9

Arrangements with Forbidden Positions

Author: John G. Michaels, Department of Mathematics, State University of New York, College at Brockport.

Prerequisites: The prerequisites for this chapter are basic counting techniques and the inclusion-exclusion principle. See, for example, Sections 4.1, 4.3, 5.4, and 5.5 of *Discrete Mathematics and Its Applications*, Second Edition, by Kenneth H. Rosen.

Introduction

In this chapter we will discuss the problem of counting arrangements of objects where there are restrictions in some of the positions in which they can be placed. For example, we may need to match applicants to jobs, where some of the applicants cannot hold certain jobs; or we may wish to pair up players to form two-person teams, but some of the players cannot be paired up with some of the other players.

Problems such as these, where we want to find the number of arrangements with "forbidden" positions, have a long history. They can be traced back to the

early eighteenth century when the French mathematician Pierre de Montmort* studied the *problème des rencontres* (the matching problem). In this problem an urn contains n balls, numbered $1, 2, \ldots, n$, which are drawn out one at a time. de Montmort wanted to find the probability that there are no matches in this process; that is, that ball i is not the ith ball drawn. This problem is really one of counting *derangements* — permutations of a set where no element is left in its own position. (The formula for D_n, the number of derangements of n objects, can be found in Section 5.5 of *Discrete Mathematics and Its Applications*, Second Edition, by Rosen.)

Another problem of arrangements, called the *problème des ménages* (the problem of the households), asks for the number of ways to arrange n couples around a table so that the sexes alternate and no husband and wife are seated together. This problem was solved in 1891 by E. Lucas**. We will solve problems such as these by defining a polynomial called a *rook polynomial* and showing how to use this to count arrangements.

Arrangements with Forbidden Positions

Example 1 Suppose an office manager places an ad for some part-time help: a keyboard operator (K), a file clerk (F), a stenographer (S), a delivery person (D), and someone to work in the warehouse (W). Five people answer the newspaper ad and are interviewed for the jobs. Figure 1 shows which jobs each of the five applicants (1, 2, 3, 4, and 5) is qualified to handle.

Each square in this figure is either shaded or unshaded. A shaded square represents a "forbidden position"; that is, the person cannot perform that job. An unshaded square represents an "allowable position". For example, Applicant 1 cannot hold the job of stenographer, but can hold any of the other jobs. In how many ways can the office manager place the five applicants in jobs for

* Pierre-Rémond de Montmort (1678–1719) was born into the French nobility, received his father's large fortune at age 22, studied philosophy and mathematics with Father Nicholas de Malebranche, and held the position of canon at Notre-Dame. He married and began his study of probability, possibly because of his contacts with the Bernoulli family. In 1708 he published his *Essai d'Analyse sur les Jeux de Hasard*. One of the games studied in this work was the matching game treize. The significance of his contributions in mathematics lies in his use of algebraic methods to study games of chance.

** Edouard Lucas (1842–1891) was a French number theorist. In 1876 he proved that the Mersenne number $M_{67} = 2^{67} - 1$ was not prime. In that year he also proved that $M_{127} = 2^{127} - 1$ was prime; for 75 years this was the largest number proven to be prime. Lucas attached the name "Fibonacci" to the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ and studied the closely-related Lucas sequence, $1, 3, 4, 7, 11, 18, \ldots$.

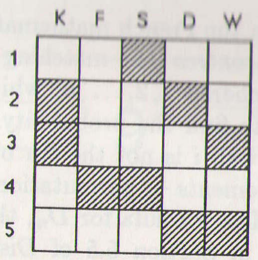


Figure 1. Job applicants and possible jobs.

which they are qualified?

Solution: A matching of the applicants with the jobs is called an **arrangement with forbidden positions**. Two possible job assignments are:

1–keyboard, 2–stenographer, 3–delivery, 4–warehouse, 5–file clerk,

1–warehouse, 2–stenographer, 3–file clerk, 4–delivery, 5–keyboard.

