic activities which are necessary to be performed at every child:

- Performing proper diagnostics of the child's problems from the point of view of its mathematical abilities, skills and knowledge, and its own dyscalculic errors.
- Respecting the fact that pieces of mathematical knowledge are not transferrable; each child achieves the knowledge on the basis of its own manipulative activities, experience and train of thought.
- Finding the "gravity point" from which it is necessary to start the re-education.
- Respecting the fact that remedial and re-educational procedures are in the first phase based on understanding the given phenomenon.
- Using manipulative activities for understanding the subject-matter so that there arises the "I SEE effect" when the child has comprehended the subject-matter thanks to its own mental activity.
- Building knowledge based not only on the mere remembering pieces of knowledge, being aware of the fact that memorizing essential subject-matter is performed in the second phase of the teaching process after the child has understood the meaning of individual notions and operations.
- It is necessary to perform the remedial procedures by small steps so that the child is not overloaded by the excessive amount of the subject-matter.
- It is important to respect the child's own strategies and trains of thought.

The timely diagnostics and detection of the causes of the child's problems in mathematics, understanding of its own trains of thought and finding suitable procedures for exactly this child are the preconditions for success. Besides, the child itself, its teachers, parents or grandparents, and teaching methods are the decisive elements at this process as well.

2 SPECIFIC LEARNING DISABILITIES

The problem of specific learning disabilities and of educating pupils with specific learning disabilities are topical both for schools and for many families. Educating children with specific learning disabilities has been considered constantly in connection with their placing into specially oriented schools or classes, or in connection with an inclusive education. The inclusive education of pupils with learning disabilities in ordinary classes of primary and secondary schools requires the teacher's qualified approach and his ability to carry out differential and individualized education of these pupils.

In the past (in the first half of the twentieth century), the problem of specific learning disabilities was not paid much attention to. Pupils with learning problems were classified to two or three groups – simply, they were not "good at it" or they were considered dull, possibly lazy. However, they were preparing for lessons properly and fulfilling school duties required the disproportionate amount of their time and effort. The problems often appeared in one subject, but in other subjects these pupils were achieving average or above-average results. Also, in that time many mathematics teachers were trying to find procedures which would facilitate the mathematics subject-matter to pupils with disabilities.

From the historical point of view, since the antiquity there have been searched the methods which could facilitate mastering the basics to the pupils who had problems with e.g. the trivium. In the following periods, many outstanding scientists and educators paid their attentions to the pupils who had learning problems and at the same time their intellect was on a high level. Among them we can name e.g. Erasmus of Rotterdam (1567 – 1636), John Amos Comenius (1592 – 1670), John Looke (1632 – 1704), Johan Heinrich Pestalozzi (1746 – 1827), Johann Friedrich Herbart (1776 – 1841), and many others. From our scientists and educators who dealt with learning disabilities, let us mention at least O. Chlup, Z. Matějček, L. Košč, J. Langmaier, Z. Žlab, from our contemporaries e.g. M. Vítková, M. Bartoňová, V. Pokorná, O. Zelinková, J. Novák. A great interest is devoted to the pupils' problems in mathematics at the Department of Mathematics of the MU Faculty of Education in Brno.

Specific learning disabilities started to be studied systematically by psychologists and special pedagogues in the last century. In 1976, the Office of Education in the USA issued the definition of specific developmental learning disabilities as: *"The term means a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, that may manifest itself in an imperfect ability to listen, think, speak, read, write, spell, or to do mathematical calculations, including conditions such as perceptual disabilities, brain injury, minimal brain dysfunction, dyslexia, and developmental aphasia."*

(Matějček, 1987)

In 1980, a group of experts from the National Health Institute in Washington together with experts from the Orton Society and other institutions formulated the following definition:

"Learning disabilities is a generic term that refers to a heterogeneous group of disorders manifested by significant difficulties in the acquisition and use of listening, speaking, reading, writing, reasoning or mathematical abilities. These disorders are intrinsic to the individual and resumed to be due to central nervous system dysfunction. Even though a learning disability may occur concomitantly with other handicapping conditions (e.g. sensory impairment, mental retardation, social and emotional disturbance) or environmental influences (e.g. cultural differences, insufficient/inappropriate instruction, psychogenic factors), it is not the direct result of those conditions or influences." (Matějček, 1987*)*

In 1992, in sections F 80 to F 89 of the 10th revision of the International Classification of Diseases, there were given Disorders of Mental Development, and specifically in part F 81 there were given Specific Developmental Disorders of Scholastic Skills:
F 81.0 Specific reading disorder
F 81.1 Specific spelling disorder
F 81.2 Specific disorders of arithmetical skills
F 81.3 Mixed disorder of scholastic skills
F 81.8 Other developmental disorders of scholastic skills
F 81.9 Developmental disorder of scholastic skills, unspecified
(International Classification of Diseases 1992).

Gradually, the disorders of reading, spelling, counting and other abilities and skills have been studied. The causes of these disorders have been examined systematically and the educational procedures helping the pupils to overcome these problems have been searched for.

The contemporary perspective on the definition of specific learning disabilities is given by e.g. Věra Pokorná (Pokorná, 2010) and in her publication she cites some approaches of the authors from the USA. *"Specific learning disabilities mean a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, which may manifest itself in the imperfect ability to listen, think, speak, read, write, spell, or do mathematical calculations. The term includes such conditions as perceptual disabilities, brain injury, minimal brain dysfunction, dyslexia, and developmental aphasia. The term does not include an individual with learning problems that are primarily the result of visual, hearing, or motor disabilities, of mental retardation, of emotional disturbance, or of environmental, cultural, or economic disadvantage."* (Pokorná, 2010, p. 18, cit. Spear-Sweling 1999).

According to our research (Blažková, Pavlíčková, 2010), at primary schools there are 16 % of pupils with diagnosed specific learning disabilities, from which there are 5–6 % of pupils with dyscalculia. From the whole examined sample of pupils there are 1–2 % of pupils with dyscalculia. The pupils' problems in mathematics could be of different reasons. These can be minimal brain dysfunctions, the improper way of teaching, the negative attitude to mathematics and to learning as the whole, the distaste for work and any activity, the insufficient preparation for lessons, and many others. The reasons are various, however children are not able to analyse them in depth and understand their problems. The adults often cannot reveal the trains of thought which happen in the child's brain while working with mathematical notions; they are trying to find the help, but it could be ineffective in many cases. They train the child in the way which would suit them, but they do not perceive the child's trains of thought. Their help is sometimes based on the mere memorizing of facts, on applying inappropriate teaching methods; sometimes it is based on incorrect prerequisites, etc.

Along with the efforts to rectify the flaws, it is also necessary to consider other child's problems in the area of deficits of partial functions of mathematical abilities, especially visual and hearing perceptions, fine and gross motor skills, laterality, concentration, and others (Pavlíčková, 2015).

2.1 TERMINOLOGY

While working with children with specific learning disabilities, we often meet different deviations which are demonstrated by lowered perception, attention, memory, speech, motor skills etc., and which are usually caused by the deviations of the nervous system function. They can, but do not have the impact on the children's success at school. They are usually represented by the following abbreviations:

MBD – minimal brain dysfunction

MEC – minimal encephalopathy in children

The syndrome of the minimal brain dysfunction appears at children who can be of average to above-average intellect and their problems at learning or behaviour are caused by the deviations of the central nervous system functions.

ADD – Attention Deficit Disorder

ADHD - Attention Deficit Hyperactivity Disorder

SLD – Specific Learning Disabilities

2.2 FORMS OF SOME DISORDERS

In the school practice we often meet children with some of the following problems which influence the level of acquiring mathematical skills and knowledge:

- **Concentration disorders**
  Children have problems to concentrate on a certain activity, they are tired easily, they are inattentive, they digress from the topic easily, and they can be disturbed by any stimulus which is not connected to the performed activity. Solving any mathematical task or problem requires the full concentration and the absence of success while solving could be caused by the inability of a child to concentrate on the problem fully. Children suffer from the lack of time, they cannot keep up; they take longer to get to the heart of the matter. They are not alert and quick enough, which is manifested by the fact that e.g. they are not successful in competitions aimed at swiftness.

- **Left-right orientation disorders (right-left confusion)** – not pronounced laterality (the preferred usage of one of the pair organs) causes problems in mathematics to children e.g. while writing numbers which are one-sidedly oriented, numbers with more figures, while understanding the relations on the real line, etc.

- **Spatial orientation disorders** – children live in a three-dimensional space and naturally, they perceive relations among objects and their distribution in the space (relations above, under, up, down, next to, at the front, at the back, in front of, behind). However, they have problems with understanding the representation of a spatial situation in the plane with the help of some projection (e.g. a parallel projection) in the picture. The child knows very well what e.g. a cube is, but it does not understand the tangle of lines on the paper which represent the cube and also it often cannot understand the drawn net of the cube and other solid figures.

- **Time orientation disorders** – children at first perceive chronological succession during the day, usually according to events and regularly repeated activities, later in longer time periods (a week, month, year). They have problems with understanding time units and their conversions mainly because there is used the number system with the base sixty (1 hour = 60 minutes, 1 minute = 60 seconds), and also some children can have difficulty understanding relations on a circular clock face and the linear flow of time. Reading of time data noted down in a digital form can cause problem to some children as well.

- **Auditory processing disorders** – the child does not have a hearing impairment, it hears properly, but does not comprehend what has just been said. It quite often asks about the fact which has just been said. The adult should welcome it because the child knows what to ask about, when it has missed it. Moreover, in the class there are surely more children who do not perceive through hearing, but they do not ask.

- **Disorders of rhythm reproduction** – the perception of rhythm and its reproduction is very important for mathematics – e.g. while counting by one, orientation in the number line, following the regularities, dependences, etc.

- **Visual processing disorders** – the child can see very well, but it does not perceive fully what it should as the mathematical subject-matter – e.g. it can see that 1 dm is divided into 10 cm, but the mathematical notion dealing with the relation of these units and their conversion does not arise in the brain. The child is not able to distinguish changes, orientate in a geometric picture etc.

- **Speech disorders** – besides logopedics problems, in mathematics the most important ability is to formulate ideas in own words. The accuracy of mathematical formulation is the reflection of the accuracy of the thought process. When the child says: "I know it, but I cannot say it,", then the child usually does not know, it merely guesses. If the child's notion is formed properly and the child understands the substance of the problem, then it is able to express it in words. Children are not supposed to perform the definitions of mathematical notions.

- **Fine and gross motor skills disorders** – are manifested mainly at manipulative activities while deducing basic notions and operations, while noting numbers, operations algorithms, especially while drawing.

- **Behavioural disorders as the result of learning disabilities** – if children are not successful in mathematics, they either draw attention to themselves in another way (by showing off as a class clown, by the lack of discipline), or they retreat within themselves and cease to communicate, which is a worse case. Getting such child to resume the communication is always quite demanding.

2. 3 CLASSIFICATION OF SPECIFIC LEARNING DISABILITIES

The child's success in mathematics is influenced also by other specific learning disabilities. For the sake of clarity, let us give all the described learning disabilities and emphasize those having impact on the pupil's performance in mathematics. In the professional literature the following ones are described:

Dyslexia – the disability can affect distinguishing individual letters, the speed of reading, the correctness of reading, or understanding the read text. For the child with dyslexia it is difficult to read and understand the defining of mathematical tasks, especially of word problems, where it is necessary to transform the text given by a Czech sentence into the mathematical language. They also have difficulties with reading the symbolic mathematical notation. However, some of them can understand the symbolic mathematical notation and are able to use it, and this can save them in mathematics.

