$z = f(x,y)$

f'_x $\quad \frac{\partial f}{\partial x}(x,y)$ = derivace podle „x" (y „zamrazíme")
y je konst.

$$f(x,y) = -5x^3y^2 + 3xy - \frac{1}{2}y^2$$

$$\frac{\partial f}{\partial x}(x,y) = -5y^2\cdot 3x^2 + 3y = -15x^2y^2 + 3y$$

$$\frac{\partial f}{\partial y}(x,y) = -5x^3\cdot 2y + 3x - y = -10x^3y + 3x - y$$

$$f''_{xx} = (f'_x)'_x = (-15x^2y^2 + 3y)'_x = -15y^2\cdot 2x = -30xy^2$$

$$f''_{xy} = (-15x^2y^2 + 3y)'_y = -15x^2\cdot 2y + 3 = -30x^2y + 3$$

$$f''_{yy} = (-10x^3y + 3x - y)'_y = -10x^3 - 1$$

$$f''_{yx} = (-10x^3y + 3x - y)'_x = -30x^2y + 3$$

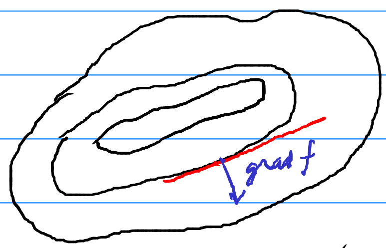
parc. der. II. ř. jsou spojité

$$H_f(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} -30xy^2 & -30x^2y + 3 \\ -30x^2y + 3 & -10x^3 - 1 \end{pmatrix}$$

Hesseova matice

$$\operatorname{grad} f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

gradient v daném bodě



(x_0, y_0) $\quad f(x_0, y_0)$

$$\lim_{t\to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} \qquad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$t \to 0$ $\qquad \frac{\partial f}{\partial \vec{u}}$ ve směru $\vec{u}$

$f(x,y) = x^2 + 2y^2$ gradient v bodě $P(1;2)$

$f'_x = 2x$, $f'_y = 4y$

$\text{grad } f = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$ $\text{grad } f(1;2) = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$x=1, y=2$

Směr největšího růstu f v bodě $(1;2)$ je určen vektorem $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$\underbrace{|\text{grad } f(1;2)|}_{\text{délka}} = \sqrt{2^2+8^2} = \sqrt{4+64} = \sqrt{68}$ největší rychlost

Vrstevnice? $f(x,y) = c$ c – konst

$f(x,y) = xy$ vrstevnice

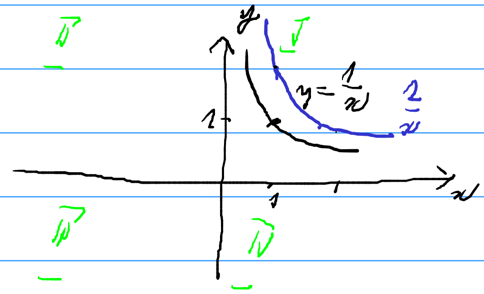
$xy = c$ c – konst. $c = 0$ osy x, y

$c \neq 0$

$y = \frac{c}{x}$

$c > 0$ I, III

$c < 0$ II, IV



$\text{grad } f = \begin{pmatrix} y \\ x \end{pmatrix}$ $P(1;2)$ $\text{grad } f(1,2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Tečná rovina a normála

$f(x,y) = y^3 - x^2y + 2xy$, bod $P(2;-1)$

$z = f(x,y)$; $P(x_0, y_0)$ – bod doteku; -1 rovnice teč. roviny:

$$\underbrace{f'_x(P)}_{2}\cdot(x-\underset{2}{x_0}) + \underbrace{f'_y(P)}_{-3}\cdot(y-\underset{-1}{y_0}) + z - \underbrace{f(P)}_{z_0} = 0$$

(x_0, y_0, z_0)