We can think of Figure 1 as a 5×5 chessboard with nine squares removed. A **rook** is a chess piece that moves horizontally or vertically and can take (or capture) a piece if that piece rests on a square in the same row or column as the rook (assuming that there are no intervening pieces). For example, a rook on square $(2, F)$ can capture an opponent's piece on any of the squares in row 2 or column F, but cannot capture a piece on square $(1, K)$. A matching of applicants to jobs corresponds to a placing of five rooks on the unshaded squares so that no rook can capture any other rook. These are called "nontaking" rooks. Thus, the number of acceptable job assignments is equal to the number of ways of placing five nontaking rooks on this chessboard so that none of the rooks is in a forbidden position.

The key to determining this number of arrangements is the inclusion-exclusion principle. To set up the problem so that we can use the inclusion-exclusion principle, we let

A_i = the set of all arrangements of 5 nontaking rooks with the rook in row i in a forbidden square,

for $i = 1, 2, 3, 4, 5$.

If we let U be the set of all possible job assignments, then the solution to our problem is $|U - (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)|$. Applying the inclusion-exclusion principle to $|A_1 \cup \cdots \cup A_5|$ yields

$$|A_1 \cup \cdots \cup A_5| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \qquad (1)$$
$$- \sum |A_i \cap A_j \cap A_k \cap A_l| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|,$$

where we sum over the appropriate sets of subscripts.

The problem of placing nontaking rooks on allowable squares has now been reduced to a series of problems of counting the number of ways of placing nontaking rooks on forbidden squares. We need to determine the size of each of the 31 sets on the right side of (1). To simplify the solution, we introduce some notation. Let

r_i = the number of ways of placing i nontaking rooks on forbidden squares of a board.

If we need to emphasize the fact that we are working with a particular board B, we will write $r_i(B)$ instead or r_i.

Each of the five expressions on the right side can be written in terms of r_i, for $i = 1, 2, 3, 4, 5$. The number $|A_i|$ counts the number of ways of placing 5 nontaking rooks, with the rook in row i on a forbidden square. For example, $|A_3| = 2 \cdot 4!$ since there are two ways to place a rook on a forbidden square of row 3 and 4! ways to place four other nontaking rooks. Therefore $\sum |A_i| = r_1 \cdot 4!$. Similar reasoning applies to $\sum |A_i \cap A_j| = r_2 \cdot 3!$, $\sum |A_i \cap A_j \cap A_k| = r_3 \cdot 2!$, $\sum |A_i \cap A_j \cap A_k \cap A_l| = r_4 \cdot 1!$, and $|A_1 \cap \cdots \cap A_5| = r_5 \cdot 0!$. Making these substitutions allows us to rewrite (1) as

$$|A_1 \cup \cdots \cup A_5| = r_1 \cdot 4! - r_2 \cdot 3! + r_3 \cdot 2! - r_4 \cdot 1! + r_5 \cdot 0!.$$

Hence, the solution to our problem can be written as

$$5! - (r_1 \cdot 4! - r_2 \cdot 3! + r_3 \cdot 2! - r_4 \cdot 1! + r_5 \cdot 0!). \qquad (2)$$

It is easy to see that $r_1 = 9$, since there are nine ways to place a rook on a forbidden square. It is also not difficult to see that $r_2 = 28$ by counting the 28 ways to place two nontaking rooks on forbidden squares. However the problems grow increasingly more difficult when we try to find the coefficients r_3, r_4, and r_5. This leads us to look for techniques to help simplify the counting process.

Our technique for simplification is one that is often used in problems of counting — relate the given problem to a series of smaller problems, each of which is easier to solve. We begin by taking the given chessboard and changing the order of the rows and the order of the columns to obtain the board B in Figure 2.

With this rearrangement, the original board B of forbidden squares can be broken into two disjoint subboards, B_1 and B_2, shown in Figure 2. (We say that two boards are *disjoint* if they have no rows or columns in common.) The problem of computing the right side of (1) by placing nontaking rooks on forbidden squares of B is reduced to two smaller problems: placing nontaking rooks on the forbidden squares of B_1 and placing nontaking rooks on the forbidden squares of B_2.

