From Brouwer to Hilbert:
the debate on the foundations
of mathematics in the 1920
Paolo Mancosu, ed.
New York: Oxford University Press, 1998

PART I: L. E. J. BROUWER

Brouwer's Intuitionist Programme

WALTER P. VAN STIGT

uitzen Egbertus Jan Brouwer (1881–1966) is a central figure in the history of con-mporary mathematics and philosophy.[1] His main contributions are in the field of pology and the foundations of mathematics. It is Brouwer's contribution to the undations of mathematics, the intuitionist programme, that has made him known the wider scientific and philosophical community. His influence is very much live today, as is witnessed by the ongoing mathematical research in intuitionist and onstructive mathematics (see Bridges, Richman 1987, and Troelstra, van Dalen, 988) and by the variety of philosophical contributions that have their roots in rouwer's intuitionism (see, for instance, Detlefsen 1990, Dummett 1973, 1977, McCarthy 1983). However, although many of these contributions take their start rom Brouwer's intuitionist approach, it is also true that they have departed to a reat extent from Brouwer's original formulation of the programme. In van Stigt 1990 I have endeavored to present Brouwer and his intuitionist programme in their historical setting. The following introduction is conceived in the same spirit. I shall begin with a section on the intuitionist–formalist controversy. Section 1.2 is about Brouwer's intuitionist philosophy of mathematics. Sections 1.3 and 1.4 present Brouwer's views on the nature of mathematics and on the relationship between math-ematics, language, and logic. Section 1.5 gives an account of Brouwer's new set theory and his conception of the continuum. Finally, Section 1.6 gives a short in-troduction to the selected contributions.

1.1 The Intuitionist–Formalist Controversy

In 1920 Hermann Weyl diagnosed "a new crisis in the foundations of mathematics" (Weyl 1921), sparked off by the publication of Brouwer's "Foundations of Set Theory Independent of the Principle of Excluded Middle" (B1918B and B1919A). In a series of lectures at the Mathematical Colloquium of Zürich, he dramatically renounced his own *Das Kontinuum* and hailed Brouwer's set theory and interpretation of the continuum as "the revolution": ". . . und Brouwer—das ist die Revolution!" (Weyl 1921, p. 99), the one mathematician who at last had solved the problem of the continuum, which since ancient times had defeated even the greatest minds. At the same time, in "Intuitionist Set Theory" (B1919D) Brouwer set out the consequences for established mathematics of his Intuitionist theses, in particular, his rejection of the logical Principle of the Excluded Middle: "the use of the Principle of the Excluded Middle is *not permissible* as part of a mathematical proof . . . [it] has only scholastic and heuristic value, so that theorems which in their proof cannot avoid the use of this principle lack all mathematical content." (p. 23)

Both Brouwer's challenge and Weyl's support raised the alarm among the Cantorian and Formalist establishment of Göttingen. Hilbert, who had recognized Brouwer's major contribution to topology and had welcomed him as a member of his inner circle, grew increasingly impatient with his old friend and alarmed by the implied threats to Cantorian set theory and his own programme. He launched a counterattack in 1922:

> What Weyl and Brouwer do amounts in principle to following the erstwhile path of Kronecker: they seek to ground mathematics by throwing overboard all phenomena that make them uneasy . . . if we follow such reformers, we run the danger of losing a large number of our most valuable treasures. (Hilbert 1922, p. 200)

The ensuing Intuitionist–Formalist "debate" dominated the foundational scene throughout the 1920s. Brouwer and Hilbert remained the main protagonists, each drawing support for his cause beyond national frontiers and an even greater audience of interested observers and commentators.

The debate centered on two different, though related, issues:

1. The nature of mathematics: either human thought-construction or theory of formal structures;
2. The role of the Principle of the Excluded Middle in mathematics and Brouwer's restrictive alternative logic.

Brouwer's main concern was the nature of mathematics as pure, "languageless" thought-construction. He had set himself the task of bringing the mathematical world around to his view, convincing them of the need for reform, and had started the programme of reconstructing mathematics on an Intuitionist basis. Most of his publications in the period 1918–1928 were part of this programme; only a few dealt directly with the "negative" aspects of his Intuitionist campaign: the misuse of logic, in particular the Principle of the Excluded Middle, and the flaws in the Formalist programme. Understandably these papers aroused greater interest and further controversy. His excursion into the field of logic ("Intuitionist Splitting of the Fundamental Notions of Mathematics," B1923C), in which he drew the immediate conclusions from his strict interpretation of negation and his rejection of the Principle of the Excluded Middle, created considerable excitement among logicians and started a debate about an alternative, "Brouwer Logic." This debate was joined by Kolmogorov, Borel, Wavre, Glivenko, Heyting, and others (see Part IV). Brouwer himself did not take a further active part, remaining true to his conviction that logic and formalization were "an unproductive, sterile exercise" with no direct relevance to mathematics and its foundations.

The main Intuitionist–Formalist "debate" was a contest between the leaders of two opposing philosophies of mathematics, each with its own programme and competing for the support of the mathematical world. Apart from the occasional direct exchange, each camp concentrated on its own programme. Hilbert's Programme, retaining the "whole treasure of classical mathematics" and basing its validity on a proof of the consistency of its formalization, attracted widening support and an able team of collaborators. Brouwer's constructive interpretation of mathematics, much in line with the natural outlook of the working mathematician, was enthusiastically received and raised early hopes. However, his austere programme of reconstruction within the Intuitionist constraints failed to gather momentum. His increasing isolation was partly due to his inability to work with others, but more important, Brouwer's and Weyl's hopes that the "natural" Intuitionist approach would lead to a simplification of reformed mathematics did not materialize. Indeed, it proved "unbearably awkward" in comparison with traditional mathematics relying on the methods of classical logic. Even Weyl had to accept this with regrets:

> Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with Intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice, which he believed to be built of concrete blocks, dissolve into mist before his eyes. (Weyl 1949, p. 54)

The Brouwer–Hilbert debate grew increasingly bitter and turned into a personal feud. The last episode was the "Annalenstreit," or, to use Einstein's words, "the frog-and-mouse battle." It followed the unjustified and illegal dismissal of Brouwer from the editorial board of the *Mathematische Annalen* by Hilbert in 1928 and led to the disbanding of the old *Annalen* company and the emergence of a new *Annalen* under Hilbert's sole command but without the support of its former chief editors, Einstein and Carathéodory.

For Brouwer it was the last straw. His failure to "simplify" Intuitionist methods and make Intuitionism the universally accepted mathematical practice had eroded his self-confidence. The conspiracy of his fellow *Annalen* editors and "lack of recognition" left him bitter and disillusioned. He abandoned his Intuitionist Programme and withdrew into silence just about the time when the Formalist Programme was shown to be fundamentally flawed. Some "books" were left uncompleted and unpublished. The 1928 "Vienna Lectures," *The Structure of the Continuum* (B1930A) and "Mathematics, Science and Language" (B1929), and his paper "Intuitionist Reflections on Formalism" (B1928A2) mark the end of Brouwer's creative life and his Intuitionist campaign. They reflect the stage his programme had reached and the mood of its founder at the time. *The Structure of the Continuum* summarizes his Intuitionist vision and analysis of the continuum. In "Mathematics, Science and Language" he returns to the pessimism of his philosophy of science and language, which had inspired his Intuitionist rebellion. "Intuitionist Reflections on Formalism" is Brouwer's final assessment of the state of play in the contest between Intuitionism and Formalism and an emotional outburst at the lack of recognition. It lists outstanding differences as well as "the Intuitionist Insights" adopted by Formalists "without proper mention of authorship," such as the notion of metamathematics.

1.2 Intuitionism and Brouwer's Intuitionist Philosophy of Mathematics

Brouwer's Intuitionist reform of mathematics and his revolutionary views on the use of logic can only be fully understood in the context of his particular philosophy of mathematics. Indeed, his Intuitionism is first and foremost a philosophy of mathematics from which these new ideas emerge quite naturally.

