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Network Survivability

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Prerequisites: The prerequisites for this chapter are graphs and trees. See, for example, Chapters 7 and 8 of *Discrete Mathematics and Its Applications*, Second Edition, by Kenneth H. Rosen.

Introduction

For the first two years after the end of World War II, there were essentially no atomic bombs available in the United States [10] and none anywhere else in the world. But by the mid 1950s, both the United States and the Soviet Union had not only fission bombs but fusion bombs ready for use. Since then, the number of countries having one or both kinds of bomb has grown alarmingly, and the arsenals have grown to frightening proportions.

At the moment, we may have no nuclear-equipped enemy, but that can change quite suddenly. Friends can become enemies overnight (witness Iraq and the United States in August, 1990). Worse, enemies can attain nuclear capabilities much faster than expected. (The Soviet Union was thought incapable of producing a nuclear bomb for several years at the time it exploded its first one.) Therefore, we should continue all possible peaceful preparations for

protection against nuclear attack.

Among the essential requirements for an effective response to a nuclear attack are command, control and communications (C^3, in military parlence). Knowing this, the enemy is certain to attack them. He has several options: to attack the commanders, to attack the control systems used by the commanders, to attack the communications links between various commanders and between commanders and their forces, or to attack a mixture of these. In the United States, responsibility for the aspects of C^3 is divided among several agencies; the Defense Communications Agency (DCA) is responsible for the design and maintenance of the Defense Communications Systems [1].

There are several aspects of a nuclear strike that the communications system must take into account. The enemy is likely to allocate several weapons to high altitude (above 100 km) explosions at the very beginning of an attack, since one such explosion will black out radio communications (including transmissions between microwave towers) over several thousand square kilometers [1]. This will, of course, do little damage to properly protected equipment*, so that communications could be restored within a few hours, but it would seriously disrupt our early response. A surface detonation has a similar effect, but over a much smaller area [1], [3]. Burying communication cables a little less than a meter underground, will protect them from all but direct hits [1], but burying cables is expensive. Most communication links will remain microwave relays between towers and between satellites and ground stations.

Let us represent a communications network by a multigraph. We introduce a vertex for each command center and each switching center. Two vertices are joined by an edge whenever the corresponding centers are directly connected by a communications link (by a cable, or by a string of microwave antennae, or through a satellite, etc.). Because the term "multigraph" is a bit cumbersome, we will refer to multigraphs in this chapter as "graphs" except in definitions and theorems. When we need to discuss a graph without multiple edges, we will call it a "simple graph". We will reserve the word "network" for the communications network and we will call the multigraph a "graph representation of the network".

A good strategy is to build our network so that it satisfies the following two criteria:

(i) It should survive a limited attack, including attacks aimed at other nearby targets.

(ii) A careful study of our network by a knowledgeable enemy should reveal that the network is **bland**, in that it has no parts especially attractive to attack.

The thought behind "blandness" is that an attack is unlikely to be purposely

* However, there is an initial electromagnetic pulse of enormous size generated by a nuclear explosion; this pulse would destroy unprotected electrical equipment as effectively as a lightning bolt [1].

made against just part of a bland network, since no part of the network would appear more worth attacking than any other part.

We will address blandness shortly, but first we give a brief discussion of survivability of networks in limited attacks.

Cut Vertices and Blocks

One obvious failure of the first criterion for survivable networks occurs if every message traveling through the network must pass through one particular switch: One bomb on the switch would totally disable the network. In the graph representation, such a switch is a **cut vertex**, that is, a vertex such that removing the vertex and all edges incident with it leaves a subgraph with more connected components than in the original graph. We are interested in the subgraphs joined together by the cut vertices.

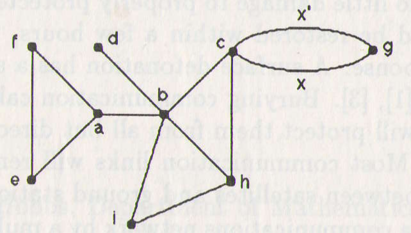


Figure 1. Graph G_1 with cut vertices a, b, and c.

Example 1 Consider the graph G_1 of Figure 1. G_1 is connected, but the erasure of any one of vertices a, b or c produces a subgraph with two or more (connected) components; thus a, b and c are cut vertices of G_1. But look at the triangle aef. Although this subgraph includes the cut vertex a of G_1, a is not a cut vertex of the triangle by itself. On the other hand, if we try to expand triangle aef to a larger connected subgraph of G_1, as for example subgraph H in Figure 2, we find that a is a cut vertex of any such larger subgraph. Thus the triangle aef is a maximal connected subgraph of G_1 without cut vertices of its own.

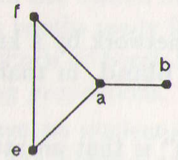


Figure 2. Subgraph H of G_1.

The cut edge $\{a, b\}$ is another such subgraph. Indeed, there are five such maximal subgraphs of G_1, as shown in Figure 3. □

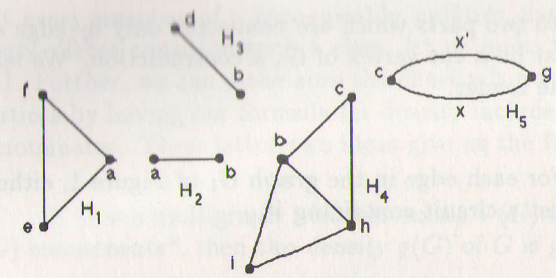


Figure 3. The five blocks of G_1.

This example suggests the following definition.

Definition 1 A *block* of a multigraph G is a maximal connected subgraph B of G such that B has no cut vertices of its own (although it may contain cut vertices of G). A multigraph is itself a *block* if it has only one block. □

There are three types of blocks. One is an isolated vertex; we will not see these again. A second type is the graph having just one edge and two vertices. The third has more than one edge; in Theorem 1 we will show that any block of the third kind contains a circuit, just as do the subgraphs H_1, H_4, and H_5 shown in Figure 3. To distinguish the third sort of block from the first two, we say that a block with more than one edge is **2-connected**. We need a preliminary lemma first.