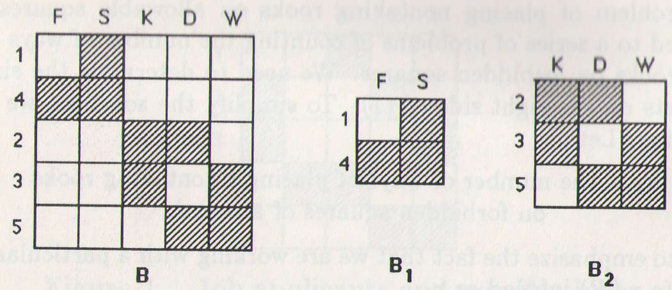


Figure 2. **Rearrangement of board of Figure 1, and two disjoint subboards.**

For example, to find $r_1(B)$ either we place 0 rooks on B_1 and 1 on B_2, or else we place 1 rook on B_1 and 0 on B_2. That is,

$$\begin{aligned} r_1(B) &= r_0(B_1)\cdot r_1(B_2) + r_1(B_1)\cdot r_0(B_2) \\ &= 1\cdot 6 + 3\cdot 1 = 9. \end{aligned}$$

To find $r_2(B)$, we observe that placing two nontaking rooks on the forbidden squares of B gives us three cases to consider: place 0 on B_1 and 2 on B_2, place 1 on B_1 and 1 on B_2, or place 2 on B_1 and 0 on B_2. That is,

$$\begin{aligned} r_2(B) &= r_0(B_1)\cdot r_2(B_2) + r_1(B_1)\cdot r_1(B_2) + r_2(B_1)\cdot r_0(B_2) \\ &= 1\cdot 9 + 3\cdot 6 + 1\cdot 1 = 28. \end{aligned}$$

Similar reasoning can be used to show that:

$$\begin{aligned} r_3(B) &= \sum_{i=0}^{3} r_i(B_1)\cdot r_{3-i}(B_2) \\ &= 1\cdot 2 + 3\cdot 9 + 1\cdot 6 + 0\cdot 1 = 35, \end{aligned}$$

$$\begin{aligned} r_4(B) &= \sum_{i=0}^{4} r_i(B_1)\cdot r_{4-i}(B_2) \\ &= 1\cdot 0 + 3\cdot 2 + 1\cdot 9 + 0\cdot 6 + 0\cdot 1 = 15, \end{aligned}$$

$$\begin{aligned} r_5(B) &= \sum_{i=0}^{5} r_i(B_1)\cdot r_{5-i}(B_2) \\ &= 1\cdot 0 + 3\cdot 0 + 1\cdot 2 + 0\cdot 9 + 0\cdot 6 + 0\cdot 1 = 2. \end{aligned}$$

Substituting the values of $r_i(B)$ into (2) yields the solution of the original problem:

$$\begin{aligned} 5! &- (r_1\cdot 4! - r_2\cdot 3! + r_3\cdot 2! - r_4\cdot 1! + r_5\cdot 0!) \\ &= 5! - (9\cdot 4! - 28\cdot 3! + 35\cdot 2! - 15\cdot 1! + 2\cdot 0!) \\ &= 15. \end{aligned}$$

Hence the original job assignment problem can be done in 15 ways. □

Rook Polynomials

The numbers $r_0(B) = 1, r_1(B) = 9, r_2(B) = 28, r_3(B) = 35, r_4(B) = 15$, and $r_5(B) = 2$ in Example 1 can be stored as coefficients of a polynomial:

$$1 + 9x + 28x^2 + 35x^3 + 15x^4 + 2x^5.$$

More generally, we have the following.

Definition 1 If B is any board, the *rook polynomial* for B, written $R(x,B)$, is the polynomial of the form

$$R(x,B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_n(B)x^n$$

where $r_i(B) =$ the number of ways of placing i nontaking rooks on forbidden squares of the board. □

The rook polynomial is not only a convenient bookkeeping device for storing the coefficients $r_i(B)$, but the algebraic properties of polynomials can also be used to help solve problems of counting arrangements.