Most of Brouwer's philosophical views on life in general and on the nature of mathematics were formed during the years of undergraduate and doctoral studies, and they remained virtually the same throughout his life. They are expressed most clearly in his early publications: his doctoral thesis *On the Foundations of Mathematics* (B1907) and *Life, Art and Mysticism* (B1905), in some of his post-1928 papers such as "Mathematics, Science and Language" (B1929), "Will, Knowledge and Speech" (B1933), and "Consciousness, Philosophy and Mathematics" (B1948C), and in unpublished papers.

This section is a brief introduction to Philosophical Intuitionism and the main aspects of Brouwer's philosophy as are relevant to his Intuitionist practice. A more detailed analysis is given in van Stigt 1990.

Intuitionism is a philosophical trend that places the emphasis on the *individual consciousness* as the source and seat of all knowledge.[2] Besides the faculty and activity of reasoning, it recognizes in the individual mind a definite faculty and act of direct apprehension, intuition, as the necessary foundation of all knowledge, both in the grasping of first principles on which a system of deductive reasoning is built and as the critical link in every act of knowing between the knower and the object known. Intuitionism stands in contrast to a more general rationalistic and deterministic trend that denies the possibility of knowing things and facts in themselves and restricts human knowledge to what can be deduced mechanically by analytical reasoning, ultimately from self-evident facts and principles that result from common sense or are based on the authority of collective wisdom.

Elements of Intuitionism can already be found in classical philosophies, for example, in the Aristotelian *νοῦσ*, a special faculty of direct apprehension, an active faculty that is indispensable in the creation of primary concepts and first principles as well as at every step of the thought process. Elements of Intuitionism are also found in the systems of some of the modern German and English philosophers such as Kant, Hamilton, Whewell, and even Russell (for the Kantian roots of Brouwer's philosophy of mathematics, see Posy 1974). But it is in the revolutionary and libertarian climate of Holland and France that Intuitionism took root and developed into a full and coherent philosophy.

Descartes, the father of modern philosophy, can rightly be claimed to be the father of modern Intuitionism. A Frenchman by birth, Descartes settled in Holland, "a country"—as he wrote to Balzac—"where complete liberty can be enjoyed." His rebellion was the fundamental break with the traditional reliance on authority, religious and otherwise, as the ultimate source of truth and placing the origin and seat of knowledge firmly in the individual mind of man. He starts from "self-awareness" and distinguishes between various faculties in the process of acquiring knowledge, but insists that every form of knowing ultimately requires an act of immediate mental apprehension, "intuition." He insists on the need for rational argument and sets out rigorous rules of correct reasoning, but points out that logical deductive reasoning does not produce any new truths, that true knowledge comes from intuition.

Descartes' intuitionist lead was followed in the nineteenth century by a number of French philosophers such as Maine de Biran, Ravaisson, Lachelier, and Boutroux. It was developed into a full and comprehensive philosophy by Henri Bergson, who raised Intuition to the faculty of grasping the spiritual and changing reality, distinct from Reason, the analytical mind, which probes the material and static reality. Bergson's living reality, however, did not include the mathematical universe; his concepts of number and the mathematical continuum are spatial, products of the analytical intellect.

As to the precise nature of Intuition as the foundation of mathematics, Descartes remains somewhat vague: The fundamental mathematical truths are "indubitable" because they are "clearly and distinctly perceived" by the mind's eye. Yet in his argument for the existence of God, for which he claims "the same level of certainty as the truths of mathematics," he concludes that these truths, such as the essence and nature of the triangle, are "immutable and eternal and not invented by me nor dependent on my mind" (Descartes, 5th Meditation). Equally vague as to the nature of Intuition are the French "New Intuitionists" Poincaré, Borel, and Lebesgue.[3] It was not until the beginning of the twentieth century that an attempt was made at a precise interpretation of mathematical Intuition, when Brouwer took Descartes' intuitionist thesis to its radical subjective and constructive conclusion.

Brouwer's outlook on life and general philosophy can best be described as a blend of romantic pessimism and radical individualism. In *Life, Art and Mysticism* (B1905) he rails against industrial pollution and man's domination of nature through his intellect and against established social structures, and promotes a return to "Nature" and to mystic and solitary contemplation.

In his *Foundations of Mathematics* (B1907), especially its original version, it is the application of mathematics in experimental science and logic that is exposed as the source of all evil and analyzed as "the causal" or "cunning act," superimposing a mathematical regularity on the physical world. Both works express his conviction of the opposition between mind and matter, the individual consciousness and the exterior world.

Reflecting on the nature of man, Brouwer identifies personal identity, the "Self" or "the Subject," with the pure-spiritual "Soul" in his later work referred to as "Consciousness in its deepest home" ("Consciousness, Philosophy and Mathematics," B1948C). The life of the Soul is the complex of thought processes in response to its awareness of the world outside. They are analyzed as distinct mental states, "phases of consciousness" in a process of evolution, each resulting from a definite "happening" and each producing its characteristic form of knowledge and human activity. It is a "deteriorative" process moving consciousness further and further away "in its exodus from its deepest home" on a sliding scale from "beautiful," that is, good, to evil.

The original preperceptional stage of "stillness" is followed by "the naive phase" of receiving images through physical sensations and reacting spontaneously to them.

The momentous event of the Subject linking isolated sensations, becoming aware of time, referred to by Brouwer as "the Primordial Happening" or "the Primordial Intuition of Time," brings about a transformation of the Naive Consciousness to the rational "Mind" and at the same time generates the fundamental concepts and tools of mathematics. The Primordial Intuition of Time is the fundamental single act of isolating and linking distinct moments in time, creating mathematical "Two-ity" and the ordinal numbers as well as the continuum. It is first mentioned in Brouwer's analysis of science, Chapter 2 of his *Foundations*, where it is used to show the priority of mathematics with respect to science and expose the ideal nature of science, no more than man's mathematical interpretation of the world. The mathematical power to generate sequences enables man to create in his individual thought-world an interpretation of "Nature," the outside world, which is manmade and mathematical. "Things," including other human beings, are no more than repeated sequences or sequences of sequences, manmade, as is indeed the so-called scientific or "causal" coherence of the world. And because of the individual nature of human thought, this universe of "things" is wholly private. Brouwer refers to it as "the Exterior World of the Subject." The scientific observation of regularity in Nature, linking things and events in time as sequences, is a creative, mathematical process of the individual Mind and is referred to as "mathematical viewing" or "causal attention." Causality is an artificial, mind-made structure, not inherent in Nature. Indeed, Brouwer rejects any universal objectivity of things as well as their "causal coherence," basing his argument on the essential individuality of thought and mind. In "Consciousness, Philosophy and Mathematics" (B1948C) he emphatically denies the existence of a collective or "plural" mind.

The Brouwerian evolutionary "exodus from its deepest home of consciousness" enters a moral phase when man takes advantage of and acts upon his causal knowledge by setting in motion a causal sequence of events, selecting a first element of the sequence in order to achieve a later element, the desired "end." Such mathematical or causal acting is "calculated" and "cunning," condemned as "sinful" and "not-beautiful," that is, morally evil.

Even more remote from the "deepest home of consciousness" is the next and final phase of "social acting," described as "the enforcement of will" in social interaction and organization, in particular by the creation of language. Brouwer's philosophy of language starts from the conviction that direct communication between human beings is impossible. His chapter on "Language" in *Life, Art and Mysticism* (B1905) starts as follows: "From Life in the Mind follows the impossibility of communicating directly with others . . . never has anyone been able to communicate directly with others soul-to-soul." (p. 37). The privacy of mind and thought and the hypothetical existence of minds in other human beings, who are no more than the Subject's mind-creations, "things in the exterior world of the Subject" rule out "any exchange of thought" (B1948C, p. 1240). In line with his "genetic" principle of ontological analysis, Brouwer searches for the nature of language in the process that brought it into being. He traces the origin of language to a particular form of cunning or mathematical acting, the "imposition of will through sounds," forcing another human being to act in pursuance of the end desired by the speaker: "At the most primitive stages of civilization . . . the transmission of will to induce labour or servitude is brought about by simple gestures of all kinds especially and predominantly the emotive natural sounds of the human voice" (B1933, p. 51). As social interaction develops and grows more complex, language becomes more sophisticated, but its essence, as of all instruments, is determined by its purpose: the transmission of will. Used as a means of communicating thought to others, language is bound to remain defective, given the essential privacy of thought and the nature of the "sign," the arbitrary association of a thought with a sound or visual object.