Dysgraphia – the disorder affects acquiring individual letters, links sound – letter, layout of the writing. In mathematics the child with dysgraphia has difficulties with acquiring individual figures and symbols, the link "number" and "number notation by figures", the distinction between notions “number” and “figure” and their notation, further the notation of numbers in lines (e.g. they do not keep the same size of all figures in the notation of the number with more digits), or in the notation of numbers in algorithms where the accuracy of the notation according to individual orders is important. The mistakes in mathematical operations can be caused by the untidiness of the notation or by the considerate slowness of writing.

Dysorthographia – the disorder of spelling. These are not gross errors caused by bad knowledge, but these are specific problems connected with e.g. the disability to distinguish sibilant consonants, the length of vowels, palatalization, omitting, adding, and confusion of letters or syllables, gaps between words in writing. It can appear at dictated five-minute tests when the child has to master too many tasks.

Dyscalculia – the disorder affects mainly forming mathematical concepts, problems connected with operations with numbers, disorders of spatial skills, etc. It will be dealt with in detail in the following text.

Dyspinxia – the disorder in the area of drawing skills, clumsiness in fine motor skills of hand and figures – it is manifested especially while drawing. Also the graphic representation of the three-dimensional situation in the plane, in the picture, causes problems to children.

Dysmusia – the lowered or completely lost sense for music – the melody and rhythm. Especially, the loss of the sense for rhythm is a big problem for mathematics.

Dyspraxia – the disorder of skillfulness; it can influence the neatness of mathematical writing, drawing pictures, which can be caused by the children’s clumsiness as well.

2.4 DEFINITION OF DYSCALCULIA

The term dyscalculia denotes a specific disorder of mathematical abilities. The child performs considerably worse than it would be expected with respect to its intellect. In the literature there are published different definitions of dyscalculia; let us give at least some of them. According to the 10th revision of the International Classification of Diseases “Mental and Behavioural Disorders”, dyscalculia belongs among “Specific Developmental Disorders of Scholastic Skills” under the code F 81.2. (1992).
“This disorder involves a specific impairment in arithmetical skills that is not solely explicable on the basis of general mental retardation or of inadequate schooling. The deficit concerns mastery of basic computational skills of addition, subtraction, multiplication, and

division rather than of the more abstract mathematical skills involved in algebra, trigonometry, geometry, or calculus."

However, let us note that if the child has problems in the area of acquiring basic arithmetical operations, these problems will manifest themselves also in other parts of mathematics, e.g. in algebra where we deal with coefficients in algebraic expressions, in equations, or with exponents at variables.

Another definition of dyscalculia was formulated by Ladislav Košč in 1985: *"The developmental dyscalculia is a structural disorder of mathematical abilities which has its origin in the genetically or perinatally influenced impairment to such parts of the brain which are the direct anatomical-physiological substrate of the age adequate maturing of mathematical functions which at the same time do not result in the lowering of general cognitive functions."*

This definition is followed up by J. Novák who gives the widened definition of dyscalculia: *"The developmental dyscalculia is a specific counting disorder which is manifested by evident difficulties of acquiring and performing arithmetical skills, while the child's sociocultural background is the usual one and its overall general intellectual precondition is at the lower limit of the average range or above. The intellect should have the characteristic internal structure within which the level of mathematical abilities is significantly reduced and its structure is disturbed, in the presence of the displays of the central nervous system dysfunctions due to hereditary or developmental influences."*. (Novák 2004).

Based on our experience from the real contact with children who have intellectual preconditions in the average range, or even above, but they have problems in mathematics, we conclude that the approach to the child should not be determined by the fact if the dyscalculia has or has not been diagnosed, but that it is important to understand the child's individuality, its specific problems in mathematics, and to seek appropriate re-educative procedures tailored exactly to this child.

Nevertheless, it is important to search particularly for the causes of the disorders and classify them at least according to the following list:

1. The causes which are conditioned by partially removable influences, like e.g. the learning style, teaching method, preparation suitability for lessons, motivation for learning, etc.
2. The causes which can be removed with more difficulties, like hereditary influences or the disruption of such parts of the brain which influence the mathematical abilities formation.
3. The causes which are induced by the low talent for mathematics or the low talent generally.

Therefore, it is different to work with the children who have the specific learning disability and the children whose problems in mathematics have been caused by another reason. The choice of remedial re-educative and compensatory exercises is for such children different because some children are able to master the subject-matter through the suitable tutoring with common teaching methods, while the other ones need the formation of such mechanisms which can replace the affected functions, or can develop them in a suitable way.

2.5 CLASSIFICATION OF DYSCALCULIA

2. 5. 1 Classification According to L. Košč

Ladislav Košč (Košč, 1978) provided the dyscalculia classification with respect to basic problems which appear at children in connection with the mathematical notions and relations development and forming, in connection with reading and writing of mathematical expressions as follows:

Practognostic dyscalculia

- the disorder of manipulation with concrete objects or symbols,
- the disorder at creating groups of objects,
- the incomprehension of the term of a natural number,
- the inability to compare numbers of elements,
- the inability to differentiate geometrical figures,
- the disorder of the spatial factor.

Verbal dyscalculia

- problems in naming the number of objects, operation symbols,
- the inability to enumerate the series of numbers in a certain ordering,
- the incomprehension of the pronounced number,
- the incomprehension of the verbal formulation of mathematical symbols and signs.

Lexical dyscalculia

- the inability to read mathematical symbols (figures, numbers, symbols for comparing, operation signs),
- the confusion of figures similar in shape,
- the disorder of spatial orientation,
- the right-left confusion.

Graphical dyscalculia

- the inability to write mathematical symbols (figures, numbers and others),
- the disorder at the notation of numbers with more digits,
- the inability to write numbers from the dictation,
- the inability to write one number below the other (figures of the same order),
- problems while drawing diagrams,
- the disorder of the right-left and spatial orientation.

Operational dyscalculia

- the impaired ability to perform mathematical operations with natural numbers (but also with other numbers),
- the confusion of individual operations
- disorders at acquiring memory connections,
- the inability to respect the priority while performing more operations of different parity,
- problems at written algorithms of different operations.

Ideognostical dyscalculia

- the disorder in the area of the concept activity,
- the disorder at understanding mathematical notions and their relationships,
- the disorder at generalizing,
- problems while solving word problems.

2.5. 2 Classification According to J. Novák

The impairment of mathematical abilities has a lot of different reasons and manifestations, and the more general classification is given by J. Novák (Novák, 2004):

Calculastenia - it means a slight impairment of mathematical knowledge caused e.g. by an insufficient stimulation at school or in the family, while the intellectual and mathematical abilities are in the average range.
The calculastenia is further classified as an emotional calculastenia (inappropriate reactions of other involved people to the problems in mathematics), a social calculastenia (the influence of the social environment, the insufficient preparation for school) and a didactogenic calculastenia (inappropriate teaching methods just for this child).

Hypocalculia - it is the disorder of essential arithmetic skills which can be caused by an uneven composition of mathematical abilities at the overall level of intellectual abilities in the average range or above.

Oligocalculia - it is distinguished by the disrupted structure of mathematical abilities and the low level of general intellectual abilities.

Developmental dyscalculia - in essence, it uses the classification according to L. Košč.

Acalculia - it is the disorder of mastering arithmetical operations and arithmetical skills which could arise e.g. on the basis of an experienced trauma, while previously the skills were developed adequately.

2. 5. 3 Classification According to the Mathematical Content (R.Blažková)

The classification concerns the area of the subject-matter where children experience problems. Understanding and mastering one area is the necessary precondition for understanding and mastering the next area. It relates mainly to the following areas:

Forming of the term number – firstly the natural number, later the decimal number, the fraction, the rational number, generally the real number. If the child has problems with understanding the notion, it will not get to the necessary abstraction (to understand e.g. number 5 without the concrete objects), and it has no chance to continue in other mathematical topics.

Reading and writing numbers, numeration, ordering, comparing of numbers, rounding of natural and decimal numbers. Due to the problems in numeration, the child cannot form the necessary concept of the set of all natural numbers and their ordering.

Operations with numbers, primarily with natural numbers, later with numbers in other number sets. The biggest problems appear in the area of operations (addition, subtraction, multiplication and division) in individual number sets and include problems with understanding every single operation, mastering counting by memory and in writing as well.

Word problems, the transformation of the word task defining into the mathematical symbolic language, solving the mathematical task, and its interpretation into the reality. Solving word problems belongs to the most problematic and feared topics of the school mathematics, it requires a thought-out teaching methodology.

Geometrical and spatial imagination, understanding the position and relations of subjects in the space and their representation in the plane. If the correct concepts of the shape, position and size are not formed, the child manages geometrical tasks with great difficulties.

Counting geometry, realizing the size of figures, estimates, calculations. The use of formulas for counting the circumference and area of plane figures, the surface and volume of solid figures requires understanding of these formulas and appropriate matching to the task. In calculations there appear problems which the child has in connection with operations in the natural and rational number sets.

Measurement units, understanding of each unit, their conversions. The tasks dealing with the unit conversions belong to the most problematic parts of mathematics at all school levels.

This classification was developed after a long-lasting work with children when there has been proved that if the child does not understand the substance of the mathematical problem, it does not know how to proceed and why to proceed this way. When the result has been arrived at only in the memory, without the support in understanding, without experiences, the remediation is not efficient. For example problems with reading (lexical dyscalculia) appear both while reading mathematical figures, numbers, symbols, expressions, and while understanding the task defining, word problems and application tasks etc. Similarly, problems with writing (graphic dyscalculia) appear in all areas of the mathematics subject-matter. Mistakes in mathematics need not be caused by the lack of mathematical knowledge, but they can source from the incorrect reading or writing, from the incomprehension of the context etc.

2. 5. 4 Basic Criteria for Dyscalculia Qualification

The basic criteria, according to which it is possible to qualify the specific developmental disorder in mathematics – dyscalculia, can be introduced as follows:

- there is a significant contrast between the child's established intellect and its success in mathematics,

- the level of intellect is not below average, the child's problems have not arisen on the base of an illness or on the social or emotional basis,

- the child is surrounded by a normal social background which provides a positive motivation,

- based on the professional examination, it is possible to identify the dysfunction of the central nervous system, the dysfunction of the brain cognitive centres.

It is necessary to realize that there is not a complete mathematical illiteracy or mathematical blindness, and that each child is able to reach the solution in a certain way. Every person uses such mathematical pieces of knowledge which are essential in his profession or everyday life.

2. 6 OTHER CAUSES OF LEARNING DISABILITIES IN MATHEMATICS

Apart from specific learning disabilities, many other factors influence the child's success in mathematics. The learning disabilities in mathematics can be caused by the subject-matter of mathematics itself, but they also can be found in the personality of the pupil, teacher or parents.

