$$\lim_{(x,y)\to(1,2)} \underbrace{\frac{x^3y - xy^3+1}{(x-y)^2}}_{f(x,y)}$$

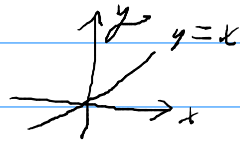
$(x,y)\to(1,2)$

$x\to 1,\ y\to 2$

$$f(1,2) = \frac{1\cdot 2 - 1\cdot 8 + 1}{1^2}$$

0 ve jmenov. bude pouze pro

$\Gamma = \{(x,y) : y = x\}$



Avšak $(1,2) \notin \Gamma$

Limita pak bude $= f(1,2) = -5.$ (dosazením)

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} = 1 \qquad \text{„}\frac{0}{0}\text{“}$$

$\boxed{x\to 0,\ y\to 0}$

$\sin(x,y)\to 0$, $xy\to 0$

$u = xy$, $u\to 0$

$$\lim_{u\to 0}\frac{\sin u}{u} = 1$$

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \underbrace{(x^2+y^2)}_{\to+\infty}\sin \underbrace{\frac{1}{x^2+y^2}}_{\to 0} \qquad \text{„}\infty\cdot 0\text{“}$$

$$(x^2+y^2)\sin\frac{1}{x^2+y^2} = \frac{\sin\left(\frac{1}{x^2+y^2}\right)}{\frac{1}{x^2+y^2}} \qquad \begin{matrix}\to 0\\ \to 0\end{matrix} \qquad \text{„}\frac{0}{0}\text{“}$$

limita je $= 1$

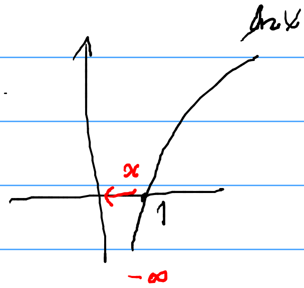
$u = \frac{1}{x^2+y^2}$,

$u\to 0$ pro $x\to+\infty$, $y\to+\infty$.

$u = \frac{1}{r^2}$,

r = vzd. mezi $[x,y]$ a $[0,0]$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)\ \nexists$, kde je $f(x,y) = \sin \ln (x^4+y^2)$



Zvolme postupnosť bodov $P_n(x_n, y_n)$, $n=1,2,\dots$
tak, aby bolo

$$x_n^4 = \frac{1}{2n}, \quad y_n^2 = \frac{1}{2n}$$

$$\left(\text{tj } x_n = \frac{1}{\sqrt[4]{2n}}, \quad y_n = \frac{1}{\sqrt{2n}} \right)$$

Potom $f(x_n,y_n) = \sin \ln \Big(\underbrace{\frac{1}{2n} + \frac{1}{2n}}_{\frac{1}{n}} \Big) = \sin \ln \frac{1}{n} = -\sin(\ln n)$

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \ ? \ = 0$$

$\lim\limits_{u\to 0} \frac{\sin u}{u} = 1$ $\quad$ [$\sin u \sim u$ pre $u \to 0$]

$u = xy$, $\quad \sin xy \sim xy$ pre $(x,y)\to(0,0)$

$$\frac{\sin xy}{y} = \overbrace{\frac{\sin xy}{xy}}^{\to 1 \text{ pre } (x,y)\to(0,0)} \cdot \underbrace{x}_{\to 0} \longrightarrow 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = f(x,y) \quad ? \quad \text{„}\frac{0}{0}\text{"}$$

Zkusme $(x,y)\to(0,0)$ podle přímek $y=kx$

$$f(x,kx) = \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$$

$x\to 0$; $x\to 0,\ x\neq 0$ — závisí na k

$k=0$ ($(x,y)\to(0,0)$ podle osy x): $f(x,0\cdot x)=1$

$k=1$ ($(x,y)\to(0,0)$ podle přímky $y=x$): $f(x,x)=0$

↳ limita existovat nemůže

Jinak: $(x_n,y_n)=\left(\frac{1}{n},\frac{1}{n}\right) \Rightarrow f(x_n,y_n)=f\left(\frac{1}{n},\frac{1}{n}\right)=0$

$(x_n,y_n)=\left(\frac{1}{n},0\right) \Rightarrow f\left(\frac{1}{n},0\right)=1$

Postupné limity:

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2} = -1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(y,x)\right) = \lim_{y\to 0}(-1) = -1$$

Naopak: $\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2} = 1$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} 1 = 1$$

$\neq$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0 \quad ?$$

$= f(x,y)$

Polární souřadnice:

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$

$$f(x,y) = f(\rho\cos\varphi, \rho\sin\varphi) = \frac{\rho^2\cos^2\varphi\cdot\rho\sin\varphi}{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} =$$

$$= \frac{\rho^3\cos^2\varphi\sin\varphi}{\rho^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}} = \underbrace{\rho\cdot\cos^2\varphi\sin\varphi}$$

v polárních souř. (ρ,φ)

$(x,y)\to(0,0) \Leftrightarrow$ vzdál. mezi (x,y) a $(0,0) \to 0 \Leftrightarrow \rho\to 0$

$$f(\rho\cos\varphi, \rho\sin\varphi) = \rho\cdot\cos^2\varphi\sin\varphi$$

$$0 \leftarrow 0 \le |\rho\cos^2\varphi\sin\varphi| \le \rho\,|\cos^2\varphi|\cdot|\sin\varphi| \le \rho \to 0 \quad \text{pro } \rho\to 0$$

(2 polic.!) $\quad \rho\cos^2\varphi\sin\varphi \to 0$ pro $\rho\to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{y-2x^2}{y-x^2} = f(x,y) \qquad \nexists$$

přímky: $y = kx$

$$f(x,kx) = \frac{kx-2x^2}{kx-x^2} = \frac{k-2x}{k-x} \to 1, \quad x\to 0$$

($2x \to 0$, $x \to 0$)

paraboly $y = kx^2$

$$f(x,kx^2) = \frac{kx^2-2x^2}{kx^2-x^2} = \frac{k-2}{k-1}$$

výsledek závisí na hodn. k

Např. $k=2$: $f(\ldots) = 0$
$k=3$: $f(\ldots) = \frac{1}{2}$ $\quad\neq$

$$\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2+y^2} = f(x,y)$$

$P_n(x_n, y_n)$ $\qquad (x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right), \quad n = 1, 2, \ldots$

$$f(x_n, y_n) = \frac{5\,\frac{1}{n}\cdot\frac{1}{n}}{2\cdot\frac{1}{n^2}} = \frac{5}{2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{5}{2}$$

$(x_n, 0) = \left(\frac{1}{n}, 0\right) \Rightarrow f(x_n, 0) = 0$ $\quad\neq$