Within the private world of the Subject, language may have a function as "an aid to memory," helping the Subject to recall his past thought. In "Will, Knowledge and Speech" (B1933), when Brouwer had to accept human frailty, the limitations of the flesh-and-blood mathematician, even the stability of such private language, was called into question: "The human power of memory . . . is by its very nature limited and fallible" (p. 58), even when it calls in the help of linguistic signs. It is at this point that Brouwer introduces his notion of the "Idealized Mathematician."

1.3 The Nature of Pure Mathematics

The mathematical nature of causality and the Primordial Intuition of Time as the fundamental creative act of mathematics are the central theses of Brouwer's analysis of science and language. In Chapter 2 of *The Foundations of Mathematics* they are treated as closely related, and the "mathematical" appears somewhat tainted by its association with "causal" or "cunning acting." There are, however, signs that as early as 1907 Brouwer had established an independent and redeeming role for pure mathematics. His *Foundations* ends with a summary that starts: "Mathematics is a free creation, independent of experience; it develops from one single a priori Primordial Intuition . . ."(B1907, p. 179). In the original plan of the thesis, moreover, there is an additional chapter entitled "The Philosophical Significance of Mathematics," in his *Preparatory Notes* referred to as "Mathematics and the Liberation of Mind." The "Liberation of Mind" is a favorite theme of *Life, Art and Mysticism*. In the mathematical context it refers to the elimination of all exterior, phenomenal elements and causal influences from the creative mathematical act. It allows the Primordial Intuition as an abstraction of pure time awareness, eliminating also the content of sensations, to be a pure and a priori basis of mathematics and its defining act. The Primordial Intuition is not only necessary and sufficient for the creation of two-ity; it also holds the continuum as "its inseparable complement" and contains the fundamental elements and tools from which and with which the whole of mathematics is to be constructed. Indeed, mathematics is identified with the whole of the constructive thought-process on and with the elements of the Primordial Intuition alone. Brouwer's preferred term is "building" (Dutch: *bouwen*) rather than "construction," a building upwards from the ground, a time-bound process, beginning at some moment in the past, existing in the present, and having an open future ahead. Indeed, mathematics is the life of what Brouwer calls "the Subject," "the Creating Subject," or the "Idealized Mathematician." Its characteristics are determined by the time-bound and individual nature of mind as the sole creator and seat of mathematical thought and by the limits of Intuition.

Mathematical Existence and Truth

Mathematical existence then in its strictest sense is "having been constructed" and remaining alive in the mind or memory. The whole of the Subject's constructive thought-activity, past and present, constitutes mathematical reality and mathematical truth: "Truth is only in reality, i.e. in the present and past experiences of consciousness" (B1948C, p. 1243). Mathematical entities are identified with the whole of their constructive pedigree, whether they be single concepts, such as, for example, an ordinal number, or more complex, such as a mathematical theorem, which combines various constructions.

Past, completed constructions consist of sequences of constructive steps and as such are finite. Mathematical existence can be claimed for "the infinite" within an interpretation that is based on completed constructions and the freedom of the Subject to proceed. In the case of a denumerably infinite sequence such as "the fundamental sequence" of ordinal numbers, the completed construction is the algorithm or "law" by which each element of the sequence is uniquely determined. The "free" power of the live Subject to proceed ensures that the elements be generated "indefinitely." The essential active role of the Subject in constructing his procedure for determining elements and in the continued generation of these elements allows the possibility of extending the traditional notion of infinite sequence. Brouwer took this step in 1917, when he introduced the "free-choice sequence" and his new set concept as the procedure for generating "points on the continuum" (see further Section 1.5). The established concept of the continuum as a set, the totality of existing points, was rejected outright in *The Foundations* (B1907) and "On Possible Powers" (B1908A). The Brouwer notion of the continuum-as-a-whole, "the Intuitive Continuum," is a primitive concept generated in the Primordial Intuition of time. It is abstracted from the time interval, the mathematical "between" that is never exhausted by division into subintervals.

1.4 Mathematics, Language, and Logic

Within the Brouwerian conception of mathematics as pure, individual thought-construction on the basis of the Primordial Intuition alone there is clearly no place for language in any form. The emphatic "languageless" in nearly all his definitions of mathematics reflects the need for express denial of the role language plays in almost all alternative philosophies of mathematics, even that of his favorite "preintuitionist," Poincaré. Freeing mathematics from its traditional reliance on language and logic was the objective of Brouwer's first Intuitionist campaign, the "First Act of Intuitionism," in his historical surveys described as "the uncompromising separation of mathematics and mathematical language and thereby of the phenomena described by theoretical logic."

In the genesis of mathematics, wholly confined to the private thought-world of the Subject, no use is made of any aspect of language, either as the carrier of common "objective" concepts or as primitive symbols with no meaning. For the sake of "aiding the memory" the flesh-and-blood mathematician may resort to recording his constructions in symbols, linking a thought-construction to a name, "an aural or visual thing"; such recording, however, is a posteriori and not part of the mathematical process itself. Moreover, it is essentially private language since both the thought-construction and the assignation to a particular symbol are exclusively acts of the individual mathematician.

As to language as a means of communicating mathematics to other individuals, there is no basis for agreement between the constructive thought-processes of different individuals represented in a "common language." To the Subject other individuals are "things," creations of his Exterior World, and the existence of other minds similar to his own "mere hypothesis." And even if the existence of other minds were to be conceded, there is no guarantee that common words would represent the same thought-construction in the private worlds of different individuals.

Brouwer's "mathematical language" is the report, the record of a completed mathematical construction; its truth and noncontradictority are due solely to the constructions it represents.

Logic

The Foundations (B1907) defines "theoretical logic" as an application of mathematics, the result of the "mathematical viewing" of a given mathematical record, seeing a certain regularity in the symbolic representation: "People who want to view everything mathematically have done this also with the language of mathematics . . . the resulting science is theoretical logic . . . an empirical science and an application of mathematics . . . to be classed under ethnography rather than psychology" (p. 129).

The classical laws or principles of logic are part of this observed regularity; they are derived from the post factum record of mathematical constructions. To interpret an instance of "lawlike behavior" in a genuine mathematical account as an application of logic or logical principles is "like considering the human body to be an application of the science of anatomy" (p. 130).

The cunning application of such principles in the verbal or symbolic domain produces nothing but "verbal edifices" outside the mathematical reality:

> Linguistic edifices, sequences of sentences which follow one another according to the laws of logic. . . . Even if it appears that these edifices can never show up the linguistic figure of a contradiction, they are only mathematics as linguistic constructions and have nothing to do with mathematics, which is outside this edifice. (B1907, p. 132)

Brouwer reiterates Descartes' observation that logic cannot generate new mathematical truths. In his "Unreliability of the Principles of Logic" (B1908C) he goes one step further and questions the validity of the principles of logic when applied to mathematics. To prove his general point he singles out the Principle of the Excluded Middle (PEM)—identified with the principle of the solvability of every mathematical problem—as flawed and an obvious misstatement of fact. The argument here is based on the lack of guarantee of a solution for an infinite system and on his strict interpretation of affirmation and negation. True mathematical statements, affirmative and negative, express the completion of a constructive proof; in particular the negative statement expresses what Brouwer calls "absurdity," the constructed incompatibility of two mathematical constructions represented, respectively, by the subject and predicate of the sentence. In B1908C Brouwer simply states that for an infinite system there is no guarantee that such a construction can be completed and "that for infinite systems the Principle of the Excluded Middle is not a reliable principle" (p. 157). His major campaign against the use of the PEM starts after the publication of "The Foundations of Set Theory Independent of the Logical Principle of the Excluded Middle" (B1918B and B1919A), when in a number of papers he challenges the proofs of certain classical theorems, analyzes the logic of negation, and tries to prove the invalidity of the PEM by means of his counterexamples of essentially unsolvable mathematical problems (see further Section 4.1).