In the previous example the coefficients were found by breaking board B into 2 disjoint subboards B_1 and B_2. Each of these subboards has its own rook polynomial:

$$R(x,B_1) = 1 + 3x + x^2, \qquad R(x,B_2) = 1 + 6x + 9x^2 + 2x^3.$$

(The reader is asked to verify this in Exercise 1.) If we multiply these polynomials, we obtain

$$\begin{aligned} R(x,B_1)\cdot R(x,B_2) &= (1+3x+x^2)(1+6x+9x^2+2x^3) \\ &= (1\cdot 1) + (1\cdot 6 + 3\cdot 1)x + (1\cdot 9 + 3\cdot 6 + 1\cdot 1)x^2 \\ &\quad + (1\cdot 2 + 3\cdot 9 + 1\cdot 6)x^3 + (3\cdot 2 + 1\cdot 9)x^4 + (1\cdot 2)x^5 \\ &= 1 + 9x + 28x^2 + 35x^3 + 15x^4 + 2x^5 \\ &= R(x,B). \end{aligned}$$

Thus, the rook polynomial for B is the product of the rook polynomials for the subboards B_1 and B_2. The fact that a similar result is always true is stated in Theorem 1.

Theorem 1 If a board B is broken into 2 disjoint subboards B_1 and B_2, then $R(x,B) = R(x,B_1) \cdot R(x,B_2)$.

Proof: We will prove that the 2 polynomials, $R(x,B)$ and $R(x,B_1) \cdot R(x,B_2)$, are equal. To do this, we will show that, for each i, the x^i term of $R(x,B)$ is equal to the x^i term of the product $R(x,B_1) \cdot R(x,B_2)$. To see that this is always true, consider the product of the two rook polynomials

$$R(x,B_1) \cdot R(x,B_2) = (r_0(B_1) + r_1(B_1)x + \cdots + r_m(B_1)x^m) \cdot (r_0(B_2) + r_1(B_2)x + \cdots + r_n(B_2)x^n).$$

Multiplying these two polynomials and combining like terms yields the x^i term

$$(r_0(B_1) \cdot r_i(B_2) + r_1(B_1) \cdot r_{i-1}(B_2) + \cdots + r_i(B_1) \cdot r_0(B_2))x^i.$$

This sum gives the number of ways of placing i nontaking rooks on B, broken down into $i+1$ cases according to the number of rooks on B_1 and the number of rooks on B_2. Therefore this coefficient is equal to $r_i(B)$, which yields the term $r_i(B)x^i$ of $R(x,B)$. Since the corresponding terms of $R(x,B_1) \cdot R(x,B_2)$ and $R(x,B)$ are equal, we have $R(x,B) = R(x,B_1) \cdot R(x,B_2)$. ■

The following example illustrates the technique of this theorem.

Example 2 A woman on a sales trip brought four skirts (blue, brown, gray plaid, green stripe) and five blouses (yellow, pink, white, tan, and blue). Some of the skirts cannot be worn with some of the blouses, as shown by the shaded squares in Figure 3. In how many ways can she make four outfits by pairing the four skirts with four of the five blouses?

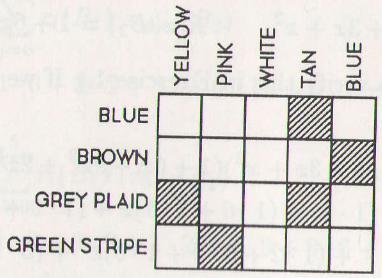


Figure 3. Possible skirt and blouse outfits.

Solution: (Note that in this example we are matching a set of four objects into a set of five objects. We will find the rook polynomial for the board B and then use the inclusion-exclusion principle to finish the counting process. Since the board is not square, we will need to suitably adjust our counting when we use the inclusion-exclusion principle.)

We observe that this board B of forbidden positions can be broken into two disjoint subboards B_1 and B_2, as in Figure 4.

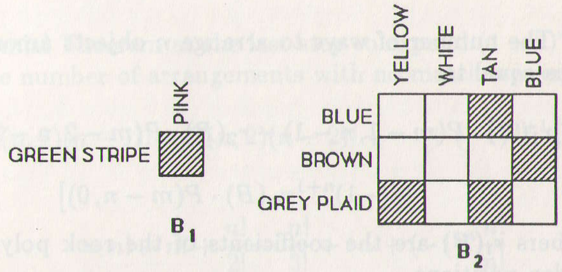


Figure 4. Disjoint subboards for the board of Figure 3.