Lemma 1 Let G be a connected multigraph, and let a be a vertex of G. Then a is a cut vertex of G if and only if there are two vertices v and v' distinct from a in one component of G such that every path in G from v to v' includes vertex a.

Proof: The proof is left as Exercise 4. ■

In Theorem 1, we prove even more than claimed previously. Notice that in subgraphs H_1, H_4, and H_5 of Figure 3 every edge is in some circuit of the subgraph. This is a general property of such blocks.

Theorem 1 Suppose G is a block with more than one edge. Then every edge of G is in a circuit of G.

Proof: The main idea of the proof of this theorem is that if G is a block, if G has more than one edge, and if an edge e of G is in no circuit of G, then G can be subdivided into two parts which are connected only by edge e. Then one of the ends of e must be a cut vertex of G, a contradiction. We leave the details of the proof to the reader. □

Example 2 For each edge in the graph G_1 of Figure 1, either state that it is a cut edge or list a circuit containing it.

Solution:

Edge	Circuit or cut edge	Edge	Circuit or cut edge
$\{a,f\}$	a,e,f,a	$\{a,e\}$	a,e,f,a
$\{e,f\}$	a,e,f,a	$\{a,b\}$	cut edge
$\{b,d\}$	cut edge	$\{b,c\}$	b,c,h,i,b
$\{b,h\}$	b,c,h,b	$\{b,i\}$	b,c,h,i,b
$\{c,h\}$	b,c,h,i,b	$\{h,i\}$	b,c,h,i,b
x	c,g,c	x'	c,g,c

□

Suppose we have represented a communications system by a multigraph with at least two edges and have found that the graph is a block. By Theorem 1 every edge in the graph is in a circuit. If a single vertex of a circuit is deleted, any two of the remaining vertices are still connected by a path in the rest of the circuit. Thus the communications system can survive at least one destroyed switch, from any cause whatsoever. This gives at least a partial solution to the first criterion of a survivable network.

As seen here, the subject of survivability of networks to limited attack is closely bound to the concept of removing vertices from a graph and thus breaking the graph into pieces. This subject is treated at greater length under the heading "connectivity" in many books, as for example in [2]. We will examine further the first criterion that must be satisfied by a survivable communications network later in this chapter.

Density

When we say a network is "bland," we intend to mean that the network offers no especially attractive targets. One interpretation of this notion is that there are no parts that are crowded together, so the network is not "dense." Going over to graphs, we want the density of a graph to be a measure that increases as the number of edges of the graph is increased. Further, if we say the density is 1, for example, we do not want the meaning of that phrase to depend on the number of vertices in the graph. In addition, a scale for the density should

be set so that every member of a recognizably uniform class of graphs has a constant density. Trees consitute such a class; let us make the density of any tree equal to 1. Further, we can make sure the density is not dependent on the number of vertices by having our formula for density include that number as a part of its denominator. These latter two ideas give us the following formula.

Definition 2 If G is a multigraph with vertex set $V(G)$ and edge set $E(G)$ and with $\omega(G)$ components*, then the *density* $g(G)$ of G is given by

$$g(G) = \frac{|E(G)|}{|V(G)| - \omega(G)}.$$ □

Notice that if T is a tree, then $\omega(T) = 1$ and $|E(T)| = |V(T)| - 1$, so $g(T) = 1$ in that case. Also, in a graph with a fixed number of components and a fixed number of vertices, it is clear that $g(G)$ will increase with increasing $|E(G)|$. Thus we can expect this measure of density to be useful in our analysis of communications networks.

Example 3 Compute $g(G)$ for C_3, C_n, K_4, K_n, $K_{2,3}$, $K_{2,n}$, and $K_{m,n}$.

Solution: Since a triangle C_3 has three edges and three vertices, $g(C_3) = 3/(3-1) = 3/2$. In general, if C_n is a circuit with $n > 1$ vertices, then $g(C_n) = n/(n-1)$.

Since K_4 has 6 edges and 4 vertices, we have $g(K_4) = 6/(4-1) = 6/3 = 2/1 = 4/2$. The general case here is that if K_n is a complete graph on $n \geq 2$ vertices, then $g(K_n) = n/2$, and this is an upper bound on $g(G)$ if G is a simple connected graph.

Because the complete bipartite graph $K_{2,3}$ has $2\times 3 = 6$ edges and $2+3 = 5$ vertices, we get $g(K_{2,3}) = 6/(5-1) = 6/4 = 3/2$. In general, $g(K_{2,n}) = 2n/(n+2-1) = 2n/(n+1) = 2 - \frac{2}{n+1}$, and even more generally $g(K_{m,n}) = mn/(m+n-1)$. □

But our comment about $g(G)$ increasing with $|E(G)|$ is not quite satisfactory. We did not intend to include a requirement that the number of components should not change. This problem is solvable, however. We first need an interesting arithmetic lemma.

Lemma 2 [6] Let $p_1/q_1, p_2/q_2, \ldots, p_k/q_k$ be fractions in which p_i and q_i are positive integers for each $i \in \{1, 2, \ldots, k\}$. Then

$$\min_{1\leq i\leq k} \frac{p_i}{q_i} \leq \frac{p_1 + p_2 + \cdots + p_k}{q_1 + q_2 + \cdots + q_k} \leq \max_{1\leq i\leq k} \frac{p_i}{q_i}.$$

Proof: See Exercise 5. ■

* ω is the lower case Greek letter "omega".

For example, $\min(\frac{1}{2}, \frac{2}{3}, \frac{7}{7}, \frac{9}{5}) = \frac{1}{2} \le \frac{1+2+7+9}{2+3+7+5} = \frac{19}{17} \le \frac{9}{5} = \max(\frac{1}{2}, \frac{2}{3}, \frac{7}{7}, \frac{9}{5})$.