Formalism and the Formalization of Mathematics and Logic

Brouwer's interpretation of mathematics and his views on language and logic are in direct opposition with the Formalist definition of mathematics as the theory of abstract structures or formal systems. In his *Foundations* (B1907) he analyzes the processes of logic and formalization as stages of linguistic engineering, further and further away from the pure mathematical thought-construction. In particular, the system as created by Hilbert's formal axiomatic method is described as a mathematical application in the domain of language, "mathematics of the second and third order." He frequently expresses his doubts about the success of the Formalist Programme of creating a language free from contradiction. His fundamental disagreement, however, is with the Formalist identification of mathematics with formal consistent system:

> It does not follow from the consistency of the axioms that the supposed corresponding mathematical system exists. Neither does it follow from the existence of such a system of mathematical reasoning that the linguistic system is *alive*, i.e., that it accompanies a chain of thought, and even less that this chain of thought is a *mathematical* construction. (B1907, p. 138)

Following his analysis in 1923 of the logic of negation, there appears to be some shift in Brouwer's views on logic and formalization. When one of his research students, Arend Heyting, expressed an interest in Intuitionism, Brouwer suggested to him the "Intuitionist Axiomatics of Projective Geometry" as the subject of his doctoral dissertation (1925). He also approved of Heyting's formalization of Intuitionist logic and supported its publication ("The Formal Rules of Intuitionist Logic," Heyting 1930b). The character and function of such formalization in Brouwer's view remained strictly limited: a post factum analysis without any contributory role in the construction of mathematics proper. In the opening lines Heyting expressly conforms to Brouwer's view:

> Intuitionistic mathematics is a mental activity [*Denktätigkeit*] and for it every language, including the formalistic one, is only a tool for communication. It is in principle impossible to set up a system of formulas which would be equivalent to intuitionistic mathematics, for the possibilities of thought cannot be reduced to a finite number of rules set up in advance. (Heyting 1930b, p. 311)

As to the value of formalization, Brouwer remained sceptical: an interesting but "sterile," that is, unproductive, exercise with no benefit to mathematics itself, as Heyting testified: "Answering your question about Brouwer's attitude towards formalization, I would like to add that he always maintained that formalizing mathematics is unproductive, since mathematics is constructing-in-the-mind, of which language, and therefore also a formal system, can give only inadequate representation." (Letter of Heyting to van Stigt; October 29, 1969).

1.5 Brouwer's New Theory of Sets and the Continuum

Brouwer's new set theory as published in 1917 was the result of years of critical study of classical set theory and a continued search for an alternative, constructive tool in the analysis of the continuum. His rejection of the standard set-theoretical treatment of the continuum is argued at length in his *Foundations* (B1907), "On Possible Powers" (B1908A), and "Intuitionism and Formalism" (B1912A). Cantor's definition of a set as "comprehension (German: *Zusammenfassung*) of definite and separate objects of our intuition or thought into a whole" is readily accepted as a legitimate mathematical construction; indeed, the primordial construction of two-ity and the further construction of what Brouwer calls "separable mathematics" are particular cases of such construction. However, for the characterization of the continuum and real number, the set–element relation is deemed inappropriate and wholly inadequate. According to Brouwer, in the genesis of mathematical sets the existence or the completed construction of the elements is primary. Any definition of continuum as "the set or totality of all its points or real numbers" would require the preexistent construction of each point. The only acceptable constructive interpretation of real number so far, however, is its identification with the constructed algorithm or law generating convergent sequences of rationals, and Brouwer argues that the totality of such constructed algorithms is denumerable.

The extension of the set concept by means of language, property, or comprehension is rejected as nonmathematical in particular in "Intuitionism and Formalism" (B1912A). Even the preintuitionists are taken to task for "not seeking an Intuitive origin for the continuum, one which lies outside the domain of language and logic," and for not realizing that such a system of real numbers "generated by an ever-unfinished and ever-denumerable system of laws" remains ever denumerable "and incapable of fulfilling the mathematical functions of the continuum for the simple reason that it *cannot have a measure positively differing from zero*" (BMS49, p. 3). Brouwer's topological investigation into the nature of dimension had confirmed his earlier conviction of the dimensional gulf between sets of points and the continuum. Recent publications by Bergson on the nature of movement and time reinforced that conviction and highlighted the "becoming" and indeterminate aspects of the live continuum.

His own editorial involvement in the re-edition of Schoenflies' standard work on set theory during the years 1912–1913 made him even more aware of the inadequacy of current set theory as a tool in the analysis of the continuum. In his "Review of Schoenflies and Hahn" (B1914) he elaborates on its deficiencies, in par-

NÁRODNÍ KNIHOVNA ČESKÉ REPUBLIKY UKF

ticular its failure as a coherent theory founded on one philosophical "Grundanschauung." The significance of the "Review," however, lies in its launching of the notion of a "well-constructed set," which contains all the new elements of Brouwer's alternative set theory of 1917.

The new set theory, "The Second Act of Intuitionism," introduces two distinct set concepts: the "Set" or "Spread" and the "Species." They are based on what Brouwer calls his "New Insights":

1. The continued conviction that the continuum is given in Intuition, that is, is a primitive concept, and that the appropriate method of investigation is "analysis," that is, decomposition into homogeneous parts; the continuum is the "between" never to be exhausted by division into subintervals. "Elements of the continuum" reflect the nature of the continuum and share its properties. In the "Review" they are introduced as "sequences of nested intervals whose measures converge to zero." Unlike the classical interpretation of real point, Brouwer's real point remains identified with the sequence of intervals itself:

> *We call such an indefinitely proceedable sequence of nested* λ intervals a point P or a real number P.
>
> We must stress that for us the point P *is* the sequence (1) $\lambda_{\nu_1}, \lambda_{\nu_2}, \lambda_{\nu_3}, \ldots$ *itself*; not something like "the limiting point" to which, according to the classical view, the λ intervals converge, or something to be defined as the unique accumulation point of midpoints of these intervals. Every one of these λ intervals (1) is therefore *part of* the point P. (BMS37, p. 3)

2. This identification of the sequence with the process of generation in progress marks a fundamental change from Brouwer's earlier concept of infinite sequence, hitherto justified by its identification with the completed algorithm or law of generation. As an ongoing process of construction in time, the infinite sequence has its own mathematical legitimacy, derived ultimately from the constructive activity of the Subject Mathematician. In the case of the "lawlike" sequence, the Subject determines each element uniquely by means of an algorithm constructed by him, a self-imposed restriction of his freedom. Acceptance of the constructive act of the Subject as the ultimate source of mathematical legitimacy, however, allows the broadening of the concept sequence to include those whose elements are determined by immediate free choice of the Subject: "sequences of mathematical systems proceeding indefinitely in complete freedom or in freedom subject to (possibly changing) restrictions" (BMS32, p. 7).

