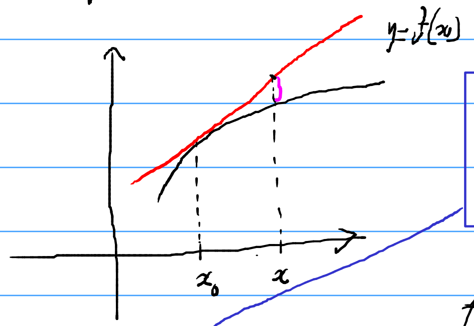
Příklad: dokažme, že

$$(1+x)^{\alpha} \doteq 1+\alpha x$$

pro x malé $\quad (x \to 0)$



Rovnice tečny v bodě $[x_0, f(x_0)]$:

$$y - f(x_0) = f'(x_0)(x-x_0)$$

$$\boxed{f(x) \doteq \underline{f(x_0)} + f'\underline{(x_0)}(x-x_0)}$$

$$f(x) = (1+x)^{\alpha} \qquad x_0 = 0$$

$$f(x_0) = (1+0)^{\alpha} = 1$$

$$f'(x) = \alpha(1+x)^{\alpha-1} \; ; \qquad f'(x_0) = \alpha(1+0)^{\alpha-1} = \alpha$$

$$(1+x)^{\alpha} = f(x) \doteq 1 + \alpha x$$

pro malá x

$\sqrt{5} \doteq ?$

$$\sqrt{5} = \sqrt{4+1} = (4+1)^{\frac{1}{2}} = \left(4\left(1+\tfrac{1}{4}\right)\right)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\tfrac{1}{4}\right)^{\frac{1}{2}} =$$

$$= 2\left(1+\tfrac{1}{4}\right)^{\frac{1}{2}}$$

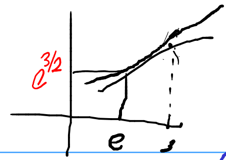
$(1+x)^{\alpha} \doteq 1+\alpha x$ pro malé x $\rightarrow$ $\left(1+\frac{1}{4}\right)^{\frac{1}{2}} \doteq 1 + \frac{1}{2}\cdot\frac{1}{4} = 1+\frac{1}{8}$

$x = \frac{1}{4}$

$\alpha = \frac{1}{2}$

Pak bude $\sqrt{5} \doteq 2\left(1+\frac{1}{8}\right) = 2 + \frac{1}{4} = 2{,}25$

$$e^{\frac{3}{2}} \doteq ?$$



$f(x) \doteq f(x_0) + f'(x_0)\cdot(x-x_0)$

$x = e,\ x_0 = 3$

$x - x_0 = e-3 \doteq 2{,}8-3 = -0{,}2$ malé

$$f(x) = x^{\frac{3}{2}} \qquad e^{3/2} \doteq 3^{3/2} + f'(3)\cdot(e-3)$$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}, \quad f'(3) = \frac{3}{2}\sqrt{3}$$

$$e^{3/2} \doteq (\sqrt{3})^3 + \frac{3}{2}\sqrt{3}\,(e-3)$$

$$\sqrt{3} = \sqrt{4-1} = \sqrt{4\left(1-\frac{1}{4}\right)} = \sqrt{4}\cdot\sqrt{1-\frac{1}{4}} = 2\sqrt{1-\frac{1}{4}} = 2\left(1-\frac{1}{4}\right)^{\frac{1}{2}}$$

$$\left(1-\frac{1}{4}\right)^{\frac{1}{2}} \doteq 1+\frac{1}{2}\left(-\frac{1}{4}\right) = 1-\frac{1}{8},$$

$$(1+x)^{\alpha} \doteq 1+\alpha x \qquad x = -\frac{1}{4}, \quad \alpha = \frac{1}{2}$$

$$\sqrt{3} \doteq 2\left(1-\frac{1}{8}\right) = 2-\frac{1}{4} = \frac{7}{4} = 1{,}75$$

$$e^{3/2} \doteq \left(\frac{7}{4}\right)^3 + \frac{3}{2}\cdot\frac{7}{4}(e-3) = \frac{7}{4}\left(\frac{7^2}{4^2} + \frac{3}{2}(e-3)\right) = \frac{7}{8}\left(\frac{49}{8} + 3(e-3)\right)$$

Zanedbáme-li hodnotu $e-3 \doteq -0{,}2$, bude

$$e^{3/2} \doteq \frac{7}{8}\cdot\frac{49}{8} = \frac{343}{64} \doteq 5{,}4$$

$$e^{3/2} \doteq \frac{7}{8}\Big(\underbrace{\frac{49}{8} - 9}_{\frac{49-72}{8} = \frac{-23}{8}} + 3e\Big) = \frac{7}{8}\left(3e - \frac{23}{8}\right) = \frac{21}{8}e - \frac{161}{64}$$

Obdržíme

$$e^{\frac{3}{2}} \doteq \frac{21}{8}e - \frac{161}{64}$$

Kalkulačka: 4,48... 4,61...

$f(x) \doteq f(x_0) + f'(x_0)\cdot(x-x_0)$ aproximace lineární funkcí

$$f(x) \doteq \boxed{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)} + \underbrace{\frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$$

aproximace polynomem stupně n

členy vyšší ř.