It is not difficult to compute the rook polynomials for each of these boards:

$$R(x,B_1) = 1 + x$$
$$R(x,B_2) = 1 + 4x + 4x^2 + x^3.$$

(This is left as Exercise 2.) By Theorem 1,

$$\begin{aligned} R(x,B) &= R(x,B_1) \cdot R(x,B_2) \\ &= (1+x)(1+4x+4x^2+x^3) \\ &= 1 + 5x + 8x^2 + 5x^3 + x^4. \end{aligned}$$

Therefore $r_0 = 1$, $r_1 = 5$, $r_2 = 8$, $r_3 = 5$, $r_4 = 1$. Now that we know the number of ways to place nontaking rooks on forbidden squares, we use the inclusion-exclusion principle to obtain the final answer:

$$\begin{aligned} |U - (A_1 \cup \cdots \cup A_4)| &= |U| - |A_1 \cup \cdots \cup A_4| \\ &= |U| - \Big(\sum |A_i| - \sum |A_i \cap A_j| \\ &\qquad + \sum |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|\Big) \\ &= 5 \cdot 4 \cdot 3 \cdot 2 - (5(4 \cdot 3 \cdot 2) - 8(3 \cdot 2) + 5(2) - 1(1)) \\ &= 39. \end{aligned}$$

(Note that $|U| = 5 \cdot 4 \cdot 3 \cdot 2$ since $|U|$ is equal to the number of ways to place four nontaking rooks on the 4×5 board. Also, $\sum |A_i| = 5(4 \cdot 3 \cdot 2)$ since $r_1 = 5$

and there are $4 \cdot 3 \cdot 2$ ways to place the three other nontaking rooks in three of the other four columns.) □

The following theorem summarizes the technique of using a rook polynomial together with the inclusion-exclusion principle to count arrangements with forbidden positions.

Theorem 2 The number of ways to arrange n objects among m positions (where $m \geq n$) is equal to

$$P(m,n)\big[r_1(B)\cdot P(m-1,n-1) - r_2(B)\cdot P(m-2,n-2) + \cdots$$
$$+(-1)^{n+1} r_n(B)\cdot P(m-n,0)\big]$$

where the numbers $r_i(B)$ are the coefficients of the rook polynomial for the board of forbidden positions.

In particular, if $m = n$, the number of arrangements is

$$n! - \big[r_1(B)\cdot(n-1)! - r_2(B)\cdot(n-2)! + \cdots + (-1)^{n+1} r_n(B)\cdot 0!\big]. \quad ■$$

Example 3 **Problème des rencontres** An urn contains n balls, numbered $1,2,\ldots,n$. The balls are drawn out one at a time and placed in a tray that has positions marked $1,2,\ldots,n$, with the ball drawn first placed in position 1, the ball drawn second placed in position 2, etc. A *rencontre*, or match, occurs when ball i happens to be placed in position i. In how many ways can the balls be drawn from the urn so that there are no matches?

Solution: We need to find D_n = the number of derangements of $1,2,\ldots,n$. We will do this by using rook polynomials. Since a match occurs when ball i is in position i, we shade the square (i,i), for $i = 1,2,\ldots,n$, of board B, as in Figure 5.

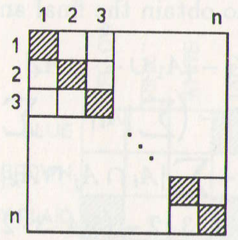


Figure 5. An $n \times n$ board.

Board B can be broken into n disjoint subboards $B_1, B_2, \ldots, B_n$, each consisting of the single square (i,i).