The admission of such "as-yet-uncompleted" constructions as legitimate mathematical objects extends the power of the range of sequences beyond the denumerable, in particular that of the sequences defined as "elements of the continuum." Brouwer was well aware that as incomplete sequences the "elements of the continuum" would be an inadequate basis for analysis and function theory, especially in the light of his strict interpretation of mathematical existence as "having been constructed." He found a solution by carefully analyzing the infinite sequence as an ongoing process in time. Viewed as such the sequence is split by the present moment of choice into a completed initial segment and a yet-to-be-determined tail. The initial finite segment is a completed construction and as such can claim full mathematical existence; the yet-to-be-constructed tail is governed by guarantees of proceedability and convergence. Brouwer hints at a solution in an added note to his unpublished *Lecture Notes 1916*. Referring to the free-choice sequence f as a possible argument of a real function, he finds: "with such a sequence one can work very well as long as at every stage [of the function assignment] one can work with a *suitable* initial segment of f" (BMS15, p. 1).

The assertion that the function assignment must be wholly dependent on an initial segment of the Indefinitely Proceeding Sequence became the Fundamental Hypothesis of Brouwer's analysis and function theory and, in particular, of his well-known Uniform Continuity Theorem. "The Foundations of Set Theory" (B1918B) describes it in terms of an integral-valued function: "A law which assigns a natural number h to each element g of C must have determined h completely after a certain initial segment α of the sequence g has become known . . ." (p. 13).

Identification of real number with the whole of the Indefinitely Proceeding Sequence of nested intervals ensures the nondenumerability and measurability of the continuum; the Fundamental Hypothesis makes it a "workable" basis of analysis. Moreover, the choice sequence highlights the continuing active role of the Creating Subject and is perhaps his most characteristic product, reflecting as it does "the Life of the Subject" both in its past, completed initial segment and in the free and open future of its yet-to-be-determined tail.

3. "Intuitionist Set Theory" (B1919D2) confirms Brouwer's earlier total rejection of Cantor's Principle of Comprehension: "The Axiom of Comprehension, on the basis of which all things with a certain property are joined into a set (also in the restricted form given to it later by Zermelo), is not acceptable and cannot be used as a foundation of set theory" (p. 23). His introduction of an alternative, constructive interpretation of "property" follows his growing awareness that the use of some form of comprehension cannot be avoided. In his *Lecture Notes 1916* he admits that in his own work he had made use of it: "a point set is in fact a set defined by comprehension . . . further in this text we shall meet a variety of point sets" (p. 23); but he is reluctant to give them the full status of constructive set; they are "pseudosets, better referred to as 'sorts' " (p. 1).

The new *Sort* or *Species* was introduced in 1917 as part of the Second Act of Intuitionism: the insight that a constructive interpretation of property can be given by restricting its domain to existing, that is, completed, mathematical constructs: "the admission as a modality of the self-unfolding of the Primordial Intuition of mathematics . . . at each stage of this construction of mathematics of properties which can be presupposed for mathematical entities [lit. thought-constructions; Dutch: *denkbaarheden*] already conceived, as new mathematical entities [thought-constructions] under the name *Species*" (BMS32, p. 7).

As any legitimate mathematical entity, "property" is a construction; in particular it is a construction considered as a substructure, part of another construction: "the fitting-in of one construction into another" (B1908C, p. 156). Candidates for elementhood are only "previously acquired" mathematical entities or systems. The Species or property does not by itself partition the domain of previously acquired mathematical entities into those that possess and those that do not possess the property. Elementhood is established by the successful fitting in of the property construction, and its negation by the absurdity of its fitting in.

On the other hand, a property is considered to be a definite mathematical entity, irrespective of whether all its elements have been or can be established. As legitimate mathematical objects they can be used in the construction of new species, in particular the construction of species of species, enabling the use of general statements in analysis and function theory. (This "species principle" was one of the points of disagreement between Brouwer and Weyl; see the introduction to Weyl in Part II.)

In order to avoid self-reference, Brouwer further introduced the notion of "higher-order species," based on a hierarchy of domains at various levels of construction. Restricting the definition of "mathematical entities" to the fundamental generating procedures and their elements, he defines the *Species of the first order* as "a property which only a mathematical entity can possess" (B1918B, p. 3). Species of higher order are then defined inductively: "By a *Species of the second order* we understand a property which only a mathematical entity or a species of the first order can possess" (ibid.). The fundamental role of species is acknowledged by Brouwer when he stresses its use "at every stage of the construction of mathematics." It is particularly evident in his definition and use of equivalence and negation.

"Equivalence" plays a role in the generation of the fundamental "mathematical entities." Even the abstraction of "two-ity" is "the common substratum of all two-ities" constructed by the Subject at different points in time. The property of coincidence of points with a given point P generates or rather is a fundamental "mathematical entity" and is a species, the "*point core P*." At first Brouwer refers to it as "a species of order zero" (BMS32, p. 8); in a letter to Heyting he later suggests "Perhaps a better way of treatment is to introduce along with the things themselves the 'species of identical things' and then mainly use the latter, in the same way that topological set theory considers point cores rather than the points themselves" (Brouwer to Heyting; July 17, 1928).

The Brouwer Negation

In the generation of the fundamental "mathematical entities," such as the natural numbers and the Brouwer set or spread and its elements, there is no place nor immediate need for negation. The question of negation only arises at the level of species construction, at the point where the Subject is attempting to establish elementhood of a species S over a given domain of existing mathematical entities. Such attempt may lead to "successful fitting in"; that is, a particular mathematical entity is established as an element of S. The alternatives to "successful fitting in" are: (1) the constructed impossibility or "absurdity" of fitting in; and (2) the simple absence of the construction of elementhood or of its absurdity. Only negation in the first sense, of constructed impossibility, meets Brouwer's strict requirements and can claim to be an act of mathematical construction.

The implications of such strict interpretation of negation are far reaching. It immediately calls into question the use of double negation and the logical principle of the excluded "third" or middle. In Brouwer's own reconstruction of set theory and mathematics the use of the Principle of the Excluded Middle is carefully and expressly avoided. In the foundational "debate," however, it becomes the most contentious issue, the clearest manifestation of the fundamental differences between the two opposing philosophies of mathematics (see further the introductions to Parts III and IV).

As to the Brouwer negation, the question still remains as to what constitutes "absurdity" or "constructed impossibility." His definitions remain somewhat vague, and in their use of terms such as "impossibility," "incompatibility," "difference," and "contradiction" they seem to be circular. "Contradiction" or "the impossibility of fitting in" is first described in B1907: "I just observe that the construction does not go further [Dutch: *gaat niet*, that is, it does not work], that in the main edifice there is no room to be found for the posited structure." (p. 127). The impossibility is due to some "incompatibility," a term Brouwer uses in his later work. But incompatibility—latent and inherent in the structures concerned—is not sufficient by itself; he insists that negation is "a construction of incompatibility" (B1954A, p. 3) or "the construction of the hitting upon the impossibility of the fitting in" (B1908C, p. 3).

Proof of the "absurdity of" or the incompatibility of two complex systems is identified, in particular, in the post-Brouwer Intuitionist tradition, with the reduction to a simple contradiction such as $1 = 0$ or the logical $p \,\&\, \neg p$. But these contradictions, as indeed all descriptions of "absurdity," make use of some notion of negation or difference. Their absurdity can ultimately only be justified by some intuitive, primitive relation of distinctness, an element of the Primordial Intuition, the fundamental recognition of the Subject of *distinct* moments in time.

Brouwer also uses other, weaker forms of negation, in particular, where he moves outside the domain of mathematics proper into the realm of "mathematical language" and mathematical "assertions," where, for example, he speaks of "unproven hypotheses," "the case that α has neither been proved to be true nor to be absurd." Negation in this case expresses the simple absence of proof, which in the world of mathematics as construction in time may well be reversed: Unsolved problems may one day become proven truth or absurdity. Moreover, "a mathematical entity is not necessarily predeterminate, and may, in its state of free growth, at some time acquire a property it did not possess before" (BMS59, p. 1), leading to further distinctions, in particular, between "cannot now" and "cannot now and ever," the latter term frequently used by Brouwer in his later work as an alternative description of "absurdity."