Since we are looking for the densest parts of the graph, we can restrict our attention to connected graphs G, as shown in the next theorem.

Theorem 2 Suppose multigraph G has components $H_1, H_2, \ldots, H_k$, and suppose l is an index such that $g(H_l) = \max_{1\le i\le k} g(H_i)$. Then $g(H_l) \ge g(G)$.

Proof: We use Lemma 2, obtaining

$$\begin{aligned} g(G) &= \frac{|E(G)|}{|V(G)|-k} \\ &= \frac{|E(H_1)| + |E(H_2)| + \cdots + |E(H_k)|}{(|V(H_1)|-1) + (|V(H_2)|-1) + \cdots + (|V(H_k)|-1)} \\ &\le \max_{1\le i\le k}\left(\frac{|E(H_1)|}{|V(H_1)|-1}, \frac{|E(H_2)|}{|V(H_2)|-1}, \ldots, \frac{|E(H_k)|}{|V(H_k)|-1}\right) \\ &= \max_{1\le i\le k}(g(H_i)) \\ &= g(H_l). \end{aligned}$$

■

Example 4 Examine the function g for the graph G_2 of Figure 4.

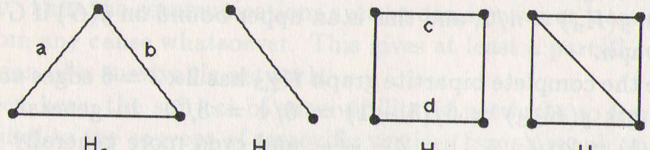


Figure 4. Graph G_2 with four components.

Solution: In Figure 4, $G_2 = H_1 \cup H_2 \cup H_3 \cup H_4$. We see that $g(G_2) = 13/(13-4) = 13/9$. But $g(H_4) = 5/(4-1) = 5/3 > 13/9 = g(G_2)$. □

Example 5 Find the densest part of G_3 of Figure 5.

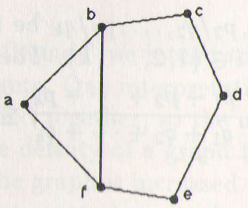


Figure 5. Graph G_3.

Solution: The connected subgraphs of G_3 (other than trees and G_3 itself) are shown in Figure 6 (up to isomorphism). Calculating g for each of the subgraphs, we have $g(T) = 1$ for any tree in G_3, while $g(G) = 7/5$, $g(H_1) = 3/2$, $g(H_2) = 4/3$, $g(H_3) = 5/4$, $g(H_4) = 6/5$, $g(H_5) = 5/4$, $g(H_6) = 6/5$, and $g(H_7) = 6/5$. Since the largest of these is 3/2, the triangle H_1 is the densest part of G_3. Thus, the part of the network whose graph representation is G_3 that is most likely to be heavily used is the triangle and that is the part that should be attacked. □

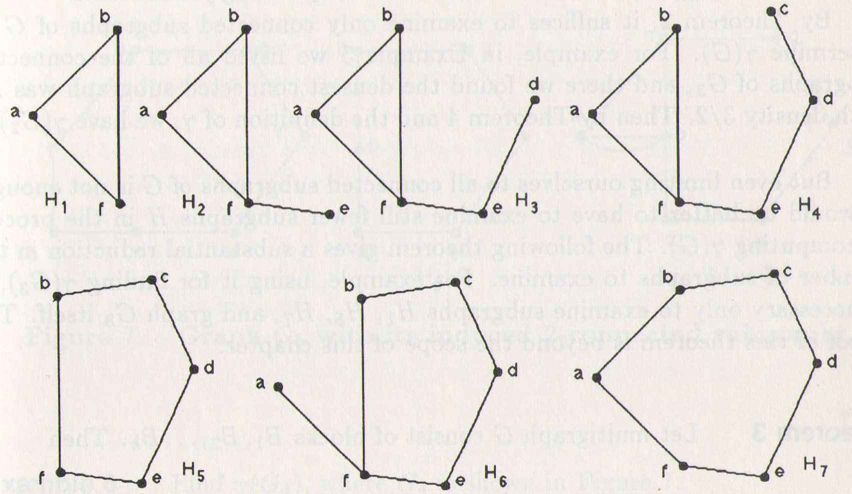


Figure 6. Connected subgraphs of G_3.

The following definition formalizes the idea of this example*.

Definition 3 Given a multigraph G,

$$\gamma(G) = \max_{H\subseteq G} g(H) = \max_{H\subseteq G} \frac{|E(H)|}{|V(H)| - \omega(H)},$$

where the maximum is taken over all subgraphs H of G for which the denominator in $g(H)$ is not zero. □

Clearly, a subgraph H of graph G which achieves the value of $\gamma(G)$ is a densest part of the graph and thus corresponds to a part of the communications network through which messages are most likely to pass.

* γ is the lower case Greek letter "gamma."

Notice that if H is a connected graph, then the number $g(H)$ is the number of edges of H divided by the number $|V(H)| - 1$ of edges in a spanning tree of H. Thus $g(H)$ is an upper bound on the number of edge-disjoint spanning trees that could appear in H. This bound is not always achieved. For example, it could not be achieved in a triangle T, where $g(T) = 3/2$. Because γ is more important than g, the bound is recognized in the name we give γ. The Latin term for a tree is *arbor*. Since, in addition, $\gamma(G)$ is a fraction, we call $\gamma(G)$ the **fractional arboricity** of G. It turns out (see [4], [8]) that $\gamma(G)$ is even more strongly associated with numbers of spanning trees in G than this simple upper bound would suggest, and so the name has a very strong justification.

By Theorem 2, it suffices to examine only connected subgraphs of G to determine $\gamma(G)$. For example, in Example 5 we listed all of the connected subgraphs of G_3, and there we found the densest connected subgraph was H_1 with density 3/2. Then by Theorem 4 and the definition of γ, we have $\gamma(G_2) = 3/2$.