$f(x) = e^x$ $\qquad e^x \doteq e^0 + e^0(x-0) = 1+x$

$x_0 = 0$

$$e^x \doteq e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 = 1 + x + \frac{1}{2}x^2$$

$x_0 = 0$

$$e^x \doteq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Taylorův polynom st. n

$f(x) = e^{-\sqrt{x}}$ $\qquad f(x) \doteq ?$

$$f'(x) = \left(e^{-\sqrt{x}}\right)' = -e^{-\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = -\frac{1}{2}x^{-\frac{1}{2}}e^{-x^{\frac{1}{2}}}$$

$$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}x^{-\frac{3}{2}}e^{-x^{\frac{1}{2}}} + x^{-\frac{1}{2}}\cdot\left(-\frac{1}{2}\right)e^{-\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\right) =$$

$$= \frac{1}{4}\left(x^{-\frac{3}{2}}e^{-x^{\frac{1}{2}}} + x^{-1}e^{-x^{\frac{1}{2}}}\right) = \frac{1}{4}e^{-x^{\frac{1}{2}}}\left(x^{-\frac{3}{2}} + x^{-1}\right) =$$

$$= \frac{1}{4xe^{\sqrt{x}}}\left(\frac{1}{\sqrt{x}} + 1\right)$$

$$f(x) \doteq f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$x_0 = \frac{1}{4}$ $\qquad f(x_0) = e^{-\frac{1}{2}}$, $\quad f'(x_0) = -\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}}e^{-\frac{1}{2}} = -\frac{1}{2}\cdot 2e^{-\frac{1}{2}} = -\frac{1}{\sqrt{e}}$

$$f''(x_0) = \frac{3}{4\cdot\frac{1}{4}\sqrt{e}} = \frac{3}{\sqrt{e}}$$

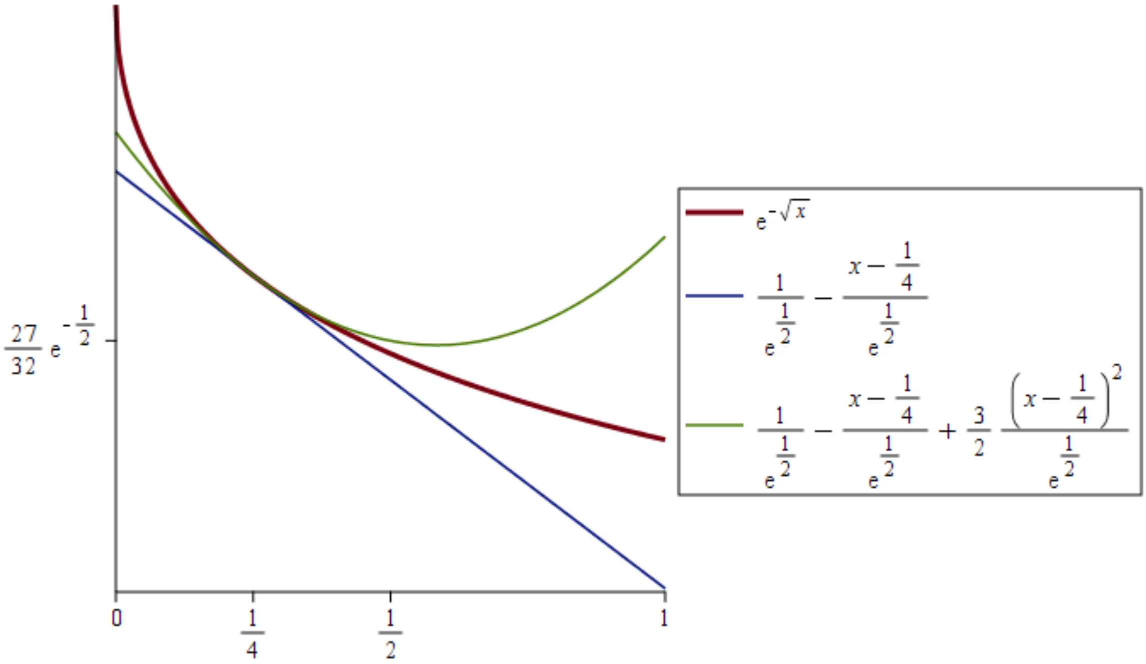
$$e^{-\sqrt{x}} \doteq \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}}\left(x-\frac{1}{4}\right) + \frac{3}{2\sqrt{e}}\left(x-\frac{1}{4}\right)^2$$

aproximace polynomem st. 2 v okolí bodu $x_0 = \frac{1}{4}$

Např. pro $x = \frac{1}{2}$: $\quad \overset{0{,}49\dots}{e^{-\frac{1}{\sqrt{2}}}} \doteq \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}}\left(\frac{1}{2}-\frac{1}{4}\right) + \frac{3}{2\sqrt{e}}\left(\frac{1}{2}-\frac{1}{4}\right)^2 =$

$$= \frac{1}{\sqrt{e}} - \frac{1}{4\sqrt{e}} + \frac{3}{2\sqrt{e}}\cdot\frac{1}{16} = \frac{3}{4\sqrt{e}} + \frac{3}{32\sqrt{e}} = \frac{3}{\sqrt{e}}\left(\frac{1}{4} + \frac{1}{32}\right) = \frac{27}{32\sqrt{e}}$$

0,51..



$f(x) = e^{-\sqrt{x}}$

$f\left(\frac{1}{2}\right) = e^{-\frac{1}{\sqrt{2}}} \doteq \frac{19}{32\sqrt{e}}$

0,49...