The Brouwer Set or Spread

The concept of Brouwer Set or Spread[4] is the result of Brouwer's search for a constructive procedure for generating elements of the continuum. "The Foundations of Set Theory" (B1918B) and the here-published "Intuitionist Set Theory" (B1919D) introduce the notion first in its generalized form. Later papers, in particular, *Real Functions* (BMS37), start with the definition of a special case, the "Point-Set," which "genetically precedes and is more easily understood than the definition of the general case," which Brouwer admits in (B1925A) "regrettably, has to be rather long-winded."

Brouwer likened the Spread to a tree, a living, growing organism that produces its elements in the form of "nodes" selected by the creator of the tree, the Subject, branching in various directions and "proceeding indefinitely." In the tree each spread element is represented by an upward path, a sequence of nodes selected by the creator of the tree, the Subject. Each node is the last element of an initial segment of such a path and leaves the Subject sufficient options to select the next node of his path. Options are labeled or "associated with" elements of a fundamental sequence such as the natural numbers, so that a path of nodes is represented by a choice sequence of these labels. The Spread then is the whole of the internal organization: the rules that govern the domain of choice: the Spread law proper, as well as its labeling: the complementary law.

In the case of *Point Spreads* the domain of choice is that of λ intervals, and the Spread law provides guarantees of indefinite proceedability and convergent nesting:

> By a *point set* we understand a *law* by means of which with the numbers n_1, n_2, n_3, ... freely and in sequence selected from the sequence of natural numbers 1, 2, 3, ... either
>
> 1. A λ square is associated in such a way that of two subsequent λ squares the second lies strictly within the first; or
> 2. *Nothing* is associated with the chosen first number n_1, or, if λ squares have been associated with $n_1, n_2, \ldots, n_{h-1}$ ($h \geq 2$), nothing is associated with the choice of the hth number n_h, and at the same time all intervals generated so far are destroyed and the process is terminated. In the last case, however, there must be at least one n_h with which, when chosen, the law does associate a λ square. (BMS37, p. 5)

The definition of Point Spread allows restrictions on the options available to the Subject. If at every stage the number of choices available is finite, the spread is called a *finite set* or *spread*.[5] Freedom of choice can also be "narrowed to the point of complete determination," resulting in a *fundamental sequence*, a sequence each term of which is uniquely predetermined. The Spread generates its elements, unlike the Species, which is a property "to be posited for entities previously acquired." "Having the same genitor," however, is a property and insofar the Spread can be considered to be a species: "A point set is always also a species, but a point species is not necessarily a point spread."

The *Brouwer-Set* or *Spread* as defined in "The Foundations of Set Theory" and "Intuitionist Set Theory" is a generalization of point set:

> A Set is a law on the basis of which, whenever an arbitrary digit-complex is chosen from the sequence 1, 2, 3, 4, 5 ..., each of these choices produces either a definite symbol or nothing, or causes the arresting of the process together with the definite destruction of its result; moreover, after every nonarrested sequence of $n-1$ choices (for every $n > 1$) at least one digit-complex can be indicated that, chosen as the nth digit complex, does not lead to the arresting of the process. Every sequence of symbols so generated by an indefinite sequence of choices (which therefore in general is unfinished in character) is called an element of the set.
>
> The common mode of generating elements of the set M is also referred to in short as the set M. (B1919D2, p. 24)

As in the case of the Point Spread, the spread law governs the choices from a given "fundamental sequence." Brouwer uses the term "digit-complexes" (Dutch: *cijfercomplexen*), that is, numerals as distinct from numbers. In BMS15 he clearly refers to numerals in their decimal composition: "We start from the sequence r of the digit-groups 1, 2, ..., 10, 11, ..., which can be continued without restriction according to a well-known law that enables us to derive from each digit-group the next one." The Complementary Law of the general spread associates with each chosen numeral "a definite symbol," or it produces "nothing." In the latter case, if at the nth selection as a result of a particular choice "nothing" is produced, the whole of this n-term sequence is eliminated. However, the spread law requires that there be at least one alternative choice at the nth selection that does not produce "nothing" and allows the sequence to proceed indefinitely.

1.6 The Selected Publications

In this section we present an anthology of Brouwer's Intuitionist Programme during the period 1920–1928, papers that set out his Intuitionist principles as well as his attempts at reconstructing set theory and mathematical analysis. The following is a synopsis of the papers presented.

"Intuitionist Set Theory" (B1919D2) This paper marks the beginning of Brouwer's Intuitionist campaign of reform of general mathematical practice. It is the first time he uses the term "Intuitionist" to describe his own conception of mathematics and its reconstruction. The "Set" and "Species" are announced as the new, improved alternative to the "classical theory of sets" as the basis of analysis; only definitions are given, and the reader is referred to "The Foundations of Set Theory" for further detail. More attention, however, is given to the reasons for the proposed change and its impact on traditional mathematics. Brouwer describes it as "a consequence of his earlier theses rejecting the Axiom of Choice and the Principle of the Excluded Middle," identified with Hilbert's axiom of the solvability of all mathematical problems. He further lists the radical changes in classical set theory and mathematics, resulting from the new constructive interpretation of set: the abandonment of some notions and theorems and the redefinition and considerable change of proofs of others.

On only a few occasions during the following 10 years did Brouwer speak out directly on the philosophical controversy of the Principle of the Excluded Middle. One such occasion was the Antwerp Congress of Natural and Medical Science in August 1923, when he delivered his paper on "The Splitting of the Fundamental Notions of Mathematics" (1923C1), starting the debate on the "Brouwer Logic" (see Part IV). Most of his publications are part of his Intuitionist programme of reconstructing mathematics: redefining fundamental notions of algebra and analysis and providing new, constructive proofs of classical results. Brouwer was well aware of what he himself describes as the "destructive and debilitating" effects of his Intuitionist restrictions on classical mathematics. He was confident, however, that his new set theory and what Weyl calls "his natural treatment of the continuum" would make up for these losses and lead to new developments and a new simplified analysis.

Promising new developments were made in particular in the following two selected papers:

"Does Every Real Number Have a Decimal Expansion?" (B1921A) In this paper Brouwer explores the effect of his new characterization of "point of the continuum" on the notion of "real number." Real number now is a special case of "linear point"; in the definition of linear point, "indefinitely proceeding sequence" is replaced by "fundamental sequence," that is, a sequence whose terms are determined by a law. He finds that this interpretation "comprises considerably more" than the classical definition of real number as Dedekind cut, in particular, in their respective representations by decimal expansion.

"Proof That Every Full Function is Uniformly Continuous" (1924D1) In his analysis of the Brouwer continuum Weyl observed: "The concept of continuous function in a bounded interval cannot be defined without simultaneously including uniform continuity and boundedness in the definition. . . . What nowadays is called a discontinuous function actually consists of several functions in separate continua" (Weyl 1921, p. 114). Brouwer marked his full approval in the margin of Weyl's manuscript: "Very true! Underline, because this is the main and most important point!" (see further Part 2 of this book). Continuity and uniform continuity of all full functions, Brouwer claimed, "is the immediate consequence of my Intuitionist view and has been frequently mentioned in my lectures ever since 1918" (B1927B, p. 62; see also Parsons 1967). The theorem that "every full function is uniformly continuous" is first stated, and a sketch of proof given in the first pages of "The Foundations of the Theory of Functions Independent of the Principle of the Excluded Middle" (B1923A). During the following years he made a number of attempts at a full proof, first in the here-published B1924D1, and further in B1927B, BMS32, and BMS37. In all these proofs uniform continuity follows directly from a proof of what we referred to above as "an insight" and "Brouwer's Fundamental Hypothesis," the assertion that the function assignment is wholly dependent on an initial segment of the Indefinitely Proceeding Sequence. A proof is given for a function defined on a "finite set" and is now called "the Fundamental Theorem of Finite Sets" (in his postwar papers also the "Bunch Theorem" or "Fan Theorem"). In BMS66 Brouwer describes it as "a most wonderful theorem whose importance would justify calling it 'the Fundamental Theorem of Intuitionism' . . ." (p. 6). The relative importance of the Uniform Continuity Theorem is clear from BMS32, where it is relegated to the third place in "the applications of the Fundamental Theorem."