But even limiting ourselves to all connected subgraphs of G is not enough. It would be better to have to examine still fewer subgraphs H in the process of computing $\gamma(G)$. The following theorem gives a substantial reduction in the number of subgraphs to examine. For example, using it for finding $\gamma(G_3)$, it is necessary only to examine subgraphs H_1, H_5, H_7, and graph G_3 itself. The proof of this theorem is beyond the scope of this chapter.

Theorem 3 Let multigraph G consist of blocks $B_1, B_2, \ldots B_k$. Then

$$\gamma(G) = \max_{1 \le i \le k} (\gamma(B_i)).$$ ■

There is one more aid available in computing γ. If H is a connected subgraph of graph G and if G contains an edge e which joins two vertices of H but is not in H, then adding e to H produces a subgraph H' with no more vertices than H has, but with another edge. Hence $g(H') > g(H)$. Thus it is not sensible for us to examine subgraphs like H when subgraphs like H' exist. For example, the subgraph H_4 in Figure 6 has all of the vertices of G_3, but it is missing the edge $\{b, c\}$. Thus it is not surprising that $g(G_3) = 7/5$, which is 1/5 more than $g(H_4)$.

To make this idea formal, we say that a subgraph H of graph G is **induced** by its vertex set $V(H)$ if every edge of G joining two vertices in $V(H)$ is in H. Thus H_4 in Figure 6 is not an induced subgraph of G_3, nor are H_6 or H_7, but subgraphs H_1, H_2, H_3, and H_5 are induced subgraphs.

In computing $\gamma(G)$ for a connected graph G, we now must find only the 2-connected induced subgraphs of G, if any, compute the function g for each of them, and choose the largest among the number 1 ($= g(T)$ for any tree T of G) and the values of g computed.

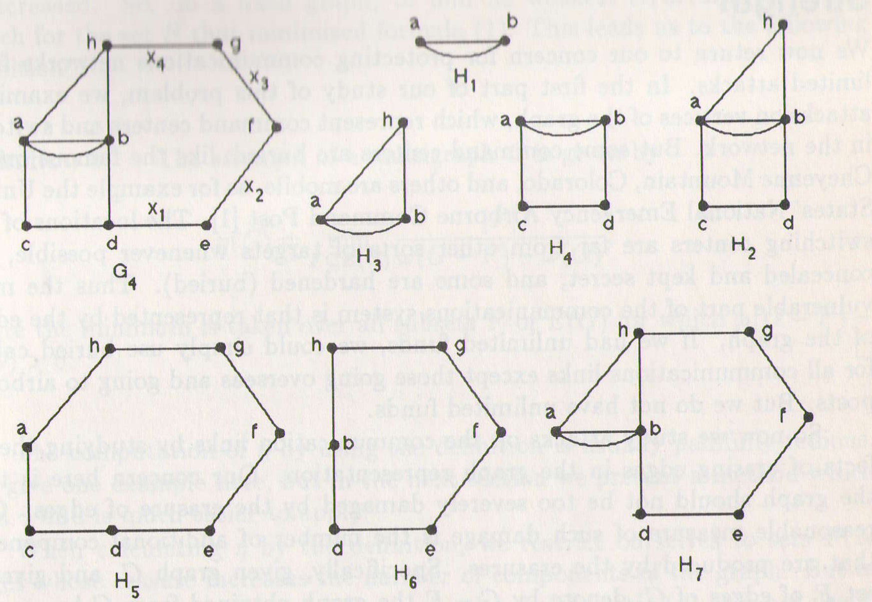


Figure 7. Graph G_4 with its induced 2-connected subgraphs.

Example 6 Find $\gamma(G_4)$, where G_4 is shown in Figure 7.

Solution: Figure 7 also shows all of the induced 2-connected subgraphs of G_4. We find that $g(G_4) = 11/7$, $g(H_1) = 2$, $g(H_2) = 7/4$, $g(H_3) = 2$, $g(H_4) = 5/3$, $g(H_5) = 7/6$, $g(H_6) = 6/5$, and $g(H_7) = 3/2$. The largest of these is 2, so $\gamma(G_4) = 2$ and this value is attained by both H_1 and H_3. Since H_3 has H_1 as a subgraph, the most attackable part of the network is that corresponding to the subgraph H_3. □

Even a slightly larger graph than G_4 would have too many induced 2-connected subgraphs for us to compute γ by hand. Worse, it turns out that the number of such subgraphs grows exponentially with increasing numbers of vertices of the graph. Thus even a computer would not be able to use this method to find $\gamma(G)$ for a graph representing a large communications system. However, several algorithms are given in [5], [7], and [9] for computing γ, and these have polynomial complexity. The descriptions of these algorithms are long, and the algorithms are hard to apply by hand, so we do not present them here.

Strength

We now return to our concern for protecting communications networks from limited attacks. In the first part of our study of this problem, we examined attacks on vertices of the graph, which represent command centers and switches in the network. But some command centers are buried, like the famous one in Cheyenne Mountain, Colorado, and others are mobile, as for example the United States' National Emergency Airborne Command Post [1]. The locations of the switching centers are far from other sorts of targets whenever possible, are concealed and kept secret, and some are hardened (buried). Thus the most vulnerable part of the communications system is that represented by the edges of the graph. If we had unlimited funds, we could simply use buried cables for all communications links except those going overseas and going to airborne posts. But we do not have unlimited funds.

So now we study attacks on the communication links by studying the effects of erasing edges in the graph representation. Our concern here is that the graph should not be too severely damaged by the erasure of edges. One reasonable measure of such damage is the number of additional components that are produced by the erasures. Specifically, given graph G, and given a set F of edges of G, denote by $G-F$ the graph obtained from G by erasing the edges in F from G. Then the number of additional components produced by the erasure of the edges in F is $\omega(G-F)-\omega(G)$.