2. 6. 1 Subject-matter of Mathematics

Mathematics is a discipline which operates with abstract notions and their proper development is demanding for the pupil's mind. It has a precise logical composition and is created deductively. The processes of generalization and abstraction require the ability to proceed gradually from the concrete images to the more general ones, and they are very complicated for children. Although in the school mathematics there are used inductive approaches, certain levels of the generalization and abstraction are necessary (e.g. while creating the notion of the natural numbers). Moreover, the school mathematics is a subject where every element of the lower level is the necessary precondition for mastering elements of the higher level. That is why children have to remember permanently all what they have learned before. For example, mastering mental arithmetic is essential for the written counting, i.e. while teaching the arithmetic operations. At the same time children use permanently the long-term, short-term and working memory. The individual reasons will be dealt with in the further text. Here we can give only the fundamental statement:

a) first of all, it is necessary to understand each of the mathematical notions,
b) according to the child's abilities it is necessary to determine the rate of knowledge and skills which the child is able to master with respect to its learning disability,
c) it is necessary to strengthen the memory constantly.

2. 6. 2 Pupil's Personality

Children do not develop at the same pace, some of their mental operations can develop a bit later. Nevertheless, their mental abilities are not impaired and they do not suffer from the specific learning disability. The child's failure can be caused by a certain immaturity with respect to the given subject-matter. It often happens that the child does not understand the topic, but after a time period (e.g. after half a year) the child understands the subject-matter without difficulties.

Other causes of the child's problems in mathematics are connected with its volitional qualities. Mathematics requires everyday systematic work (in small amounts). If the child is not able to make itself do this work, and if there is nobody in its surroundings who can help it, the child has no chance to succeed in mathematics. Mostly there appears the problem in the certain section of the subject-matter and the child itself is not able to continue and master the following subject-matter. The child's low successfulness is closely connected with its lack of attention, interest, and also its low self-confidence, anxiety, the loss of the hope in the success, the role of the outsider among children, etc.

It is very important to follow the mental barriers like e.g. the white paper syndrome – the fear of written assignments, five-minute tests, further the fear of columns of exercises, word problems, a certain topic etc. These mental problems are very serious and it is important to perceive them as warning signals in the teaching job while communicating with the child. Suspecting the child of simulating something could be quite dangerous.

2. 6. 3 Teacher's Personality

The most often causes of children's problems in mathematics connected with the teacher's personality are incurred by the teacher's insufficient professional knowledge both in the area of mathematics and in the pedagogical, psychological and special pedagogical areas. Further, the causes could be found in the teaching style which can be good for most pupils, but it is not suitable just for this child, in the choice of the methods, in the area of the communication with the child, in the lack of the teacher's patience, in the formal approach to the work with these children. Also, the causes could be in the insufficient children's motivation to learning, and in the insufficient motivation by the mathematics subject-matter, or in the teacher's not mastering the evaluation and classification etc. The child is hardly motivated if the teacher expects the poor performance from the child with the learning disability without the hope of improvement, or if the teacher shows the lack of empathy for children with dyscalculia.

2. 6. 4 Parents' Influence

The parents' reactions to the learning problems in mathematics vary and we can differentiate several groups according to the relation to the child. The first group represents the parents who fully understand their child, cooperate with the pedagogical psychological counselling centre and with the mathematics teacher and try to help their child with respect to its handicap. They help the child to overcome problems in mathematics and they do not expect unrealistic results. The second group is formed by the parents who are ambitious, too aspiring and are unable to reconcile themselves to the child's problems in mathematics. These parents either disapprove their child or they take a martyred stand (why it is just us who have such a child), or they overload the child with the permanent tutoring and excessive requirements. Some parents discipline the child, not in the physical way but in the mental one. Another group represents the parents who try to help the child at all costs by making up various procedures and didactic simplifications which further in the future prove faulty and cause the child additional problems. The next group is formed by the parents who are interested in the child's problems, but they resign and leave the child without the professional help (it cannot be helped, we were not good at mathematics either). There is also a group of parents who cooperate neither with the counselling centre nor with the teacher, and they do not care about the child. Sometimes, the work with parents is more demanding than the work with children. It is quite complicated to persuade parents that the help provided by them and their tutoring are completely inefficient and they can only lead to the increase of the child's aversion to mathematics.

2. 6. 5 Personalities with Specific Learning Disabilities

Dyscalculia is a specific learning disability, but the child has an average or above-average intellect and it often does not need to influence its university studies even in the branch of engineering or natural sciences. The individual's social status can be influenced by their development in the childhood and the relation to mathematics. While choosing the future job, they either search for the one where they do not meet mathematics very often – e.g. artistic or humanitarian branches, or on the contrary their developmental disorder does not need to influence them in the branches within the natural sciences. Many outstanding personalities had problems with mathematics in the childhood, and in spite of that they achieved excellent results, some of them just in mathematics.

For example, in the publication My World Line about the physicist George Gamov we can read that a famous astronomer Vera Rubin, his student, said about him:

„He cannot write or count. It took him some time before he told you how much 7 times 8 is. But his brain was able to comprehend the space.“ (Gamov 2000, pg. 153).

The mathematician N. N. Luzin belonged to people with a slow reaction time. He also developed slowly, he had bad results at school, even in mathematics.

David Hilbert, one of the greatest mathematicians of the 20th century, gave the impression of a dull, slowly thinking person who had difficulties understanding what is being told to him. (Blažková et al., 1995).

Albert Einstein, the biggest physicist of the 20th century, used to fail at school and had great difficulties with reading.

Thomas Alva Edison belonged to the worse part of the class, he never mastered skills like writing, spelling and also arithmetic.

Also many writers are said to have been unsuccessful at school e.g. in the Czech language – one of the examples can be the outstanding writer Bohumil Hrabal.

We would be able to give many examples when a seemingly "dull" pupil who was bad at school proved to be a genius in the future**.**

Therefore, it is necessary to approach the children with disabilities sensitively, to try to understand their problems and seek the ways how to facilitate their learning. The person with the reading disorder can somehow cope with problems in the adulthood, but whenever they come to the situation where the problematic areas are dominant they always realize the problems and have to make an effort to manage them. Most adults conceal their problems for the fear of the social degradation. With the help of the compensatory tools (calculator, computer) it is possible to eliminate the array of problems, mostly those ones from the area of numerical calculations and the notation of mathematical tasks. But the problems from the area of the operations with natural numbers are transferred to other mathematical topics, e.g.

counting with powers and with algebraic expressions, solving equations, solving word problems, where the problems with dyscalculia appear in a higher level of the mathematical subject-matter.

Questions for the self- study:

1. Study some of the books which deal with dyscalculia.
2. Study some of the diagnostic tools which can be used for diagnosing dyscalculia.
3. Characterize dyscalculia symptoms.
4. Explain the influence of dyslexia and dysgraphia on the pupil's successfulness in mathematics.
5. Familiarize with the development of opinions on dyscalculia.
6. Try to analyse and describe problems in mathematics at the particular pupil.

3 PUPILS' PROBLEMS IN CERTAIN SUBJECT-MATTERS AND POSSIBILITIES OF REMEDIATION

3.1 NATURAL NUMBERS, UNDERSTANDING THE NOTION

If the notion of a natural number is not formed on the certain level of abstraction, we will meet the following problems:

- The child cannot form a group of the certain number of elements.
- The child cannot determine the number of the elements of this group.
- While counting the elements by one, the child cannot name the series of the numbers in the real ordering; they miss numbers, confuse them, and repeat.
- The change in the element configuration leads to incorrect counting.
- The child is not able to determine the number of elements by counting by one.
- The child cannot understand the substance of the decimal system.
- The child does not understand the meaning of number 0.

Remedial procedures

We concentrate particularly on non-mathematical activities which are connected with different games, e.g. the board game Man, don't get annoyed, dominoes, games with boxes, cards, fairy tales, rhymes and songs, where there are numerical data etc.

The aim is to reach the certain level of abstraction while forming the notion of a natural number. For example, when forming number 5 we proceed as follows:

a) First, we give to children various groups of objects in which there are 5 elements, e.g. dolls, cars, beads, apples, bananas, children, marbles etc.
b) We will assign symbols to a certain group of element e.g. 5 sticks to five dolls, or we will draw 5 dots. The aim is to let the child understand that we assign the same symbol (the 1st stage of abstraction) to any objects.
c) The objects and symbols are assigned number 5 (the 2nd stage of abstraction – the visible qualities of objects are not important, it is only their number which is important).

3.2 READING AND WRITING NUMBERS

The children's problems while reading and writing numbers relate on one hand to the differentiation and writing figures – symbols, and on the other hand to writing the single digit and more digit numbers. There can be the influence of dyslexia or dysgraphia; the problems with right-left orientation can also play an important role. Among the frequent problems there belong:

- Writing figures in the appropriate size, keeping the lines.
- Differentiating figures which are similar in shape, e.g. 6 - 9, 3 – 8, 2 – 5; it deals also with figures noted digitally.
- Problems with writing figures oriented to one side, e.g. figures 1, 3, 7 etc., when the children do not know to which side to write the figure.
- Not distinguishing the order of the figure in double-digit numbers, e.g. the child does not distinguish numbers 25, 52.
- Problems while writing numbers with zeros, e.g. the child writes number 205 as it hears it – two hundred and five, so 2005, or it misses zero – it writes 25.
- The inability to write numbers from the dictation.
- Not mastering reading of numbers with more digits, reading by individual figures.
- Bad orientation at numbers of higher orders.

Remedial procedures

- Understanding the correct shape of the figure – modelling form the wire, using the sense of touch and visual imagination.

- To understand double-digit numbers it is useful to use bundles of sticks or straws by tens and units, with the help of which numbers e.g. 24, 42 can be modelled.
- For modelling numbers consisting of more digits it is suitable to use cards with all orders, e.g. for modelling number 4 586 we place cards 4 000, 500, 80, 6 successively on top of each other.
- For the correct understanding of the natural numbers progression and the transitions between individual orders we use cards for filling in numbers:

78	__	__	81

396	__	__	__	400	__

742	___	___	___	738

- We use different abacuses – the tens, the twenties, the hundreds, the orders abacuses, the hundreds table etc.
- We teach children to name the ascending and descending sequence of numbers (from the given number).

3.3 COMPARING OF NATURAL NUMBERS

To understand relations connected with comparing of natural numbers it is first of all necessary for the children to understand the relations less than, greater than, equal in the groups of objects (without numbers) and to understand the concept of ordering of the elements in the group. Only then the groups of objects are assigned numbers and are compared. Children can have problems:

- with distinguishing the comparing of the size of objects from their number.
- with distinguishing groups with the same number of objects
- with using symbols for comparing >, <, =.
- with comparing numbers on the real line.
- with comparing numbers with more digits.

Remedial procedures

First of all, we will practise relations “more than, less than, equally as many as” on the concrete objects or pictures without numbers (children match only objects or pictures, make pairs and state conclusions about what is more, less or equally as many), e.g.:

There are more stars than moons in the picture. There are 5 stars, 3 moons, 5 is more than 3,

$$5 > 3$$

There are less stars than moons in the picture. There are 3 stars, 4 moons, 3 is less than 4,

$$3 < 4$$

There are equally as many stars as moons in the picture. There are 5 stars, 5 moons,

$$5 = 5$$

- Children fill in pictures so that there are more, less or equally as many objects, e.g. to the picture of children they paint balloons to get the same number of them, to the picture of rabbits they paint carrots so that there are more carrots etc.
- Children fill in the picture to the given inequality, e.g. $6 < 8$.
- While comparing numbers on the real line, they compare numbers according to their mutual position, not according to the distance of their images from the beginning of the real line – from zero (this fact does not apply in the negative part of the real line). From the two numbers plotted on the real line, the bigger number has the image to the right from the other one.
- Numbers with more digits are compared according to the notation in the decimal system, e.g.
    - 1075 > 986 because in the first number there are thousands, in the second one there are only hundreds.
    - 42 189 < 42 564 because both numbers have the same number of tens of thousands and thousands, but they differ in the number of hundreds $1 < 5$.