$F(x,y,z) = 0$ v bodě $(x_0, y_0, z_0) \in R^3$

$$F'_x(P)(x-x_0) + F'_y(P)(y-y_0) + F'_z(P)(z-z_0) = 0$$

$F(x,y,z) = f(x,y) - z$

$2(x-2) + 3(y+1) + z - 1 = 0$

$2x - 4 + 3y + 3 + z - 1 = 0$

$2x + 3y + z = 2$

teč. rov. v bodě $(2;-1;-1)$

$f'_x = -2xy + 2y$, $f'_y = 3y^2 - x^2 + 2x$

$P(2,-1)$; $x = 2$, $y = -1$

$f'_x(2,-1) = -2\cdot 2(-1) - 2 = 2$,

$f'_y(2,-1) = 3 - 4 + 4 = 3$

$f(x,y) = y^3 - x^2y + 2xy$

$f(P) = f(2,-1) = (-1)^3 + 4 - 4 = -1$

Normála: $$\frac{x-x_0}{f'_x(x_0,y_0)} = \frac{y-y_0}{f'_y(x_0,y_0)} = \frac{z-z_0}{-1}$$

$$\frac{x-2}{2} = \frac{y+1}{3} = \frac{z+1}{-1}$$

Pak $\frac{x-2}{2} = t$, $\frac{y+1}{3} = t$, $\frac{z+1}{-1} = t$,

$x = 2t + 2$, $y = 3t - 1$, $z = -t - 1$ (parametrické rovnice normály)

Tot. diferenciál

$f(x)$ $\quad df(x) = f'(x) \cdot dx$

$f(x,y)$ $\quad df(x,y) = f'_x(x,y) \cdot dx + f'_y(x,y) \cdot dy$

$$df(x,y)[h_1, h_2] = f'_x(x,y) \cdot h_1 + f'_y(x,y) \cdot h_2$$

Př.: $\quad 1{,}05^{2{,}01} \doteq ?$

$f(x,y) = x^y \qquad P(1,2) \quad x_0 = 1, \; y_0 = 2$

$df(x,y) = y x^{y-1} \cdot dx + x^y \ln x \cdot dy$

$f'_x = (x^y)'_x = y x^{y-1} \qquad \underbrace{\Delta f}_{f(x,y) - f(x_0,y_0)} \doteq df(x_0,y_0)[x - x_0, y - y_0]$

$f'_y = (x^y)'_y = x^y \ln x$

$x_0, y_0 = \ldots \qquad df(1,2) = 2dx + 0 \cdot dy = 2\,dx$

$1{,}05^{2{,}01} - \underbrace{1^2}_{f(x_0,y_0)} \doteq 2 \cdot (1{,}05 - 1) = 2 \cdot 0{,}05$

$1{,}05^{2{,}01} \doteq 1 + 2 \cdot 0{,}05 = 1{,}1$

Stac. body; lokální extrém

$f(x,y) \to$ extrém (max / min)

Stac. body: (x,y), kde platí

$$f'_x(x,y)=0,\quad f'_y(x,y)=0 \qquad (\text{nebo } \nexists)$$

$$f(x,y) = 3x^2 + xy + 2y^2 - x - 4y$$

St. body:

$$\frac{\partial f}{\partial x} = 6x+y-1, \quad \frac{\partial f}{\partial y} = x+4y-4$$

$$6x+y-1=0, \quad x+4y-4=0 \qquad (x,y)=?$$

$$y = 1-6x \qquad x+\cancel{4}-24x-\cancel{4}=0 \Rightarrow x=0$$

$$\hookrightarrow y=1$$

Stac. bodem je pouze $(0;1)$

Hessova matice:

$$f''_{xx}=6,\ f''_{xy}=1,\ f''_{yy}=4$$

$$H_f = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$$ (v daném př. – konst., obecně ne)

$$H_f(\underbrace{x_0,y_0}_{\text{stac. bod}}) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\underbrace{\det H_f(x_0;y_0)}_{|H_f(x_0;y_0)|} = AC - B^2$$

$$\boxed{\underbrace{AC-B^2}_{|H_f|} \begin{cases} >0 & \begin{cases} A>0 & \text{min} \\ A<0 & \text{max} \end{cases} \\ <0 & \text{extrém není} \\ =0 & ? \end{cases}}$$

postačující podmínka pro lok. extrém

Pro $f(x,y) = 3x^2 + xy + 2y^2 - x - 4y$:

stacion. bod $(0;1)$

$$H_f(0,1) = \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$$ (zde konstantní, neboť $f(x,y)$ má jen kvadratické členy; obecně závisí na x, y)

$$\det H_f(0,1) = 24-1 = 23 > 0$$

Lok. extrém je; $A=6>0 \Rightarrow$ lokální minimum