Example 7 Remove $F=\{a,b,c,d\}$ from G_2.

Solution: Removing the edges of $F=\{a,b,c,d\}$ from the graph G_2 of Figure 4, we obtain the graph G_2-F of Figure 8, which has 6 components. Thus $\omega(G_2)-\omega(F)=6-4=2$. □

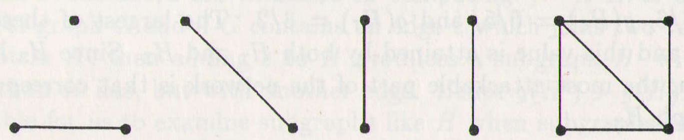


Figure 8. Graph $G_2-\{a,b,c,d\}$.

But $\omega(G-F)-\omega(G)$ is not a good measure of the resistance of graph G to edge erasure because it does not take into account the number of edges erased. The fact that erasing all of the edges of G assures $\omega(G-F)-\omega(G)=|V(G)|-\omega(G)$ does not say that G is necessarily weak. In order to take the size of F into account, we use a ratio, namely

$$\frac{|F|}{\omega(G-F)-\omega(G)}. \qquad (1)$$

Notice that formula (1) is reduced if $|F|$ is reduced or if $\omega(G-F)-\omega(G)$

is increased. So, on a fixed graph, to find its weakest structure, we would search for the set F that minimized formula (1). This leads us to the following definition*.

Definition 4 The *strength* of a multigraph G is given by

$$\eta(G)=\min_{F\subseteq E(G)}\frac{|F|}{\omega(G-F)-\omega(G)},$$

where the minimum is taken over all subsets F of $E(G)$ for which $\omega(G-F)-\omega(G)>0$. □

The computation of η by using the definition is usually painfully tedious. We give one example here, but in the next section we present a method which uses γ and is much easier to apply.

When calculating η by the definition, we restrict ourselves to sets F of edges whose erasure increases the number of components of the graph. But in addition, since we are seeking a minimum value for the ratio (1), it is not useful to include an edge e in set F if e joins two vertices in the same component of $G-F$. In other words, the components of $G-F$ should be the subgraphs induced by the vertex sets of the components. We incorporate these two observations into the next example.

Example 8 Given the graph G_5 of Figure 9, use the definition of η to find $\eta(G_5)$.

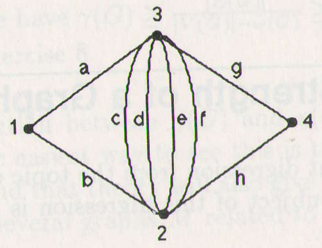


Figure 9. Graph G_5.

Solution: In Table 1 we list values of ratio (1) for various subsets F of edges of G_5. This list is organized by the number of components in $G-F$. Since the components involved are induced by their sets of vertices, they are listed by their vertex sets.

* η is the lower case Greek letter "eta."

Number of components	Vertices of components	Edge set	Formula (1)
2	1 234	$\{a,b\}$	$2/1=2$
	2 134	$\{b,c,d,e,f,h\}$	$6/1=6$
	3 124	$\{a,c,d,e,f,g\}$	$6/1=6$
	4 123	$\{g,h\}$	$2/1=2$
	12 34	$\{a,c,d,e,f,h\}$	$6/1=6$
	13 24	$\{b,c,d,e,f,g\}$	$6/1=6$
3	1 2 34	$\{a,b,c,d,e,f,h\}$	$7/2$
	1 3 24	$\{a,b,c,d,e,f,g\}$	$7/2$
	1 4 23	$\{a,b,g,h\}$	$4/2=2$
	2 4 13	$\{b,c,d,e,f,g,h\}$	$7/2$
	3 4 12	$\{a,c,d,e,f,g,h\}$	$7/2$
4	1 2 3 4	$\{a,b,c,d,e,f,g,h\}$	$8/3$

Table 1. **Finding $\eta(G_5)$.**

We notice that the minimum occurs three times, with edge sets $F_1=\{a,b\}$, $F_2=\{g,h\}$, and $F_3=\{a,b,g,h\}=F_1\cup F_2$. Thus $\eta(G_5)=2$. The maximum damage at minimum cost comes in three sets, but the larger one does the damage of both of the smaller ones, so it would be sensible to carry out the attack indicated by the set F_3 of edges. If our job is to redesign the network to make an attack less attractive, then this is the place in the network we need to improve. □

Computing the Strength of a Graph

We begin with an apparent digression from the topic of this section, but it will be seen shortly that the subject of this digression is exactly what we need to compute η with some ease.

Example 9 Calculate $\gamma(G_5)$ in Figure 9.

Solution: In Figure 10, we see the induced 2-connected subgraphs of this graph. There, $g(H_1)=4/1$, $g(H_2)=6/2=3$, $g(H_3)=6/2=3$, and $g(G_5)=8/3$. Since the largest of these is 4, we see that $\gamma(G_5)=4$. Recall that $\eta(G_5)=2$, so $\gamma(G_5)>\eta(G_5)$. □

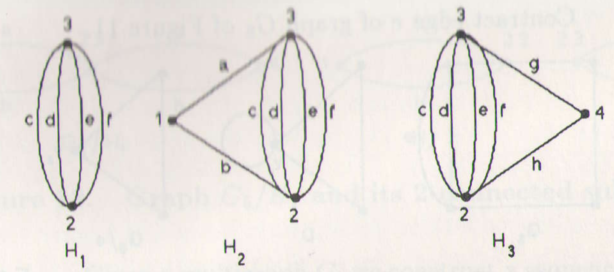


Figure 10. **The induced 2-connected subgraphs of G_5.**

It is not a coincidence that $\gamma(G_5)>\eta(G_5)$. In fact, the next theorem shows that there is always a similar relationship between $\gamma(G)$ and $\eta(G)$. The proof of Theorem 4 exploits the fact that $\omega(G-E(G))=|V(G)|$.