Each subboard has the rook polynomial $R(x,B_i) = 1 + x$. Theorem 1 applies here (using n instead of 2), and we have

$$\begin{aligned} R(x,B) &= R(x,B_1)\cdot R(x,B_2)\ldots R(x,B_n) \\ &= (1+x)(1+x)\ldots(1+x) \\ &= (1+x)^n \\ &= C(n,0) + C(n,1)x + C(n,2)x^2 + \cdots + C(n,n)x^n \end{aligned}$$

using the Binomial Theorem at the last step to expand $(1+x)^n$. Therefore, by Theorem 2, the number of arrangements with no matches is equal to

$$\begin{aligned} n! - &[C(n,1)(n-1)! - C(n,2)(n-2)! + \cdots + (-1)^{n+1}C(n,n)0!] \\ &= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n\frac{n!}{n!} \\ &= n!\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right). \end{aligned}$$
□

The method of simplifying the counting process by breaking a chessboard into two or more disjoint subboards works well when it can be done, as in Examples 2 and 3. But suppose it is impossible to break up a given board? The following example illustrates how to handle such a problem.

Example 4 Suppose a person has four gifts (1, 2, 3, 4) to give to four people — Kathy (K), Fred (F), Dave (D), and Wendy (W). The shaded squares in Figure 6 show which gifts cannot be given to the various people. Assuming that each person is to receive a gift, find the number of ways the four gifts can be given to the four people.

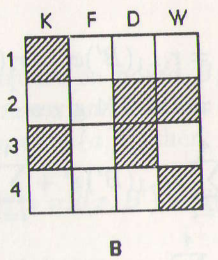


Figure 6. Possible distributions of gifts.

Solution: In this case it is not possible to break the board into two distinct subboards. (To see why, consider row 1. If square $(1,K)$ is in a subboard B_1, this would force the forbidden square $(3, K)$ to also be in board B_1. This forces the forbidden square $(3, D)$ to be in B_1. This forces the forbidden squares in row 2 to be in B_1, which in turn forces square $(4, W)$ to be in B_1. Therefore B_1 is the entire board B and we have not simplified the problem.)

To solve a problem involving such a board, we simplify the problem by examining cases. We want to find the rook polynomial $R(x, B)$. To find $r_i(B)$ we begin by choosing a forbidden square, such as $(3, K)$. Either we place a rook on this square (and hence no other rook in row 3 or column K) or else we do not place a rook on this square. In either case we are left with smaller boards to consider.

If we place a rook on square $(3, K)$, then the remaining $i-1$ rooks must be placed on forbidden squares of the board B' in Figure 7. This can be done in $r_{i-1}(B')$ ways. If we do not place a rook on square $(3, K)$, then the i rooks must all be placed on forbidden squares of the board B'' in Figure 7. This can be done in $r_i(B'')$ ways.

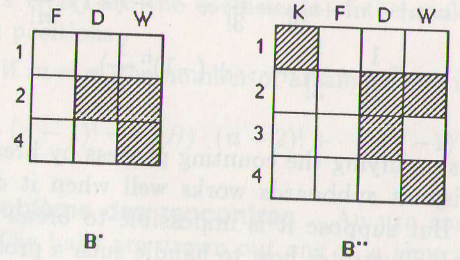


Figure 7. Subboards of the board of Figure 6.

Since these two cases exhaust all possibilities, we have

$$r_i(B) = r_{i-1}(B') + r_i(B''). \qquad (3)$$

This recurrence relation can be used to build the rook polynomial for B. Since $r_i(B)$ is to be the coefficient of x^i in the rook polynomial for B, multiply equation (3) by x^i to obtain

$$r_i(B)x^i = r_{i-1}(B')x^i + r_i(B'')x^i. \qquad (4)$$

Summing equations (4) with $i = 1, 2, 3, 4$ gives

$$\begin{aligned}\sum_{i=1}^{4} r_i(B)x^i &= \sum_{i=1}^{4} r_{i-1}(B')x^i + \sum_{i=1}^{4} r_i(B'')x^i \\ &= x\sum_{i=1}^{4} r_{i-1}(B')x^{i-1} + \sum_{i=1}^{4} r_i(B'')x^i \\ &= x\sum_{i=0}^{3} r_i(B')x^i + \sum_{i=1}^{4} r_i(B'')x^i.\end{aligned}$$