The proof of the Fundamental Theorem in turn is based on what Brouwer later calls the "Bar Theorem," which asserts that if a function f assigns each element of a spread M to a natural number, then M is split into subspreads M_α so that all elements of one M_α have the same initial segment and are all assigned by f to the same natural number β_α. In the proof of the Bar Theorem Brouwer introduces a form of regressive induction, which he justifies on the basis of "profound intuitionist reflection." That this proof of the Fundamental Theorem did not wholly satisfy Brouwer is clear from a remark he made in a lecture in 1952: ". . . a wonderful theorem . . . but one whose absolutely rigorous proof till now has not been sufficiently simplified" (BMS66, p. 6).

The remaining three selected papers are major statements made at the end of Brouwer's creative period, each concentrating on a particular aspect of his Intuitionist Programme: his philosophical views on the nature of mathematics and its relation to logic and science, his reconstruction of mathematical analysis, and his campaign to have Intuitionism universally accepted as the right interpretation and practice of mathematics.

"Intuitionist Reflections on Formalism" (B1928A2) This paper is Brouwer's summary of the state of play in the Intuitionist–Formalist debate by the end of 1927, "the battle about the Foundations of Mathematics," which was turning into a bitter personal feud between the two main protagonists. It lists the Intuitionist achievements, Brouwer's "insights," some wholly or partly adopted, "copied," by the opposition, others still to be ceded. There are, however, signs of battle fatigue, resentment at the lack of universal recognition, disappointment at the falling support for his cause, and, possibly, the progress of his Intuitionist Programme. Particularly resented is the annexation of the notion of "meta-mathematics" "without proper mention of authorship."

Brouwer's terms for an end to the foundations "battle," however, are uncompromising; they include the total surrender of the fundamental hypothesis of the Formalist Programme, the justification of mathematics by a proof of its consistency. As a peace offering he concedes a modified Principle of the Excluded Middle and its usefulness for those "endeavoring to work out a proof of the consistency of Formalist meta-mathematics."

"Mathematics, Science and Language" (B1929) The first of two lectures given by Brouwer in Vienna on March 10 and 14, 1928. It focuses on the philosophical theses that underlie his Intuitionism and clearly reflects his mood and concerns at the time. There is a return to the pessimism and isolationism of his student years and to his moralistic stance on science and language. New is the strong Schopenhauerian emphasis on the role of the will in the interpretation of both language and science. "Imposition of will" is the origin and nature of language. Science is the causal interpretation of the world; both the causal, mathematical attention and man's acting upon this knowledge, the mathematical or "cunning act," are acts of the will. The major part of the paper is devoted to the analysis of language. Its origin and therefore its nature are determined by its function of "will transmission," the imposition of one's will on others. Even as an instrument of will transmission language lacks exactness and certainty; it is wholly inadequate as a carrier of thought and in particular as a representation of the pure mathematical thought-construction. Both language and logic, the analysis of mathematical language, are fundamentally flawed. ("Will, Knowledge and Speech" (B1933) restates substantial parts of "Mathematics, Science and Language," but also includes new elements such as the notion of the "Idealized Mathematician.")

"The Structure of the Continuum" (B1930A) In his second Vienna Lecture Brouwer starts with a brief critical survey of the traditional interpretations of the continuum. He then introduces his notions "species," "set," "finite set," and continuum and proceeds with an investigation of its properties, in how far the topological properties as defined for the classical continuum apply to his Intuitionist continuum. All the properties investigated are found to be inapplicable. "Discreteness" in any form is rejected out of hand. Most properties, however, can be "re-established" after some "logical" transformation of definition, some though in substantially weaker form.

Notes

1. Brouwer was professor of mathematics at the University of Amsterdam from 1912 until his retirement in 1951. For further details of his life and career, see van Stigt 1990. A Brouwer biography by D. van Dalen is in preparation. Franchella 1994 contains an extended bibliography on Brouwer and intuitionism.
2. On the topic of this section, see also Largeault 1993b.
3. On the French Intuitionists, see references given in the introduction to Part 2 and Largeault 1993a and 1993b.
4. In his German and Dutch publications Brouwer continues to use the general word "Set" (German: *Menge*, Dutch: *verzameling*); the term "Spread" is first used in the Dutch version of (BMS32). In this introduction we refer to it as "Brouwer-Set" or "Spread" except in quotations, where we give the literal translation "Set."
5. Note that "finite" here does not refer to the number of elements of the spread. To avoid confusion modern Intuitionists have adopted the term *finitary spread*, introduced by Kleene. Finite spreads were used by Brouwer, in particular, in his proof of uniform continuity.

Bibliography

Works by L. E. J. Brouwer

Publications

(All listed books and publications, except (B1905), (B1923C1), and (B1933), are republished in A. Heyting, *Brouwer Collected Works I*, North-Holland, Amsterdam, 1975)

B1905, *Leven, Kunst en Mystiek*, Waltman, Delft 1905. An introduction and English translation by W. P. van Stigt in *Notre Dame Journal of Formal Logic* **37**, 1996, p. xx.

B1907, *Over de Grondslagen der Wiskunde*, Maas & van Suchtelen, Amsterdam, 1907. English translation in CWI.

B1908A, Die möglichen Mächtigkeiten, *Atti IV Congr. Intern. Mat. Roma III*, pp. 569–71.

B1908C, De onbetrouwbaarheid der logische principes, *Tijdschrift voor Wijsbegeerte* **2**, pp. 152–58. English translation in CWI.

B1912A, *Intuitionisme en Formalisme*, Clausen, Amsterdam. English translation: Intuitionism and Formalism, *Bull. Am. Math. Soc.* **20**, pp. 81–96.

B1914, [Review of] A. Schoenflies und H. Hahn, Die Entwickelung der Mengenlehre und ihrer Anwendungen, Leipzig und Berlin 1913, *JDMV* **23**, pp. 78–83.

B1918B, Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil: Allgemeine Mengenlehre, *KNAW Verhandelingen*, 1e Sectie, deel XII, no. 5, pp. 1–43.

B1919A, Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Zweiter Teil: Theorie der Punktmengen, *KNAW Verhandelingen*, 1e Sectie, deel XII, no. 7, pp. 1–33.

B1919D1, Intuitionistische Mengenlehre, *JDMV* **28**, pp. 203–8.

B1919D2, Intuitionistische Verzamelingsleer, *KNAW Verslagen* **29**, 1921, pp. 797–802. English translation in this volume, Chapter 1.

B1921A, Besitzt jede reelle Zahl eine Dezimalbruchentwickelung? *KNAW Verslagen* **29**, pp. 803–812; also in *KNAW Proceedings* **23**, p. 955, and *Mathematische Annalen* **83**, pp. 201–210. English translation in this volume, Chapter 2.

B1923A, Begründung der Funktionenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil, Stetigkeit, Messbarkeit, Derivierbarkeit, *KNAW Verhandelingen*, 1e sectie, deel XIII, no. 2, pp. 1–24.

B1923C1, Intuitionistische splitsing van mathematische grondbegrippen, *KNAW Verslagen* **32**, pp. 877–80. English translation in this volume, Chapter 18.

B1923C2, Intuitionistische Zerlegung mathematischer Grundbegriffe, *JDMV* **33**, 1925, pp. 251–56. English translation in this volume, Chapter 19.

B1924D1, Bewijs dat iedere volle functie gelijkmatig continu is, *KNAW Verslagen* **33**, pp. 189–93. English translation in this volume, Chapter 3.

B1924D2, Beweis dass jede volle Funktion gleichmässig stetig ist, *KNAW Proceedings* **27**, pp. 189–93.