Theorem 4 For any multigraph G having at least one edge,

$$\gamma(G)\geq\frac{|E(G)|}{|V(G)|-\omega(G)}\geq\eta(G)\geq 1. \tag{2}$$

Proof: Since G has an edge, $\omega(G)<|V(G)|$, or $|V(G)|-\omega(G)>0$. By their definitions,

$$\gamma(G)\geq\frac{|E(G)|}{|V(G)|-\omega(G)} \tag{3}$$

and

$$\eta(G)\leq\frac{|E(G)|}{\omega(G-E(G))-\omega(G)}=\frac{|E(G)|}{|V(G)|-\omega(G)}. \tag{4}$$

Combining (3) and (4) we have $\gamma(G)\geq\frac{|E(G)|}{|V(G)|-\omega(G)}\geq\eta(G)$.

For $\eta(G)\geq 1$, see Exercise 8. ■

Actually, the connection between $\gamma(G)$ and $\eta(G)$ is even stronger than Theorem 4 suggests. The easiest way to see this is through the theory of contractions of subgraphs, and that theory will also give us a way to compute $\eta(G)$ by computing $\gamma(M)$ for several graphs M related to G.

Definition 5 Let G be a multigraph, and let e be an edge of G. We *contract* e by replacing e and its two ends by a single vertex v_e, letting each edge that met either end of e now be incident with v_e. For our purposes, we will allow multiple edges to be created by this process, but any loops generated will be erased. The resulting multigraph is denoted by G/e. □

Example 10 Contract edge e of graph G_6 of Figure 11.

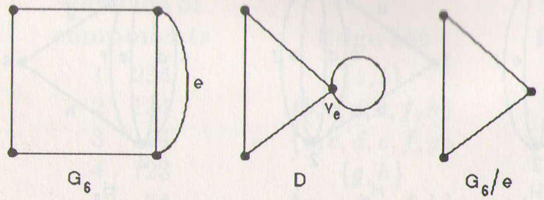


Figure 11. Graph G_6 and its contraction to G_6/e.

Solution: When contracting edge e, the intermediate step is the pseudograph D shown in Figure 11. After erasing the loop of D, we obtain the graph G_6/e also shown in Figure 11. □

To speed the process of contraction, we next define the contraction of a subgraph H of G.

Definition 6 Let G be a multigraph, and let H be a subgraph of G. We *contract* H by contracting every edge of H. The multigraph obtained by contracting subgraph H of multigraph G is denoted G/H and is called the *contraction* of H in G. □

The graph G/H is independent of the order in which the edges of H are contracted, so our definition and notation does not need to mention that order. In our examples, we label each vertex formed by contraction with a concatenation of the labels of the vertices that became that vertex.

Example 11 Form G_5/H_1 using H_1 shown in Figure 10, and find the value of $\gamma(G_5/H_1)$.

Solution: The graph G_5 is shown in Figure 9 and the subgraph H_1 of G_5 shown in Figure 10. After contraction of H_1, we obtain the graph G_5/H_1 shown in Figure 12. Also in Figure 12 are the 2-connected induced subgraphs H_4 and H_5 of G_5/H_1. Notice that $g(H_4) = g(H_5) = g(G_5/H_1) = 2$. Thus $\gamma(G_5/H_1) = g(G_5/H_1) = 2$. □

In the following definition, for brevity we let* $\Gamma(G)$ stand for a connected subgraph of G such that $\gamma(G) = g(\Gamma(G))$.

* Γ is the upper case Greek letter "gamma."

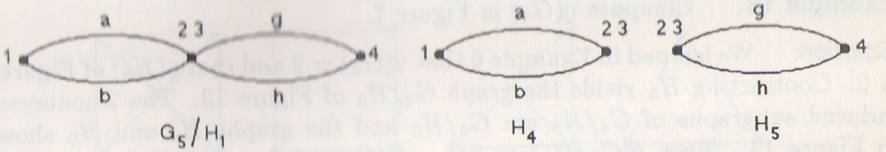


Figure 12. Graph G_5/H_1 and its 2-connected subgraphs.

Definition 7 Given a multigraph G, we construct a sequence of multigraphs $H_1, H_2, \ldots, H_k$ by the following rules:

(i) $H_1 = G$;
(ii) For $i \geq 1$, if $\gamma(H_i) \neq g(H_i)$, we let $H_{i+1} = H_i/(\Gamma(H_i))$; and
(iii) H_k is the first multigraph reached by this contraction process for which $\gamma(H_k) = g(H_k)$.

By Theorem 2, there is always a connected subgraph H of G for which $g(H) = \gamma(G)$. Thus H_k is defined for any multigraph G for which $\gamma(G)$ is defined. Because of its importance in calculating η, we call H_k an **η-reduction** of G, and we use G_0 to denote any η-reduction of G. When $G = G_0$, we say that G is **η-reduced.** □

For example, we found in Example 12 that the η-reduced graph G_0 for G_5 is G_5/H_1. The following theorem justifies the operation described here. The proof of this theorem is beyond the scope of this chapter.

Theorem 5 There is only one η-reduction G_0 of a multigraph G. Further, G_0 satisfies $\eta(G) = \eta(G_0)$, and the edge set $E(G_0)$ is the largest edge set F of G such that

$$\eta(G) = \frac{|F|}{\omega(G-F) - \omega(G)}.$$ ■

This theorem gives us a way of computing $\eta(G)$ for any graph G — we simply find the η-reduction G_0 of G as described, and then compute $g(G_0)$.

Example 12 For the graph G_5 of Figure 9, compute $\eta(G_5)$.

Solution: In Example 9 we learned that $g(H_1) = \gamma(G_5)$ and in Example 11 we contracted that graph to obtain G_5/H_1 shown in Figure 12, determining that $\gamma(G_5/H_1) = 2$. Since $\gamma(G_5/H_1) = g(G_5/H_1)$, so that $G_0 = G_5/H_1$, it follows that $\eta(G_5) = 2$ and the largest subset of edges achieving this value in the definition of η is $\{a, b, g, h\}$. We learned this directly (the hard way) in Example 8. □

Example 13 Compute $\eta(G_4)$ in Figure 7.