- 12 345 = 12 345

- In the case of wrong writing we will ask the child to demonstrate the inequality or to draw a picture.

Questions for the self- study:

- Monitor what problems children can have with understanding the amount and with natural numbers. Which numbers cause them difficulties?
- What specific problems do children have when writing figures and numbers? What influence can dysgraphia have?
- What problems can you observe while reading numbers? What influence can dyslexia have?
- How does the child understand the real line?
- What problems can you see while comparing natural numbers?

3.4. OPERATIONS IN NATURAL NUMBERS SET

3.4.1 Adding Natural Numbers

- **Mental addition**

The operation addition of natural numbers is derived on the basis of the union of two groups of objects (sets) without common elements, which in practice means that we group objects, put them together, add, etc. For the children to understand the operation addition, it is necessary to feel the need of adding; they should be properly motivated for this activity (otherwise they can determine the sum e.g. by counting the objects by one).

The procedure of deriving the operation addition should respect several principles:

1. We start from the manipulative activities with real objects, e.g.
   In the bowl there are 3 apples; we will add 2 more apples. How many apples are there in the bowl?
2. We will illustrate the situation with the help of pictures (e.g. on the board or on the paper).
3. We will represent the situation with the help of symbols (dots, line segments, etc.).

ooo oo

3 2

4. We will write the problem: 3 + 2 = (we will explain the meaning of the symbol "+")
5. We will solve the problem: 3 + 2 = 5
6. We will express and write the answer: There are five apples in the bowl.
7. We will check the correctness of the calculation. At first when children know neither the properties of the operation addition nor those of the operation subtraction, we will perform the true-false test by the "step back" – e.g. we will check by counting by one that there are really 5 apples.

Numbers which we add are called addends, the result of the operation is called a sum. While deriving the addition, both addends and the sum should have the same name (3 apples and 2 apples equals 5 apples), only later we will formulate the problem of the type: 4 boys and 3 girls were playing at the playground. How many children were there at the playground? The name of the sum is the hypernym of the names of the addents.

Be careful! Do not use the incorrect graphic representation of the following type:

ooo + oo = ooooo

3 + 2 = 5

Although it seems illustrative, it does not comply with the reality, because the child requires 10 objects to illustrate the sum 3 + 2. Very often there happens that children write 3 + 2 = 10 to this illustration, because they put on the table 10 objects and count them by one. Such a picture represents models of individual numbers, not the operation addition. Moreover, in the everyday life we do not add objects (we add them to each other, we match them), so the symbol of addition is not used between objects, but between numbers. Similarly it is with the use of the symbol for the equality (here the equality of sets and the equivalence of sets are not understood correctly). It is useful to imagine the concrete situation when e.g.: In the carpark there were 3 cars and 2 lorries. How would the situation be represented on the right side of the equation?

The procedure of deriving individual addition links for children with learning disabilities is divided into very subtle methodological steps. The child always has to understand the situation on the basis of the manipulative activity connected with an experience, and only then we will proceed to the mental mastering of individual addition links. The mere mechanical practising of the addition links is not very effective because children forget the mechanically learned subject-matter very quickly.

1. **Deriving addition in the set up to five.**
   In this case there are only several basic links which children are able to learn by heart with the help of a concrete illustration.

2. **Addition in the set up to ten**.
   Here it is necessary to take into account the difficulty of individual links, because e.g. 8 + 2 is for the child easier than the problem 2 + 8. In this period the child learns how to add zero, such as the following examples 6 + 0 = 6, 0 + 6 = 6.

3. **Adding to number 10.**

   Some children need to practise especially problems of the type 10 + 7, 9 + 10.

4. **Addition in the set up to twenty without crossing the boundary of ten.**
   Exercises of the type 13 + 5.

   One of the possibilities is to use the parallel to the addition in the set up to ten:

   3 + 5 = 8, so 13 + 5 = 18.

   Another possibility is to use the decomposition:

   $$\underset{10\quad 3}{13} + 5 =$$

   We will decompose number 13 into 10 and 3 and count: 3 + 5 = 8, 10 + 8 = 18.

5. **Addition in the set up to twenty with crossing the boundary of ten.**
   Exercises of the type 7 + 8. If the child formulates its procedure and it is mathematically correct, we will leave it up to it.

   Usually, we use the decomposition of the second addend so that we fill up the first addend to ten:

   $$7 + \underset{3\quad 5}{8}$$

We count: 7 + 3 = 10, 10 + 5 = 15, so 7 + 8 = 15.

A lot of children with the learning disorder consider this decomposition very difficult, they do not understand it; they cannot find the number to fill up the first addend to ten. Some children decompose both addends with respect to number 5 and they count:

$$\underset{5\quad 2}{7} \quad + \quad \underset{5\quad 3}{8}$$

2 + 3 = 5, 5 + 5 = 10, 5 + 10 = 15, so 7 + 8 = 15.

Considering that the addition of natural numbers is commutative, i.e. we can interchange the addends and the sum is not changed, we will leave up to children to decide if they count 2 + 8 or 8 + 2.

While adding more addends, we can also use the associativity of the addition, i.e. grouping of addends. For example, it is more convenient to count the sum 4 + 9 + 6 as follows: 4 + 6 + 9.

- **Addition in the set up to one hundred**

While deriving the mental addition in the set up to one hundred, we use a very delicate procedure in choosing exercises, so that each type of examples serves as the prerequisite for mastering exercises of a higher demand. In this stage we use many tools for the graphic representation. It is e.g. the hundreds board, bundles of objects by tens, dummy banknotes, a real line etc.

- adding of tens – exercises of the type 40 + 30
- adding of a double-digit number and a one-digit number – exercises of the type:
  40 + 3, 42 + 3, 47 + 3, 46 + 7
- adding double-digit numbers - exercises of the type
  40 + 30, 42 + 30, 42 + 34, 48 + 32, 48 + 36.

Be careful: In the last case make sure that the child decomposes only one addend, not both of them, because the habit to decompose both addends can cause additional problems while subtracting numbers when crossing the boundary of ten.

We will then count: $42 + \underset{30\quad 4}{34} =$ 42 + 30 = 72, 72 + 4 = 76

$48 + \underset{30\quad 2}{32} =$ 48 + 30 = 78, 78 + 2 = 80

$48 + \underset{30\quad 6}{36} =$ 48 + 30 = 78, $78 + \underset{2\quad 4}{6} = 84$

For the children with learning disabilities we prepare such exercises which they can manage. If the child is not able to learn how to perform the mental addition of double-digit numbers regardless all effort, we will either teach it how to add in writing (if it suits to the child), or we will use a calculator as a tool for motivation and re-education. When we add numbers with more digits, which could cause even bigger problems, we do not add them mentally, but either in writing or using a calculator.

Note:

We will respect also other ways of the decomposition performed by children, if they understand it and count correctly, e.g..

46 + 34   46 + 4 = 50   50 + 30 = 80

4   30

28 + 36   28 + 2 = 30   30 + 34 = 64

2   34

Which other decompositions can we meet?

16 + 9   6 + 4 = 10   10 + 5 = 15   10 + 15 = 25

10   6   4   5

9 + 7   9 + 4 = 13   13 + 3 = 16

4   3

Some children can even imagine numbers with the help of the difference from number 10, e.g.

7 + 9 =

It writes   (3)   (1)   and counts   20 – 4 = 16

So the child counts: 7 = 10 – 3   9 = 10 – 1   (10 – 3) + (10 – 1) = 20 – 4 = 16

Similarly   8 + 5

2   5

8 + 5 = (10 – 2) + (10 – 5) = 20 = 7 = 13

- **Children's problems at mental addition**

1. Children do not understand the difference between the notation of the number and the operation addition, they write the numbers next to each other e.g.:
   1 + 4 = 14,   32 + 4 = 324,   42 + 51 = 4251

2. When introduced to the operation, children remember wrong links and they use them further on e.g..:
   3 + 4 = 9,   6 + 7 = 14, 8 + 7 = 13, 8 + 7 = 14, 9 + 8 = 18,   6 + 8 = 15,

   26 + 27 = 51

3. They do not comprehend the positional number set and add figures of different orders e.g.:
   7 + 20 = 90,   3 + 13 = 43,   3 + 13 = 34,   300 + 20 = 500

   They similarly add e.g. 44 + 32 = 67, because 4 + 2 = 6, 4 + 3 = 7

4. They use the written addition in the line (although they were not introduced to the written addition) and they do not master the work with orders e.g.:
   576 + 4 = 5 710

   They count 4 + 6 = 10, they write 10 and copy the next figures of the first addend, or they copy all other figure of the first addend: 576 + 4 = 57 610.

5. They use specialized procedures when they group figures without great sense, or with a strange procedure e.g.:
   36 + 30 = 363,   24 + 40 = 82 (the dominant link is 4 + 4), 532 + 8 = 534,

   23 + 35   = 5 800 - they count 2 + 3 = 5, 3 + 5 = 8, we write two zeros, because both addends have together 4 figures, so the sum has to have 4 figures as well.

6. They use incorrect parallels and reason them e.g. 8 + 6 = 18, because: I have 8, there are 2 remaining to ten, 8 + 2 = 10, 10 + 6 = 16, 16 + 2 = 18.

7. While adding numbers "by one" using fingers, children make mistakes when the sum is always lesser by one e.g. they count 6 + 4 : six, seven, eight, nine, 6 + 4 = 9.

Remedial procedures

1. We derive the basic links of addition on the basis of real objects and representations so that the child can see the substance of addition. We do not rely only on mental mastering of addition without checking the understanding of the operation.
2. If the child makes mistakes, we search with the child for the reasons and try suitable models which the child can understand.
3. For the addition with crossing the boundary of ten we try to find models and practical tools which the child can understand.
4. We respect the mathematical procedure so that children will not have problems in the future (e.g. while adding double-digit numbers we do not decompose both addends).
5. We choose suitable didactic games (Blažková 2007, Krejčová 2009).

- **Written addition**

The written addition differs from the mental addition by the fact that while adding in writing we start from the units, but while adding mentally we start from the highest orders (i.e. not only by the fact that while the mental addition we say the result and while adding in writing we write the sum).

The written addition algorithm is derived on the double-digit numbers and is then generalized. Nowadays, we use the system of writing the addends one below each other (in the past there was also used the system of writing the addends next to each other in the line). First of all, there is derived the addition without crossing the boundary of ten. It is suitable to follow precisely the procedure of the algorithm, so that children learn one procedure and then they can use it both at the written addition and the written subtraction. We always perform the true-false test by replacing the addends. We offer the exercise-book with bigger squares to the children having problems with writing numbers, so that they learn how to write figures of individual orders bellow each other correctly, and we denote the orders (T – tens, U – units of individual addends and the sum as well). For example

The addition without crossing the boundary of ten:

$$\begin{array}{r} 42 \\ \underline{36} \\ 78 \end{array}$$

Elementary steps: 6 + 2 = 8 8 will be written below units

3 + 4 = 7 7 will be written below tens.

The true-false test will be performed by replacing the addends (using the commutativity of addition):

$$\begin{array}{r} 36 \\ \underline{42} \\ 78 \end{array}$$

The addition with crossing the boundary of ten:

$$\begin{array}{r} 48 \\ \underline{36} \\ 84 \end{array}$$

Elementary steps: 6 + 8 = 14, 4 will be written below units, 1 ten will be added to tens 1 + 3 = 4, 4 + 4 = 8, 8 will be written below tens.