B1924F1, Bewijs van de onafhankelijkheid van de onttrekkingsrelatie van de versmeltingsrelatie, *KNAW Verslagen* **33**, pp. 479–80.

B1924F2, Zur intuitionistischen Zerlegung mathematischer Grundbegriffe, *JDMV* **36**, 1927, pp. 127–29. English translation in this volume, Chapter 20.

B1925A, Zur Begründung der intuitionistischen Mathematik I, *Mathematische Annalen* **93**, pp. 244–257.

B1927B, Über Definitionsbereiche von Funktionen, *Mathematische Annalen* **97**, pp. 60–75. English translation of §1–3 in van Heijenoort 1967.

B1928A2, Intuitionistische Betrachtungen über den Formalismus, *KNAW Proceedings* **31**, pp. 371–74; also in *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, 1928, pp. 48–52. English translation in this volume, Chapter 4.

B1929, Mathematik, Wissenschaft und Sprache, *Monatshefte für Mathematik und Physik* **36**, pp. 153–64. English translation in this volume, Chapter 5.

B1930A, *Die Struktur des Kontinuums*, Wien 1930 (Sonderabdruck). English translation in this volume, Chapter 6.

B1933, Willen, Weten, Spreken, *Euclides* **9**, pp. 177–93. English translation in van Stigt 1990 (Appendix 5).

B1948C, Consciousness, Philosophy and Mathematics, *Proceedings of the 10th International Congress of Philosophy, Amsterdam 1948*, III, pp. 1235–49.

B1954A, Points and Spaces, *Canadian Journal for Mathematics* **6**, pp. 1–17

Unpublished

BMS3B, The Rejected Parts of Brouwer's Dissertation; published in Dutch original and English translation by W. P. van Stigt in *Historia Mathematica* **6**, 1979, pp. 385–404; also in van Stigt 1990; pp. 404–415.

BMS15, Notes for a course on set theory given in Amsterdam between 1914 and 1916.

BMS32, *Berliner Gastvorlesungen*. A course on Intuitionism given by Brouwer in Berlin in 1927 and intended for publication. Published by D. van Dalen in *L. E. J. Brouwer Intuitionismus*, Wissenschaftsverlag, Mannheim, 1991. English translation of the last part of Chapter 1 p. 7, in van Stigt 1990, pp. 481–85.

BMS37, *Reelle Funktionen*. A 126-page manuscript of a partly completed book. English translation of the table of contents and extracts in van Stigt 1990, pp. 469–480.

BMS49, Disengagement of mathematics from logic. A 5-page manuscript of a lecture probably given in 1947. Published as an appendix in van Stigt 1990.

BMS51, *The Cambridge Lectures*. A course on Intuitionism given by Brouwer at Cambridge University regularly between 1946 and 1951. Now published in van Dalen 1981.

BMS59, The Influence of Intuitionist Mathematics on Logic (alt. Changes in the Relation between Classical Logic and Mathematics); a lecture given in London on November 2, 1951, published as an appendix in van Dalen 1981. A slightly different version in a different manuscript published as an appendix in van Stigt 1990, pp. 453–58.

BMS66, Second Lecture. A 9–page manuscript dated May 1952 [University College London], published as an appendix in van Stigt 1990, pp. 459–68.

Secondary Literature

Bridges, D., Richman, F., 1987, *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes Series, Cambridge University Press, Cambridge.

Detlefsen, M., 1990, Brouwerian Intuitionism, *Mind* **99**, pp. 501–34.

Dummett, M., 1973, The Philosophical Basis of Intuitionistic Logic, in *Truth and Other Enigmas*, Harvard University Press, Cambridge (Mass.).

Dummett, M., 1977, *Elements of Intuitionism*, Oxford University Press, Oxford.

Franchella, M., 1994, *L. E. J. Brouwer Pensatore Eterodosso. L'intuizionismo tra matematica e filosofia*, Guerini, Milano.

Heyting, A., 1930b, Die formalen Regeln der intuitionistischen Logik, *Sitzungsberichte der Preussischen Akademie der Wissenschaften, phys. math. Kl.*, pp. 42–56. English translation in this volume, Chapter 24.

Heyting, A., 1975, *Brouwer Collected Works I*, North-Holland, Amsterdam.

Hilbert, D., 1922, Neubegründung der Mathematik, *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* **1**, pp. 157–77. English translation in this volume, Chapter 12.

Largeault, J., 1993a, L'Intuitionisme des mathématiciens avant Brouwer, *Archives de Philosophie* **56**, pp. 53–68.

Largeault, J., 1993b, *Intuition et Intuitionisme*, Vrin, Paris.

McCarthy, C. D., 1983, Introduction, *Journal of Philosophical Logic* (special issue on intuitionism) **12**, pp. 105–49.

Parsons, C., 1967, Introductory note to B1927B in van Heijenoort 1967, pp. 446–57.

Posy, C. J., 1974, Brouwer's Constructivism, *Synthese* **27**, pp. 125–59.

Troelstra, A. S., 1977, *Choice Sequences*, Oxford University Press, Oxford.

Troelstra, A. S., and van Dalen, D., 1988, *Constructivism in Mathematics. An Introduction*, Vol. I, North-Holland, Amsterdam.

van Dalen, D., 1981, *Brouwer's Cambridge Lectures*, Cambridge University Press, Cambridge.

van Heijenoort, J., 1967, *From Frege to Gödel*, Harvard University Press, Cambridge (Mass.).

van Stigt, W. P., 1990, *Brouwer's Intuitionism*, North-Holland, Amsterdam.

Weyl, H., 1921, Über die neue Grundlagenkrise in der Mathematik, *Mathematische Zeitschrift* **10**, 1921, pp. 39–79. English translation in this volume, Chapter 7.

Weyl, H., 1949, *Philosophy of Mathematics and Natural Science*, Princeton University Press, Princeton.

1

Intuitionist Set Theory[1]

LUITZEN EGBERTUS JAN BROUWER*

The following is intended as a report and introduction to the two parts of my paper "Foundations of Set Theory Independent of the Logical Principle of the Excluded Middle," which I presented to the Academy in November 1917 and October 1918.

Since 1907 I have in various publications[2] defended the following theses:

1. The Axiom of Comprehension, on the basis of which all things with a certain property are joined into a set (also in the restricted form given to it later by Zermelo[3]), is not acceptable and cannot be used as a foundation of set theory. A reliable foundation of mathematics is only to be found in a *constructive* definition of a set.
2. The axiom of the *solvability of all problems* as formulated by Hilbert in 1900[4] is equivalent to the logical Principle of the Excluded Middle; therefore, since there are no sufficient grounds for this axiom and since logic is based on mathematics—and not vice versa—the use of the Principle of the Excluded Middle is *not permissible* as part of a mathematical proof. The Principle of the Excluded Middle has only scholastic and heuristic value, so that theorems that in their proof cannot avoid the use of this principle lack all mathematical content.

In the publications quoted in note 2 I have so far mentioned only some of the consequences of the Intuitionist conception of mathematics as condensed in these two theses. In my philosophy-free mathematical papers published during this period I still made regular use of the old methods, but I did take as much care as possible to derive only such results as could be expected also to find a place in the new system after the systematic construction of an intuitionist set theory, perhaps with some modification, but in essence the same [*met behoud van hun waarde* lit. "retaining their value"].

Only with the publication quoted at the beginning of this paper did I start this systematic construction of an Intuitionist set theory. In the following I shall give a

*"Intuitionistische verzamelingsleer," *KNAW Verslagen* **29**, 1921, pp. 797–802 (paper presented to the Royal Academy of Sciences on December 18, 1920). Translated from the Dutch by Walter P. van Stigt. The German version, "Intuitionistische Mengenlehre," was published in *JDMV* **28**, 1921, pp. 203–8 and in the *KNAW Proceedings* **23**, 1922, pp. 949–54. Published by permission of Koninklijke Nederlandse Akademie van Wetenschappen.