Solution: We learned in Example 6 that $\gamma(G_4) = 2$ and that $g(H_3)$ of Figure 7 is 2. Contracting H_3 yields the graph G_4/H_3 of Figure 13. The 2-connected induced subgraphs of G_4/H_3 are G_4/H_3 and the graphs H_8 and H_9 shown in Figure 13. Since $g(G_4/H_3) = 7/5$, $g(H_8) = 3/2$, and $g(H_9) = 5/4$, we have $\gamma(G_4/H_3) = g(H_8)$ and $\Gamma(G_4/H_3) = H_8$. Contracting H_8 results in the circuit C_4 having vertices $abcdh$, e, f, and g also shown in Figure 13. Since C_4 has only one 2-connected subgraph, we are done. We find that $\eta(G_4) = g(C_4) = 4/3$ and that the largest set of edges achieving this value in the definition of η is $\{x_1, x_2, x_3, x_4\}$ of Figure 7. □

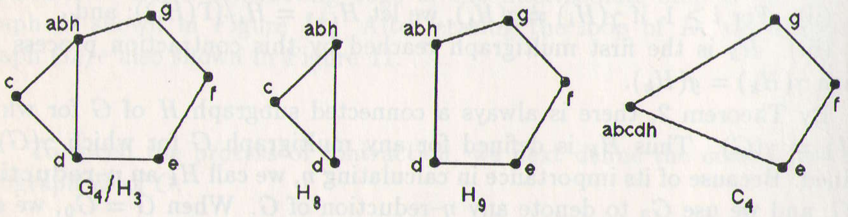


Figure 13. G_4/H_3, **its 2-connected subgraphs, and** $(G_4/H_3)/H_8 = C_4$.

Bland Networks

We have now seen three conditions in graphs that we would like to see considered in designing a communications network represented by the graph G:

(i) The graph should be a block, so that the network will be protected against a limited attack or collateral damage,

(ii) We should have $\gamma(G) = g(G)$ so that the network will not have any attractively dense parts, and

(iii) We should have $\eta(G) = \frac{|E(G)|}{|V(G)|-\omega(G)}$ to assure that the network is adequately strong against direct attack on the communication links.

But $g(G) = \frac{|E(G)|}{|V(G)|-\omega(G)}$, so our second and third conditions are satisfied if $\gamma(G) = \eta(G)$.

Definition 8 A multigraph G is *uniformly dense* if $\gamma(G) = \eta(G)$. A communications network is *bland* if its graph representation is uniformly dense. □

Example 14 Show that graph G_7 of Figure 14 is uniformly dense.

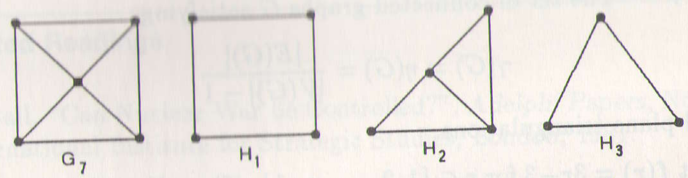


Figure 14. Graph G_7 **and its induced 2-connected subgraphs.**

Solution: The graph G_7 is shown in Figure 14 together with its induced 2-connected subgraphs H_1, H_2, and H_3. Since $g(G_7) = 2$, $g(H_1) = 4/3$, $g(H_2) = 5/3$, and $g(H_3) = 3/2$, we see that $\gamma(G_7) = 2 = g(G_7)$. But it follows that the η-reduced graph $G_0 = G_7$. Hence $\eta(G_7) = \gamma(G_0) = \gamma(G_7) = 2$ and G_7 is uniformly dense. Note also that G_7 is a block, so it satisfies all three of our graph conditions. □

The following theorem characterizes uniformly dense graphs.

Theorem 6 Let G be a multigraph with v vertices and e edges. The following are equivalent:

(a) $\gamma(G)(v - \omega(G)) = e$;

(b) $\eta(G)(v - \omega(G)) = e$;

(c) $\gamma(G) = \eta(G)$;

(d) G is η-reduced;

(e) There is a function $f : \{1, 2, \ldots, v - \omega(G)\} \to \mathcal{R}$ such that

(i) $\frac{f(r)}{r} \le \frac{f(v-\omega(G))}{(v-\omega(G))}$ for $1 \le r \le v - \omega(G)$,

(ii) $f(v - \omega(G)) = e$, and

(iii) $|E(H)| \le f(|V(H)| - \omega(H))$ for each subgraph H of G that has $|V(H)| > \omega(H)$.

Proof: We leave the details of this proof to the reader. It is not difficult to prove that each of (a) and (d) is equivalent to (c), that (e) is equivalent to (a), and that (c) → (b). However, the proof that (b) → (c) is beyond the scope of this chapter; its proof can be found in [4]. ■

For our final example, we need the following corollary, in which we apply condition (e) with $f(r)/r$ nondecreasing. A **plane triangulation** is a connected simple graph drawn on the plane such that every face has exactly three edges on its boundary. It is known that if G is a connected planar simple graph with e edges and v vertices with $v \ge 3$, then $e \le 3v - 6$. It is easy to show that if G is a plane triangulation with e edges and v vertices, then $e = 3v - 6$ (see Exercise 9).

Corollary 1 The set of connected graphs G satisfying

$$\gamma(G) = \eta(G) = \frac{|E(G)|}{|V(G)| - 1}$$

includes all plane triangulations.