Using the fact that $r_0(B)x^0 = 1$ for any board B, we add $r_0(B)x^0$ to the left side and $r_0(B'')x^0$ to the second sum on the right side, obtaining

$$\sum_{i=0}^{4} r_i(B)x^i = x\sum_{i=0}^{3} r_i(B')x^i + \sum_{i=0}^{4} r_i(B'')x^i,$$

or

$$R(x, B) = xR(x, B') + R(x, B''). \qquad (5)$$

It is easy to see that

$$R(x, B') = 1 + 3x + x^2.$$

It is also not difficult to find the rook polynomial for B'', since its board already appears as disjoint subboards:

$$\begin{aligned}R(x, B'') &= (1 + x)(1 + 4x + 3x^2) \\ &= 1 + 5x + 7x^2 + 3x^3.\end{aligned}$$

Substituting these in equation (5) gives

$$\begin{aligned}R(x, B) &= xR(x, B') + R(x, B'') \\ &= x(1 + 3x + x^2) + (1 + 5x + 7x^2 + 3x^3) \\ &= 1 + 6x + 10x^2 + 4x^3.\end{aligned}$$

By Theorem 2, the number of ways to distribute the four gifts is

$$4! - (6 \cdot 3! - 10 \cdot 2! + 4 \cdot 1!) = 4.$$

□

The analog of equation (5) holds for all boards, which gives the following theorem.

Theorem 3 If (a, b) is a square on board B, if board B' is obtained from B by removing all squares in row a and column b, and if board B'' is obtained from B by removing the one square (a, b), then

$$R(x, B) = xR(x, B') + R(x, B'').$$

■

Example 5 A tennis coach wants to pair five men (1, 2, 3, 4, 5) and five women (6, 7, 8, 9, 10) for some practice sessions in preparation for a mixed doubles tournament. Based on the players' schedules and levels of ability, the

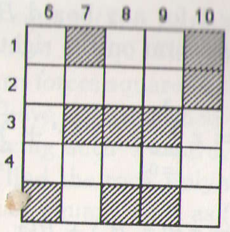


Figure 8. Possible pairings of tennis players.

coach knows that certain pairs cannot be formed, as shown by the shaded squares in Figure 8. In how many ways can the five men and the five women be paired?

Solution: Since it is not possible to break board B into disjoint subboards (the reader should check this), we use Theorem 3 to find $R(x, B)$.

If we begin with square $(3,8)$ in Theorem 3, we obtain the boards B' and B'' of Figure 9.

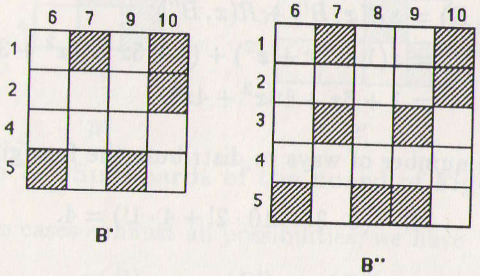


Figure 9. Subboards of board of Figure 8.

Board B' can be broken into two disjoint subboards (using squares $(5,6)$ and $(5,9)$ as one board), and its rook polynomial is

$$\begin{aligned} R(x, B') &= (1+2x)(1+3x+x^2) \\ &= 1+5x+7x^2+2x^3. \end{aligned}$$

However, it is not possible to break board B'' into disjoint subboards.

To find the rook polynomial for board B'', we need to use Theorem 3 again. Using square $(5,9)$ in the theorem, we obtain

$$\begin{aligned} R(x, B'') &= x(1+4x+3x^2) + (1+2x)(1+5x+6x^2+x^3) \\ &= 1+8x+20x^2+16x^3+2x^4. \end{aligned}$$

(The details are left as Exercise 8.) Therefore,

$$\begin{aligned} R(x, B) &= xR(x, B') + R(x, B'') \\ &= x(1+5x+7x^2+2x^3) + (1+8x+20x^2+16x^3+2x^4) \\ &= 1+9x+25x^2+23x^3+4x^4. \end{aligned}$$

From Theorem 3, it follows that the tennis coach can pair the five men and five women in 12 ways. □

Suggested Readings

1. R. Grimaldi, *Discrete and Combinatorial Mathematics*, Addison-Wesley, Reading, Mass., 1985.
2. J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
3. F. Roberts, *Applied Combinatorics*, Prentice-Hall, Englewood Cliffs, N.J., 1984.
4. A. Tucker, *Applied Combinatorics*, Second Edition, Wiley, New York, 1985.