Test:

$$\begin{array}{r} 36 \\ \underline{48} \\ 84 \end{array}$$

- **Children's problems at written addition**

1. Children cannot write the addends below each other correctly according to the individual orders e.g..

$$\begin{array}{l} 528 \\ \underline{45} \\ 978 \end{array} \qquad \begin{array}{l} \;\,350 \\ \underline{4279} \\ 7779 \end{array}$$

2. While performing the addition with crossing the boundary of ten, children do not understand the basis of the decimal system and they do not perform the crossing e.g.

$$\begin{array}{r} 59 \\ \underline{36} \\ 815 \end{array} \qquad \begin{array}{r} 176 \\ \underline{209} \\ 3715 \end{array}$$

3. Children do not understand the basis of the algorithm and they add partial sums e.g.

$$\begin{array}{r} 396 \\ \underline{528} \\ 3354 \end{array}$$

they count: 8 + 6 = 14, they write 4 correctly,
but then they count 14 + 2 = 16,
16 + 9 = 25, they write 5 correctly and then follow

25 + 5 = 30, 30 + 3 = 33.

4. They add all figures in both addends without respect to order e.g.

$$\begin{array}{r} 59 \\ \underline{67} \\ 27 \end{array}$$

they count 7 + 9 + 6 + 5 = 27

5. They add all figures in both addends and they further continue according to the algorithm e.g.

   59 they count 7 + 9 + 6 + 5 = 27, write 7 below the units and continue

   <u>67</u> 2 + 6 + 5 = 1

   137

6. They add the second addend to both figures of the first addend e.g.

   58 they count 7 + 8 = 15, 7 + 5 = 12, and write both sums

   <u>7</u>

   1215

7. At numbers written in one line, they use partially the procedure of the written addition and of the mental addition e.g.

   378 + 2 = 3710 they count 2 + 8 = 10, they write 10 and copy other figures.

8. They use strange procedures e.g.

   24 + 35 = 5900 they count 2 + 3 = 5, 4 + 5 = 9 and add two zeros because both addends have 4 figures together.

Remedial procedures

1. We derive the written addition algorithm precisely.
2. We continuously repeat the basic links for addition within the set of twenty.
3. We use squared exercise-books so that the child has one square for every order.
4. We use colours for individual orders e.g. units are red, tens are blue etc.
5. We always require the child to perform the true-false test.
6. For simpler procedures we will use the commutativity of addition (e.g. instead of 2 + 8, it is easier for the child to count 8 + 2) and the associativity of addition (e.g. instead of (12 + 9) + 8, it is easier to count (12 + 8) + 9).
7. If the result is not achieved despite all the effort and all the child's hard work, we consider the use of a suitable compensation tool like e.g. a calculator.

Note: *The written addition presupposes mastering mental addition. We always perform the true-false test by replacing the addends.*

Questions for the self- study:

1. How will you explain to the child the substance of the natural numbers addition?
2. What properties does the operation addition have in the set of natural numbers and how can they be used while working with dyscalculic children?

3. Monitor the children's own approaches while adding natural numbers with crossing the boundary of ten.
4. Which decompositions are favourable for children?
5. Which problems can you face at the written addition?
6. How much are the addition tables contributing for children?
7. When is it suitable to recommend to the children the use of the calculator?

3.4.2 Subtracting Natural Numbers

- **Mental subtraction**

Natural numbers subtraction is defined as the inverse operation to addition, i.e. if for natural numbers *a, b, c* there applies $a + b = c$, then $c - a = b$, $c - b = a$.

(e.g. 1 + 2 = 3, 3 – 2 = 1, 3 – 1 = 2).

In the school mathematics, subtraction is derived as a dynamic operation which is connected with taking away, decreasing, separating etc. Children should be sufficiently motivated to understand the sense of the operation subtraction and the meaning of the symbol "–"

The procedure of deriving the operation subtraction should respect several principles:

1. We start from the manipulative activities with real objects e.g.
   In the bowl there were 5 nuts, Jirka came and ate 2 nuts. How many nuts there remained in the bowl?

2. We will represent the situation in pictures (e.g. on the board or paper).
3. We will represent the situation with the help of symbols (dots, line segments etc.).

o ~~o~~ or o o o ~~o~~ ~~o~~

o o

~~o~~

When working with real objects, we will separate two of them, or cross them out in the picture. Objects can be represented either arranged in the line or loosely like in the pile. We will leave up to the child which two objects it will cross out or remove.

4. We will write the exercise (with a plentiful word commentary – how many nuts we had, how many nut we ate, how we will write that there is less of them, how many

nuts remained, …, so that the child can see the meaning behind any written number and symbol):

$$5 - 2 = 3$$

The names of the individual numbers are: minuend, subtrahend, difference.

5. The exercise is written, read aloud and the true-false test is performed. Because in this phase children do not know the connection between addition and subtraction, it is suitable to check the correctness by a "step back" – repeat the situation again.

Be careful: Avoid the incorrect graphic representation of the following type:

ooooo – oo = ooo

5 – 2 = 3

when the child will place 5 objects, write "–", add next 2 objects, write "=", add next 3 objects, so the child will place 10 objects on the table to subtract 5 – 2. Subtraction is not performed in this way in the real life.

Subtraction in the set up to five contains ten links which children memorize after understanding them (they can represent each example by objects or a picture):
5 – 4, 5 – 3, 5 – 2, 5 – 1,
4 – 3, 4 – 2, 4 – 1,
3 – 2, 3 – 1,
2 – 1.

Further, children will learn to subtract numbers in the set up to ten. It is necessary to realize that exercises are not equally difficult, e.g. 8 – 2 is simpler than 8 – 6, or 10 – 3 is simpler than 10 – 8. We revise more often those subtraction links which are more difficult for children, and we always require the representation by real objects. It is not possible to rely only on memorizing because children with learning disabilities usually have problems with their memory and forget very quickly.

Children also learn how to solve the exercises when the subtrahend is 0, like 7 – 0 = 7. Children always learn how to subtract in the period when they practise addition, but here we list operations separately, so that we can see the sequence of individual parts of the subject-matter while deriving the same operation.

- **Mental subtraction procedure**

1. Subtraction in the set up to five
2. Subtraction in the set up to ten

3. Subtraction in the set up to twenty without crossing the boundary of ten, exercises of the type 17 – 4 .

We will decompose the minuend to the ten and the units

17 – 4
10 7

We count: 7 – 4 = 3, 10 + 3 = 13, so 17 – 4 = 13

4. Subtraction with crossing the boundary of ten, exercises of the type 12 – 5.

We will decompose the subtrahend in such a way that will enable us to subtract the units from the minuend:

12 – 5 =
2 3

We count: 12 – 2 = 10, 10 – 3 = 7, so 12 – 5 = 7

While solving exercises of this type we have to respect:

- Children need to revise the number decompositions permanently.
- It is possible that the child forms its own subtraction procedure, and if it is correct and usable even in other exercises within the set up to hundred etc., we should let the child use it. It can be e.g. the following type (they decompose the minuend and subtract from ten):

12 – 4 =
10 2

We count: 10 – 4 = 6, 2 + 6 = 8, so 12 – 4 = 8.

- It is not very useful when children subtract "by one" with showing on fingers because they count e.g. 12 – 4 as follows: twelve, eleven, ten, nine, 12 – 4 = 9

5. Subtraction in the set up to hundred

In all following types of exercises we always use the application tasks which illustrate the use in practice, we use the graphic representations and we respect the fine methodological sequence when with each new exercise we always introduce only one new phenomenon.

a) First of all, the multiples often are subtracted, exercises of the type 50 – 20.

We can represent the example graphically with the help of the coordinate grid when children mark (e.g. they colour the appropriate tens and they cross out those which they subtract). Further, they can use bundles of straws by tens, mock money, objects packed by tens (e.g. tissues, egg cases etc.).

It is also possible to use the analogy when children use the previously learned subject-matter:

6 – 2 = 4

6 tens – 2 tens = 4 tens

60 – 20 = 40

b) Subtraction of the single digit number from the double-digit one

We will start from the simplest type of tasks: 64 – 4,

Then gradually there follow the exercises of the type: 68 – 3, 60 – 3, 64 – 8.

Children can use decompositions or the analogy with subtraction in the set up to 20:

68 – 3	60 – 3	64 – 8
60 8	50 10	4 4

If children do not need the decompositions, we do not require them. If they choose their own procedure and it is mathematically correct, we will let them use it.

c) Subtraction of double-digit numbers

We solve exercises of the type 64 – 20, 65 – 25, 65 – 23, 63 – 28

If children use the decomposition while solving these exercises, it is useful to teach them to decompose only the subtrahend. If they decomposed both the minuend and subtrahend, while subtracting with crossing the boundary of ten they could make mistakes as follows: 60 – 20 = 40, 3 – 8 is impossible, so they subtract 8 – 3 = 5, as if they were solving the exercise 68 – 23.

We count: 65 – 23: 65 – 20 = 45, 45 – 3 = 42

63 – 28: 63 – 20 = 43, 43 – 8 = 35.

We mentally subtract the numbers of more than two digits only if they contain one or two non-zero figures in their notation e.g. 30 000 – 20 000, 1 500 – 300 etc. If children cannot manage the mental subtraction, we will use the written subtraction.

Note:

Again, we will respect children's own procedures if they understand them and they are correct e.g.

31 – 3 = 1 – 3, 2 is missing, we will replace this exercise by 30 – 2 = 28

Other possible children's decompositions:

14 – 8 = 8 – 8 = 0 14 – 8 = 6
6 / \ 8

19 – 7 17 – 7 = 10 10 – 2 = 8
/ \
17 2

19 – 7 14 -7 = 7 7 + 5 = 12
/ \
14 5

19 – 7 19 – 4 = 15 15 – 3 = 12
/ \
4 3

16 – 9 6 – 6 = 8 10 – 3 = 7
/ \ / \
10 6 6 3

16 – 9   9 – 9 = 0   0 + 7 = 7

/ \

9 7

16 – 9   10 – 4 = 6   6 – 5 = 1   6 + 1 = 7

/ \ / \

10 6 4 5

54 – 26   54 – 4 = 50   50 – 22 = 50 – 20 – 2 = 28

/ \

4 22

In some cases we observe whether the child has not reached the correct result using the incorrect procedure, which can rarely happen e.g. 14 – 9 = 5, when the child counts 9 – 4 = 5.

- **Children's problems at the mental subtraction**

1. The child does not understand the operation subtraction at all, and either adds the numbers or replaces them randomly. The child does not mind if it writes 5 – 3 or 3 – 5.

2. While subtracting by one, the difference is always greater by one than the correct result e.g. they count 16 – 5 and show on fingers until they point at 5 fingers: sixteen, fifteen, fourteen, thirteen, twelve,

so 16 – 5 = 12.

3. If they subtract by one, they cannot name the descending sequence of numbers confidently, they miss some of them e.g. the count 15 – 6 and say: fourteen, twelve, eleven, ten, nine, eight,

so 15 – 6 = 8.

4. They do not understand the mental subtraction, they count e.g. 44 – 5 = 11 as

5 – 4 = 1, 5 – 4= 1.