Proof: Let $f(r) = 3r - 3$ for $r \in \{1, 2, \ldots, v-1\}$. Then $f(v-1) = 3(v-1) - 3 = 3v - 6 = e$, so (ii) of part (e) of Theorem 6 is satisfied. Also, $f(r)/r = 3 - 3/r$, which increases in value as positive r increases in value. Since $r \leq v - 1$, we have

$$\frac{f(r)}{r} \leq \frac{f(v-1)}{v-1},$$

which is (i) of part (e) of Theorem 6. We leave the proof of (iii) of part (e) of Theorem 6 as Exercise 10. ■

Example 15 Is the graph G_8 shown in Figure 15 a good choice for a graph representation of a survivable network by the criteria described in this chapter?

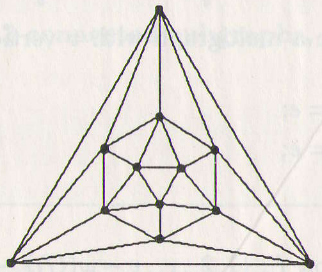


Figure 15. Graph G_8, the icosahedron.

Solution: We notice first that G_8 is connected and has no cut vertices, so it is a block. Further, it is a plane triangulation, so $\gamma(G_8) = \eta(G_8)$. Thus it satisfies all conditions that we have placed on the graph representation of a network. It is a good choice. □

We have arrived at one of the cutting edges of modern mathematical research. The following questions are not exercises; rather they are the questions being asked by some professional mathematicians in their research.

Given the graph G of an already existing communications network, and given that $\gamma(G) \neq \eta(G)$, what would be the best edge to add to G to reduce $\gamma(G) - \eta(G)$? What does "best" mean in the answer? Is it the cheapest edge whose addition will reduce $\gamma(G) - \eta(G)$, or does it reduce $\gamma(G) - \eta(G)$ by the largest amount? Would it be better to eliminate some edges already present and replace them by other edges? Which ones should be eliminated? Many other questions can be asked, and their answers could be critical to the survival of our communications system if we are ever subject to an attack on our homeland.

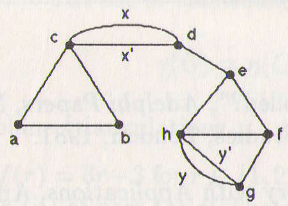
Suggested Readings

1. D. Ball, "Can Nuclear War be Controlled?", *Adelphi Papers*, No. 169, The International Institute for Strategic Studies, London, 1981.
2. J. Bondy and U. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976.
3. R. Cannell, *Live: A Handbook of Survival in Nuclear Attack*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.
4. P. Catlin, J. Grossman, A. Hobbs, and H.-J. Lai "Fractional arboricity, strength, and principal partitions in graphs and matroids" *Discrete Appl. Math*, (to appear).
5. W. Cunningham, "Optimal attack and reinforcement of a network", *J. Assoc. Comp. Mach.*, Vol. 32, 1985, pp. 549–561.
6. G. Hardy, J. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1952.
7. A. Hobbs, "Computing edge-toughness and fractional arboricity", *Contemporary Math.*, Vol. 89, 1989, pp. 89–106.
8. C. Nash-Williams, "Decompositions of finite graphs into forests", *J. London Math. Soc.*, Vol. 39, 1964, p. 12.
9. N. Tomizawa "Strongly irreducible matroids and principal partition of a matroid into strongly irreducible minors", *Elect. and Comm. in Japan*, Vol. 59A, No. 2, 1976, pp. 1–10.
10. A. Tucker "Communications black-out: Electromagnetic effects" *Nuclear Attack: Civil Defence*, ed. by the Royal United Services Institute for Defence Studies, Brassey's Publishers Ltd., Oxford, England, 1982, pp. 70–82.

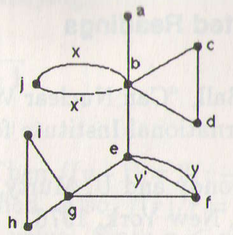
Exercises

1. Draw the blocks of each graph. Then, for each edge, either state that it is a cut edge or list a circuit containing it.

a)

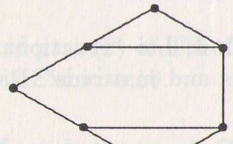


b)

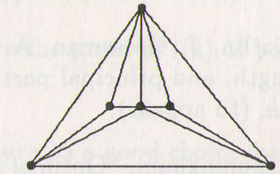


2. Find the values of γ and η for each of these graphs.

a)

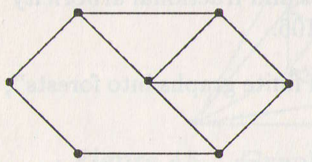


b)

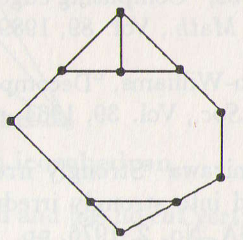


3. For each graph, find the value of γ. Then use contractions to find the η-reduction G_0 and the value of η.

a)



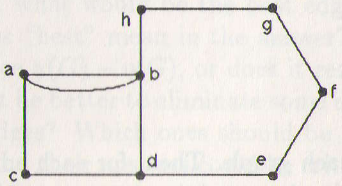
b)



$\star$4. Prove Lemma 1.

$\star$5. Prove Lemma 2. *Hint*: Prove it first for $k = 2$ and then use induction.

6. Why are only eight subgraphs of G_4 (including G_4 itself) examined in determining $\gamma(G_4)$? For example, why not also look at the following subgraph? Give another 2-connected subgraph of G_4 that was omitted for the same reason.



7. For any forest F with at least one edge, prove that $\gamma(F) = \eta(F) = 1$.

8. Prove that $\eta(G) \geq 1$ for any multigraph having at least one edge.

9. Prove that, if G is a plane triangulation with e edges and v vertices, then $e = 3v - 6$.

10. Prove that (iii) of part (e) of Theorem 6 is satisfied by the function f defined in the proof of Corollary 1.

Computer Projects

1. Write a computer program to calculate $\gamma(G)$ for a multigraph G.

2. Write a computer program which uses the program of Computer Project 1 to find the value of $\eta(G)$ for a multigraph G.