Exercises

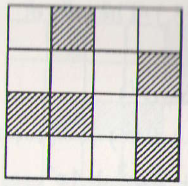
1. Verify that $R(x, B_1) = 1+3x+x^2$ and $R(x, B_2) = 1+6x+9x^2+2x^3$ for the boards of Figure 2.

2. Verify that $R(x, B_1) = 1+x$ and $R(x, B_2) = 1+4x+4x^2+x^3$ for the boards of Figure 4.

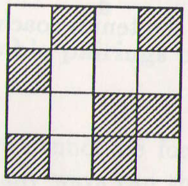
3. Prove that it is impossible to break the board B of Figure 8 into disjoint subboards.

In Exercises 4–7 find the rook polynomial and the number of arrangements with no object in a forbidden position for the given board.

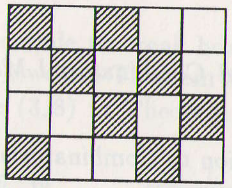
4.



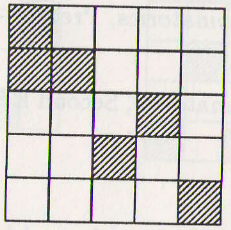
5.



6.



7.



8. Carry out the details to find $R(x, B'')$ in Example 5.

9. Find the number of permutations of 1, 2, 3, 4 where 1 is not in position 3, 2 is not in positions 3 or 4, and 4 is not in position 1.

10. A professor has divided a discrete mathematics class into four groups. Each of the groups is to write a biography on one of the following mathematicians: Boole, DeMorgan, Euclid, Euler, Hamilton, and Pascal. Group 1 does not want to write on Euler or Pascal, Group 2 does not want to write on DeMorgan or Hamilton, Group 3 does not want to write on Boole, DeMorgan, or Pascal, and Group 4 does not want to write on Boole or Euler. If the professor wants each group to write on a different mathematician, in how many ways can the professor assign a different mathematician to each group?

11. Suppose B is a 4×4 board with four forbidden positions, all in the last column. Use the method of rook polynomials to prove that there are no possible arrangements.

12. Suppose B is a 4×4 board with no forbidden positions. Use the method of rook polynomials to find the number of arrangements of the four objects.

13. Let $A = \{1, 2, 3, 4\}$. Find the number of 1-1 functions $f : A \to A$ such that $f(1) \neq 3$, $f(2) < 3$, and $f(4) > 1$.

14. The head of a mathematics department needs to make summer teaching assignments for five courses: numerical analysis, mathematical modeling, discrete mathematics, precalculus, and applied statistics. Professor Bloch does not want to be assigned to either mathematical modeling or precalculus, Professor Mahoney will not teach applied statistics, Professor Nakano does not want to teach numerical analysis or discrete mathematics, Professor Rockhill will teach any course except applied statistics, and Professor Sommer is willing to teach anything except numerical analysis. In how many ways can the department head match the five faculty to the five courses so that the wishes of the faculty are followed?

⋆15. (A *problème des ménages*) Four married couples are to be seated around a circular table so that no two men or two women are seated next to each other and no husband is to sit next to his wife. Assuming that arrangements such as $123\cdots78$ and $234\cdots81$ are different, how many arrangements are possible? (*Hint*: First determine the number of ways in which the four women can be seated. Then set up a board to determine the forbidden positions for the four men.)

Computer Projects

1. Write a computer program that takes a board of forbidden positions as input and determines whether the board can be written as two disjoint subboards.

2. Write a computer program that uses Theorem 3 to find the rook polynomial for a board of forbidden positions.

3. Write a computer program that takes a board of forbidden positions as input and gives as output the number of arrangements such that no object is in a forbidden position.