Or they count e.g. 18 – 13 = 50, because 1 – 1 = 0, they write 0 and they continue

8 – 3 = 5. The number cannot start with zero, therefore 50.

5. They count with figures of different orders e.g.

They count 80 – 6 = 20 as 8 – 6 = 2 and they add a zero,

they count 64 – 40 = 60 as 4 – 4 = 0 and copy 6,

they count 45 – 3 = 12 as 4 – 3 = 1, 5 – 3 = 2,

they count 56 – 2 = 36 as 5 – 2 = 3, and copy 6,

they count 93 – 3 = 60 as 9 – 3 = 6, 3 – 3 = 0

they count 300 – 50 = 200.

6. They replace figures in the minuend and subtrahend; they always subtract the lesser number from the greater one, although it is in the subtrahend.

62 – 28 = 46, because 6 – 2 = 4, 8 – 2 = 6,

640 – 350 = 310, because 600 – 300 = 300, 50 – 40 = 10.

7. There can appear also the right-left orientation disorders, when they count exercises of the type 74 – 26 as: 20 – 70 = 50, 6 – 4 = 2 and instead of the result 52 they write 25.

7. When subtracting double-digit numbers with crossing the boundary of ten, they constantly decompose both the minuend and subtrahend, and they always subtract the lesser number from the greater one,

They count 82 – 57 as 80 – 50 = 30, 2 – 7 is not possible, so 7 – 2 = 5, 82 – 57 = 35.

8. Exercises of the type 70 – 8 cause great problems to children, when they have difficulties with orientation in orders.

9. If they do not understand the operation subtraction, they subtract part of the subtrahend, and then add the second part of it e.g. they count 45 – 12 as 45 – 10 = 35, 35 + 2 = 37

10. They cannot see subtraction in exercises formulated with the "anti-signal" when subtraction is not formulated directly. For example, "On the wire there were 8 swallows, some of them flew away and there remained 5 swallows. How many swallows flew away?" they count

8 + 5 = 13.

Remedial procedures

1. The most important rule is to derive the operation subtraction and the symbol "–" on the basis of real situations.

2. The fundamental links within the set up to 20 are revised constantly.

3. Teachers search for suitable communication ways so that the child understands subtraction with crossing the boundary of ten.

4. The mistake is actively used to illustrate both the incorrect and correct procedures in a suitable way.

5. Appropriate motivational and application exercises are used.

- **Written subtraction**

The written subtraction algorithm is derived primarily for double-digit numbers and then it is generalized for numbers with more digits. In textbooks it is possible to find several different procedures of deriving the written subtraction, either by "counting up" or by subtracting from the top (from the numbers written in individual orders of the minuend we subtract the numbers written in the corresponding orders of the subtrahend). With respect to the further counting with numbers of more digits and numbers with zeros in their notations, it is suitable to derive subtraction with the help of "counting up".

a) The written subtraction without crossing the boundary of ten.

Subtract in writing 68 – 25. We will write the numbers below each other (e.g. into a chart)

$$\begin{array}{r} 68 \\ \underline{-25} \\ 43 \end{array}$$

We count: 5 plus how much is it 8? 5 + **3** = 8, we will write **3** units.

2 plus how much is it 6? 2 + **4** = 6, we will write **4** tens.

We will perform the true-false test by adding the difference and the subtrahend, the sum is the number written in the minuend of the given exercise:

$$\begin{array}{r} 43 \\ \underline{25} \\ 68 \end{array}$$

Note: Although in this type of the example children could subtract 8 – 5 and 6 – 2, it is not convenient to apply this procedure, because while subtracting with crossing the boundary of ten there would appear mistakes, when children would always subtract the lesser number from the greater one without respect to their position in the minuend and subtrahend.

b) The written subtraction with crossing the boundary of ten.

While performing the written subtraction with crossing the boundary of ten, we use the fact that the difference does not change if we expand both the minuend and subtrahend by the same number e.g. 8 – 5 = 3, then

18 – 15 = 3, 13 – 10 = 3, 28 – 25 = 3, etc. To be able to subtract the numbers in writing, we expand the minuend and subtrahend by ten so conveniently that we expand the minuend by 10 units and the subtrahend by 1 ten.

We will perform the written subtraction 62 – 28. We will write the numbers below each other

$$\begin{array}{r} 62 \\ \underline{-28} \\ 34 \end{array}$$

We count: 8 plus how much is it twelve? (We will add 10 units to the units of the minuend,

2 + 10 = 12) 8 + **4** = 12 We will write **4** units to the result.

Further, we will add 1 ten to the tens and count:
2 + 1 = 3, 3 plus how much is it 6? 3 + **3** = 6, we will write **3** tens to the result.

We will perform the true-false test by adding the difference to the subtrahend:

$$\begin{array}{r} 34 \\ \underline{28} \\ 62 \end{array}$$

c) The written subtraction of numbers with the zero in the notation e.g.

$$\begin{array}{r} 86 \\ -\underline{50} \\ 36 \end{array}$$

is calculated similarly as in previous examples: 0 and how much is it 6, 0 + **6** = 6, 5 plus how much is it 8, 5 + **3** = 8.

$$\begin{array}{r} 70 \\ -\underline{46} \\ 24 \end{array}$$

We count: 6 plus how much is it 10? 6 + **4** = 10, 1 + 4 = 5, 5 plus how much is it 7, 5 + **2** = 7.

- **Children's problems at the written subtraction**

1. While subtraction with crossing the boundary of ten, children keep subtracting the lesser number from the greater one e.g.

$$\begin{array}{r} 62 \\ -\underline{38} \\ 36 \end{array}$$

As 2 – 8 is impossible, they count 8 – 2 = 6, 6 – 3 = 3, as if they were counting 68 – 32.

2. Children subtract in one part of the exercise and add in the other part e.g.:

$$\begin{array}{r} 43 \\ -\underline{29} \\ 74 \end{array} \qquad \text{or} \qquad \begin{array}{r} 612 \\ -\underline{348} \\ 964 \end{array}$$

They count: 9 plus how much is it 13, 9 + 4 = 13, they write 4 correctly, further they count 2 + 1 = 3, 3 + 4 = 7,

or 8 plus 4 is 12, 1 + 4 = 5, 5 + 1 = 6, 3 + 6 = 9.

3. Children subtract from "the top" and cannot perform the crossing correctly. For example, they count the difference 7 036 – 867 (they write number 1 above each figure of the minuend):

$$\begin{array}{r} {\scriptstyle 1\,1\,1} \\ 7\,036 \\ -\underline{867} \\ 7\,279 \end{array}$$

They count: 16 – 7 = 9,   13 – 6 = 7,   10 – 8 = 2, and copy 7. They do not mind at all that the difference is greater than the minuend.

4. They apply the crossing the boundary of ten also where it is not used e.g.

$$\begin{array}{r} 7\,912 \\ -\underline{\,657} \\ 6\,255 \end{array}$$

Remedial procedures

1. We will derive the written subtraction procedure correctly.

2. We will choose the motivational exercises from the real life where the subtraction is obvious.

3. We keep revising the mental subtraction constantly.

4. We always guide children to consider if the result conforms to the reality and further we guide them to perform the true-false tests.

5. If the child is still unsuccessful although it tries hard, we choose the compensational tool, the calculator.

Questions for the self- study:

1. How will you explain to children the substance of the subtraction of natural numbers?
2. Observe the children's own approaches while subtracting natural numbers when crossing the boundary of ten.
3. What decompositions are favourable to children?
4. What problems can you face at the written subtraction of natural numbers?
5. When is it suitable to recommend the calculator to children?

3.4.3 Multiplication of Natural Numbers

- **Multiplication within the multiplication table**

Mastering the operation multiplication and the fundamental links of the multiplication is the good starting point for mastering the further subject-matter, which is division, division with remainder, written multiplication and division, counting with fractions, and the practical usage in the application exercises. First of all, children should understand what multiplication is, and just then they should try to master individual links of the multiplication tables.

Therefore, we derive the multiplication table of two, three, four, five, then of the other ones (six, seven, eight, nine). When children have understood the multiplication principle, we teach them how to multiply by number one, number zero and number ten, because children cannot understand the multiplication principle on these specific numbers. If we limit ourselves to memorizing, children are not able to use multiplication in word problems.

Multiplication of natural numbers is derived on the basis of addition of several equal addends. When deriving this operation, we start from the dramatization and the real situations which are close to children.

For example: Mother gave two oranges to each of her four children. How many oranges altogether did mother give to her children?

children:	A	B	C	D
oranges:	oo	oo	oo	oo

$$2 + 2 + 2 + 2 = 8$$

$$4 \; . \; 2 = 8$$

The names of the individual numbers are: factor, factor, product.

(Note: while this type of deriving of multiplication, it is not possible to use this example for the link 2 . 4 – here it is necessary to represent two groups by four elements.)

While deriving multiplication, we use real situations which will appeal to children e.g.

- While deriving the multiplication tables of numbers 2, 4, 6, 8, we use animals, e.g. a parrot has 2 legs, a dog has 4 legs, a bee or a fly has 6 legs, and a spider has 8 legs.
- When baking cakes or Christmas sweets we follow and count how the individual kinds are placed on the baking tin.
- We use the coordinate grid.
- We will show the children how to use the "finger multiplication" (for the detailed description see Blažková et al. 2007).
- We teach children to name the numbers' multiples in the increasing and decreasing way.
- We highlight multiples of numbers in the hundred table.
- We use board games e.g. lotto, domino, pexeso, bingo.
- We play the "shop" game and buy goods e.g. 4 yoghurts 8 Kč each, 3 chewing gums 6 Kč each, 5 lollipops 4 Kč each etc. and we count how many crowns we will pay.
- We use pictures of different kinds of goods e.g. vegetables or fruit (8 bunches of bananas, 6 pieces in each, a box of peaches with 5 rows and 6 peaches in each row, 9 packets of onions with 10 pieces in each etc.), we count how many pieces there are altogether.
- We use of the products of the same factors as the support e.g. 6 . 6, 8 . 8, 4 . 4 etc.

Multiplication of natural numbers has many properties:

Multiplication of natural numbers is commutative. We can replace factors, the product is the same e.g.

$3 . 4 = 12, \quad 4 . 3 = 12$ generally $a . b = b . a$

We will illustrate the commutativity of multiplication on one object e.g. we have a box of chocolates where the pieces are arranged:

in three rows and four columns $3 . 4 = 12$

O	O	O	O
O	O	O	O
O	O	O	O

Then we will move the box of chocolates and we get the pieces arranged in four rows and three columns $4 . 3 = 12$

O	O	O
O	O	O
O	O	O
O	O	O

Multiplication of natural numbers is associative. We can associate the factors, the product is the same e.g.

$(4 . 2) . 5 = 4 . (2 . 5)$ generally $a . (b . c) = (a . b) . c$

$8 . 5 = 4 . 10 = 40$

We use the associativity of multiplication for more convenient calculating, or for calculating outside the tables of multiplication.

Multiplication by number 1

If we multiply a natural number by number 1, the given number does not change e.g.

$6 . 1 = 6, \quad 1 . 6 = 6$ generally $a . 1 = 1 . a = a$

Multiplication by number 0

If we multiply a natural number by number 0, the product equals 0, e.g.

$5 . 0 = 0, \quad 0 . 5 = 0$ generally $a . 0 = 0 . a = 0$

- **The mental multiplication outside the tables of multiplication**

1. Examples of the type $4 . 30$

It is suitable to use the decomposition of number 30 and then the associativity of multiplication, i.e.

$$4 . 30 = 4 . (3 . 10) = (4 . 3) . 10 = 12 . 10 = 120$$

It is enough to multiply the number of tens and then multiply the product by ten.

2. Examples of the type $5 . 12$

We will use the decomposition of number 12 to the ten and the units, and we will multiply the brackets:

$$5 . 12 = 5 . (10 + 2) = 5 . 10 + 5 . 2 = 50 + 10 = 60$$

- **Children's problems at the mental multiplication**

- Children do not understand the meaning of the operation multiplication of natural numbers at all, they do not know what to do with the numbers.

- Children confuse the operation multiplication and the notation of the number e.g.
  $4 . 4 = 44, \quad 6 . 5 = 65$

- They make mistakes at deriving multiplication, one factor is dominant for them e.g.
  5 . 7 = 5 + 5 + 5 + 5 + 5

- Children keep using only the sequence of multiples and they are not able to learn the links without the sequence of multiples.

- Children confuse some products e.g.
  7 . 8 = 54, 9 . 6 = 56, 8 . 9 = 80,

  7 . 8 = 64, 7 . 7 = 53, 5 . 7 = 37, 8 . 4 = 34

- There prevails the dominance of some number e.g.
  2 . 9 = 19, 4 . 4 = 14, 8 . 8 = 68

- Children confuse operations multiplication and addition e.g.
  50 . 4 = 54

- They do not distinguish between the expansion of the number in the decimal system and multiplication:
  13 . 2 = 1 . 10 + 3 . 2 = 16

  32 . 3 = 30 + 2 . 3 = 36

Remedial procedures

1. We try to make children understand the substance of multiplication, to know what happens with numbers while multiplying. The biggest problems are caused by the fact that children do not know what to do with the factors, so they mostly write the number which comes in their mind as the product.

2. Mental mastering of multiplication links should be based on the real concept. We teach multiplication tables in small steps and practise them constantly.

3. While deriving, we usually start with the multiplication tables of 2, 3, 4, etc. The seemingly easy cases of multiplying by numbers 1, 0 and 10 cannot be explained as the first examples, because they do not illustrate the meaning of multiplication adequately.

4. Deriving multiplication is primary; only then there follows the mental practice.

5. As much as possible we use practical examples which are interesting for children.

6. We choose suitable didactic games (Blažková et al.2007, Krejčová 2009).

- **The written multiplication**
  Mastering the written multiplication algorithm requires both the knowledge of the mental multiplication and the knowledge how to proceed correctly and write figures into the multiplication scheme. The written multiplication requires integrating all types of the child's memory. Let us remind what the child has to manage while multiplying in writing e.g.

$$\begin{array}{r} 157 \\ \underline{.\ \ 8} \\ 1256 \end{array}$$

Firstly, the child recollects from the long-term memory the link 8 . 7 = 56. It writes 6, and saves 5 in the working memory. It further multiplies 8 . 5 = 40 – it again uses the long-term memory, then it adds 5, which is stored in its working memory, 40 + 5 = 45, it writes 5 and multiplies 8 . 1 = 8, adds 4, 8 + 4= 12 and writes.

These calculations present a great pressure on the child's mental activity. At the same time, the child improves the concentration, because performing this algorithm requires full concentration on the performed operations and procedures while writing down figures and the child cannot think about anything else. It is necessary to be aware of the fact that if the child has problems with tables of multiplication, it either concentrates fully on the correct multiplication while making mistakes in the algorithm notation, or it writes down the algorithm correctly while making mistakes in multiplying. Some children are not able to concentrate on both activities at the same time.

First of all, we will derive the written multiplication by a single-figure factor, namely in a fine methodological sequence when in each new example there is only one new phenomenon. If it is possible, we will show the children how to proceed at the mental multiplication and how the calculation is made easier by the written algorithm.

Multiply 123 . 3

At the mental multiplication we will start from the hundreds:

$$123 . 3 = (100 + 20 + 3) . 3 = 300 + 60 + 9 = 369$$

At the written multiplication we will start from the units:

123 elementary steps: 3 . 3 = 9
<u>. 3</u> 3 . 2 = 6
369 3 . 1 = 3

First examples are chosen so that the multiplication is performed without crossing the boundary of ten and the children manage the procedure of individual products.

We will choose other examples so that

a) there is the crossing between units and tens 125
<u>. 3</u>

b) there is the crossing between tens and hundreds 162
<u>. 3</u>

c) there is the crossing among all orders 265
<u>. 7</u>

The multiplication by a double-digit factor is derived in two phases; first there are multiples of number 10, e.g. 123 and then multiples of a double-digit factor e.g.
123 123
<u>. 30</u> <u>. 32</u>

We will respect the similar procedure to the one with multiplying by a single-digit factor.

Examples of the type 123
<u>. 30</u>

It is suitable to illustrate them as follows: 30 = 3 . 10, first we will multiply by ten (we will write the zero) and then by three: 123
<u>. 30</u>
3690

Examples of the type 123
<u>. 32</u>

They are solved with the use of previously learned procedures.

123
<u>. 32</u>
we multiply by number 2 246
we multiply by number 30 <u>3690</u> (later we will not write zero, we will shift
3936 the partial product one place to the left).

In recent years in some textbooks there is presented the procedure of the written multiplication according to the Indian way (see e.g. textbooks of the publishing house Fraus).

- **Children's problems at the written multiplication**

1. Children transfer the procedure of the written addition, they multiply units and tens with each other e.g.:

$$\begin{array}{r} 42 \\ \underline{.23} \\ 86 \end{array}$$

They multiply: 3 . 2 = 6,   2 . 4 = 8.

2. They write the product into one line e.g.

$$\begin{array}{r} 42 \\ \underline{.21} \\ 8442 \end{array}$$

they multiply 1 . 2 = 2, 1 . 4 = 4, 2 . 2 = 4, 2 . 4 = 8 or   1 . 42 = 42,   2 . 42 = 84

3. They multiply only by one figure of the second factor, they do not finish the multiplication e.g.

$$\begin{array}{r} 42 \\ \underline{.\ 23} \\ 126 \end{array}$$

they multiply 3 . 2 = 6,   3 . 4 = 12.

4. They do not manage crossing the boundary:

$$\begin{array}{r} 45 \\ \underline{.\ \ 8} \\ 3240 \end{array}$$

they count   8 . 5 = 40, 8 . 4 = 32

5. They have problems with zeros:

They calculate 304 . 2 = 68 as 34 . 2 and 564 . 205 they calculate as 564 . 25

$$\begin{array}{r} 304 \\ \underline{.\ \ 2} \\ 68 \end{array} \quad \text{as} \quad \begin{array}{r} 34 \\ \underline{.\ 2} \end{array} \qquad \text{and} \quad \begin{array}{r} 564 \\ \underline{.\ 205} \end{array} \quad \text{they calculate as} \quad \begin{array}{r} 564 \\ \underline{.\ 25} \end{array}$$

6. They do not write the partial products correctly:

$$\begin{array}{r} 257 \\ \underline{.\ \ 35} \\ 1285 \\ \underline{771} \\ 2056 \end{array}$$

7. In the crossing they always add the second factor e.g.:

$$\begin{array}{r} 75 \\ \underline{.\ \ 5} \\ 405 \end{array}$$

they count 5 . 5 = 25, 5 . 7 = 35, 35 + 5 = 40

8. They multiply the individual numbers and add the products e.g.

$$\begin{array}{rll} 608 & & \\ \underline{.\ \ 65} & & \\ 40 & 5 . 8 & \\ 30 & & 5 . 6 \\ 48 & 6 . 8 & \\ \underline{36} & & 6 . 6 \\ 154 & & \end{array}$$

9. They confuse the algorithms of addition and multiplication, because they add numbers but proceed according to the multiplication algorithm e.g.:

$$\begin{array}{r} 48 \\ \underline{.\ 39} \\ 8247 \end{array}$$

they count: $9 + 8 = 17$, they write 7 below units, they add 1 ten to the next calculation

$1 + 9 + 4 = 14$, they write 4 below tens

$1 + 3 + 8 = 12$, they write 2 below hundreds

$1 + 3 + 4 = 8$.

Remedial procedures

1. We keep checking if children understand the meaning of the operation multiplication on real examples correctly, e.g.: I will buy 5 yoghurts, each of them costs 12 Kč. How many crowns will I pay?

2. We revise constantly (every day) the basic links of multiplication.

3. The remedial procedures for the written multiplication consist in the processing of suitable and very fine methodological series of examples, primarily with smaller numbers.

4. If children have problems with tables of multiplication, they can use the charts with multiples and look for the required multiples. However, it is necessary to realize that children will not learn the tables of multiplication by using the charts of multiples – they will only learn how to search in the chart.

5. It is suitable for the children to perform the true-false test using the calculator, if they are able to enter the numbers so that they are represented on the display correctly.

Questions for the self- study:

1. Demonstrate how the multiplication of natural numbers is derived.

2. What are the properties of the operation multiplication of natural numbers in the set of natural numbers, and how is it possible to use them while multiplication?

3. What didactic games can be used for mastering the multiplication of natural numbers?

4. What problems do we observe at children in connection with the mental and written multiplication?

3.4.4 Division of Natural Numbers

- **Mental division**

Division of natural numbers is defined as the inverse operation to the operation multiplication. If for natural numbers *a, b, c* there applies $a \,.\, b = c$, then for $a \neq 0, b \neq 0$ there applies $c : a = b$, $c \,.\, b = a$.

As division is the most difficult operation for children, we derive division on the basis of dividing the real objects. Even in the pre-school age, children are able to divide several objects among the given number of children so that all children have the same amount. When deriving division, we start from the real situation when children divide real objects, when they can divide them to parts e.g. among several children, or according to the content i.e. by several objects. Therefore, we form two tasks.

1. Division to parts

Divide 20 marbles among five children so that all of them have the same amount and you have divided all marbles. How many marbles does each child have?

a) dramatization – the real performance
b) graphic representation of the situation – we successively draw marble to each child by one.

children	A	B	C	D	E
	o	o	o	o	o
	o	o	o	o	o
	o	o	o	o	o
	o	o	o	o	o

c) the example notation: 20 : 5 = 4

Each child will have 4 marbles.

True-false test: (e.g. by adding each child's marbles together) 4 + 4 + 4 + 4 + 4 = 20.

The names of numbers are: dividend, divisor, and quotient.

In this example the dividend is 20, the divisor is 5, the quotient is 4 and the quotient expresses the number of the elements in each part.

2. Division according to the content

Divide 20 marbles into groups by five. How many groups will you make?

c) dramatization – here children work on their own – each of them has 20 marbles and forms piles by five marbles.
d) graphic illustration

o o o o o   o o o o o   o o o o o   o o o o o

c) notation of the example: $20 : 5 = 4$

We will form four piles.

True-false test. $5 + 5 + 5 + 5 = 20$

Also in this example the dividend is 20, the divisor is 5, the quotient is 4; the quotient expresses the number of the formed parts.

It is important to realize that one example expresses two entirely different situations and we should perform them with children so that they will be able to solve the word problems where there appears the operation division.

Special cases at division:

a) division by number 1 $5 : 1 = 5$
we will derive this case on the basis of the example: Divide five candies by one. How many children will be given?

b) the dividend equals the divisor $5 : 5 = 1$

we will derive this case on the basis of the example: Divide five candies among 5 children. How many candies will each child get?

c) division of zero $0 : 5 = 0$

we will derive this case on the basis of the example: Divide zero marbles among 5 children. How many marbles will each child have?

d) division by zero $5 : 0 = ?$

Children are introduced to the statement "We do not divide by zero", but often without any justification, so then they make mistakes in examples and write either $5 : 0 = 0$ or $5 : 0 = 5$. It is useful to show children that there is not such a natural number for which we would be able to perform the true-false test after the division by zero.

If e.g. $5 : 0 = 0$, there would have to be $0 . 0 = 5$. This is not true, because $0 . 0 = 0$.

If 5 : 0 = 5, there would have to be 5 . 0 = 5. This is not true, because 5 . 0 = 0.

We can follow this way and look for the number for which the true-false test will be satisfied. Such number will not be found.

(Note: Generally, if there applied $a : 0 = x$ for $a \neq 0$, then there would have to apply $x . 0 = a$. Such equality does not hold true, because $x . 0 = 0$ for every natural x.)

Gradually, children master the basic mental division links and if they make mistakes, they should have the chance to illustrate the situation with real objects.

Further, children will be introduced to the connection between the operation multiplication and division in the set of natural numbers e.g. if 5 . 7 = 35, then 35 : 7 = 5 and 35 : 5 = 7.

- **Children's problems while dividing within the tables of multiplication**

1. Children do not understand the meaning of the operation division, especially if they do not have enough real activities and the practice is based only on mastering the mental links of division.
2. Children confuse some examples of division (basic links), e.g. 54 : 9 = 7, 56 : 8 = 9, etc. It mostly deals with numbers 42, 48, 54, 56, 63, 64 etc.
3. Mistakes due to absence of mind e.g. 40 : 5 = 10
4. In word problems the child does not understand when to use the operation division.
5. They replace the dividend and divisor e.g. 2 : 8 = 4

Remedial procedures

1. First of all we derive division based on real examples, we divide objects among children, or piles by several objects.
2. We teach the basic links by heart gradually (in small steps).

3. We always perform the true-false test by multiplying.

4. We choose suitable didactic games. (Blažková et al.2007, Krejčová 2009).

- **Division outside the tables of multiplication**

  Division with a remainder is performed as follows: If we have two different natural numbers *a, b* such that *a* is not the multiple of *b* and *b* is different from zero, then there exist natural numbers *q, z* such that there applies $a = b . q + z$.

The number *a* is called a dividend, *b* is a divisor, *q* is an incomplete quotient, *z* is a remainder. The remainder always has to be lesser than the divisor.

Division with a remainder is derived similarly as division without a remainder.

Firstly, we will set the example: We should divide 17 exercise books among 5 children. How many exercise books will be given to each child and how many exercise books will remain?

A B C D E
î î î î î î
3= 5: 17 î î î î î î (r 2)
2 î î î î î

True-false test: 3 . 5 + 2 = 17 or 3 . 5 = 15 15 + 2 = 17

Every child will get 3 exercise books and 2 exercise books will remain.

The next exercise: We should divide 17 exercise books to piles by five pieces. How many complete piles will we create and how many exercise books will remain?

î î î î î î î î î î î î î î î î î

17 : 5 = 3 (r 2)
2

True-false test: 3 . 5 + 2 = 17

We will create 3 piles and 2 exercise books will remain.

It is necessary for children to see the meaning of each number, i.e. which number plays the role of the dividend, divisor, partial quotient and remainder.

It is suitable to use multiples of numbers and to indicate the closest lesser multiple of the given number.

- **Children's problems while dividing with remainder**

1. Children do not master the basic multiplication and division links which are essential.
2. If the dividend is close to the next multiple of the divisor, children count e.g.
   41 : 7 = 6 (r 1)
   1

They will write the greater multiple and the remainder represents the number which this greater multiple lacks.

3. Children write the multiple e.g.: 38 : 7 = 35 (r 3)

4. They cannot deal with examples when the dividend is lesser than the divisor e.g.

   3 : 5 = there is no solution

   But 3 : 5 = 0 (r 3) – this is important to understand because of the written division.

5. They write the true-false test incorrectly, e.g.: 3 . 5 = 15 + 2 = 17. Here, the transitivity of the equality is disrupted. In the course of the computation we cannot add or subtract anything, and thus write incorrect equalities.

- **Mental division outside the tables of multiplication**

There are examples of the type 72 : 4.

It is important to find the suitable decomposition of number 72 to two numbers so that they are divisible by number 4, if possible. In this case these are numbers 40 and 32.

We count: 72 : 4 = (40 + 32) : 4 = 40 : 4 + 32 : 4 = 10 + 8 = 18

Short notation: 72 : 4 = 18
40 32

True-false test: 18 . 4 = (10 + 8) . 4 = 10 . 4 + 8 . 4 = 40 + 32 = 72

The examples of such type are mentally calculated only in easier cases, possibly they can be omitted within the individual study plan.

Remedial procedures

1. We simulate the division with a remainder on real situations, we choose the dramatization, we point out the meaning of individual numbers.

2. We work with a mistake actively.

- **Written division**

  The written division differs from other written algorithms of operations firstly by the fact that the algorithms for the written addition, subtraction and multiplication start all from the units, but the division algorithm starts from the highest order, and secondly that children have to manage the division algorithm both in the horizontal and vertical directions. Moreover, for the children to be able to perform successfully the written division it is necessary to have mastered all mental operations – especially division with a remainder and subtraction. For the written division drill it is suitable to compile a very fine methodological sequence when in each successive example there appears only one new phenomenon.

Division by a single-digit divisor

1. The first set of examples is devised in the way that children divide a double-digit number by a single-digit one, so that the number of the dividend's tens is the multiple of the divisor and the division is without a remainder. Children learn successive steps of the algorithm (what to divide by what, what to write where). We perform the true-false test at each example immediately. By this activity we revise multiplication and teach children to verify the correctness of the calculation in their own capacity as well. For example:

```
  9 : 3
  6 : 3
 69 : 3 =  23                Test:     23
 09                                  .  3
  0                                    69
```

2. In the second set of examples we choose such examples where the number of the dividend's tens is greater than the divisor, but it is not its multiple. It is important for children to master how to note the remainder during the division and how to form the new partial dividend e.g.:

```
 25 : 5
  7 : 5
 75 : 5 =  15                Test:     15
 25                                  .  5
  0                                    75
```

3. The third set contains examples when on the place of the highest order of the dividend there is a number which is lesser than the divisor e.g.:

```
  36 : 6
  15 : 6
 156 : 6 = 26          Test:      26
  36                             . 6
   0                             156
```

4. Division with a remainder e.g.

```
  34 : 4
  23 : 4
   6 : 4
 634 : 4 = 158          Test:   158      632
 23                             . 4      + 2
  34                            632      634
    2(remainder)
```

5. Division of numbers with zeros. In this case it is important to guide children to apply consistently the learned procedure and not to omit any step or number.

```
        34 : 5
         3 : 5
        10 : 5

 1 034 : 5 = 206        Test:   206        1 030
   03                           . 5         +  4
    34                         1030        1
     4 (remainder)
```

Division by double-digit divisor

The procedure of division by a double-digit divisor copies the methodological sequence of division by a single-digit divisor. Nevertheless, for children with learning disabilities it is rather difficult. These children are worse at estimating partial quotients, they get oriented in the algorithm with a greater difficulty. It is a remarkable achievement if they master simpler examples. In the opposite case we choose the calculator as a compensatory tool. However, it is essential for children to master counting with the help of the calculator confidently and to have a certain idea about the order of the quotient, i.e. to be able to determine the estimate of the result correctly.

- **Children's problems at written division**

1. Numerical mistakes follow from the insufficient knowledge of mental operations.
2. Children perform the true-false test formally.
3. Children do not follow the precise procedure of the algorithm e.g.

   2535 : 5 = 57,   422149 : 7 = 639

4. Children do not master examples where there are numbers with zeros e.g.:

   2 408 : 6 = 41, r 2   82 000 : 4 = 205

   3 000 : 10 = 30

Remedial procedures

1. We choose easier examples of division for children with problems in mathematics – it is more contributing for the child if it manages easy examples than if it does not cope with the more complicated ones.

2. We always perform the true-false test.

3. We revise mental counting constantly.

4. We include the calculator in lessons suitably.

Questions for the self- study:

1. Explain the significance of division to parts and division according to the content.

2. What problems do children have while mastering basic division links?

3. Which mistakes occur most often while the written division?

4. What is the significance of the choice and of the difficulty level of examples at the written division?

5. In which cases do we suggest the calculator?

3.4.5 Use of Brackets and Order of Operations

- **Theoretical Bases**

In practice, we often solve examples where we work with more numbers (e.g. while solving complex word problems) and we need to determine the process of the calculation in numerical expressions. Children use the fixed rules dealing with the use of brackets and the different parity of individual operations.

If there are brackets in the numerical expressions, then the expressions in brackets are performed first e.g.

$26 – (12 -8) = 26 – 4 = 22$

$(3 + 5) . 6 = 8 . 6 = 48$

If there are only addition and subtraction in the numerical expression and there are no brackets, we will proceed from the left to the right while calculating e.g.

$42 + 14 – 16 = 56 – 16 = 40$

$100 – 25 – 30 = 75 – 30 = 45$

If there are the operations addition, subtraction, multiplication and division in the numerical example and there are no brackets, then there applies that multiplication and division are of the higher priority than addition and subtraction e.g.

$3 + 5 . 6 = 3 + 30 = 33$

$28 – 6 : 3 = 28 – 2 = 26$

$3 . 9 + 8 . 4 = 27 + 32 = 59$

- **Children's problems**

1. Children calculate the expression in the brackets first, but they forget about the first number e.g. $60 – (50 – 30) = 20$.
2. They calculate the expression in the brackets first, they write it as the first one and then they do not know how to finish the example e.g. $60 – (50 – 30) = 20 – 60$.

3. Children do not respect the rule about the order of operations and always calculate from the left to the right e.g. 3 + 5 . 6 = 8 . 6 = 48

   48 – 8 : 4 = 40 : 4 = 10.

4. They calculate according to their rules e.g. they count 6 . 5 + 4 : 2 as follows
   5 + 4 = 9,   6 . 9 = 54,   54 : 2 = 27.

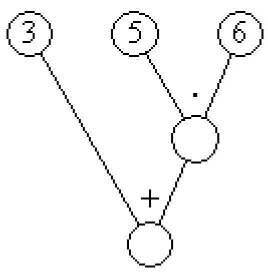
Remedial procedures

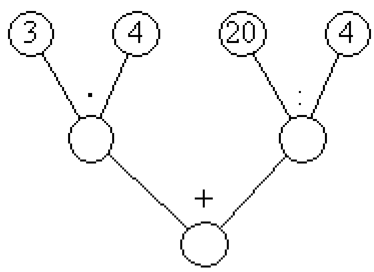
1. We will enable children to note the result of the operation above the brackets and guide them to write all numbers:

   $$60 - \overset{20}{(50 - 30)} = 60 - 20 = 40.$$

2. We will illustrate the procedure of performing operations with the help of the tree e.g.

   3 + 5 . 6   3 . 4 + 20 : 4





   In the first level from the top we will multiply or divide, in the second level we will add or subtract.

3. We will use brackets also in expressions with multiplication and division e.g.
   5 + (6 . 7)   or (3 . 4) + (20 : 4)

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