8

Ramsey Numbers

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Prerequisites: The prerequisites for this chapter are the pigeonhole principle and basic concepts of graphs. See, for example, Sections 4.2, 7.2 and 7.3 of *Discrete Mathematics and Its Applications*, Second Edition, by Kenneth H. Rosen.

Introduction

Suppose there are 6 people at a party, where each pair of people are either friends or strangers. We can show that there must be either 3 mutual friends or 3 mutual strangers at the party. To do this, we set up a graph G with 6 vertices (corresponding to the 6 people), where edges represent friendships. We need to show that there must be a triangle (i.e., a simple circuit of length 3) in G or else a triangle in $\overline{G}$, the complement of G.

This problem can be rephrased as the following edge-coloring problem: Show that if every edge of K_6 is colored red or green, then there must be either a red triangle or a green triangle. To see that these two problems are equivalent, take G to be the subgraph of K_6 consisting of the 6 vertices and the red edges; therefore $\overline{G}$ consists of the 6 vertices and the green edges. A red triangle in K_6 corresponds to a set of 3 mutual friends in G, whereas a green triangle corresponds to a set of 3 mutual strangers.

The proof that there is either a red or a green triangle in K_6 is a simple application of the pigeonhole principle. Begin by choosing a vertex v in K_6. We let the edges on v represent pigeons and the colors red and green be the names of two pigeonholes. Since $d(v) = 5$ and each edge on v is either red or green, we have 5 pigeons and 2 pigeonholes. Therefore some pigeonhole holds at least 3 pigeons; that is, there are at least 3 red edges or 3 green edges incident on v. Assuming that v has at least 3 red edges, say $\{v,a\}, \{v,b\}$, and $\{v,c\}$, we examine the edges joining a, b, c. If any of these 3 edges is red, we obtain a red triangle. If none of the 3 edges is red, then we have a green triangle joining a, b, c. (A similar argument works if v has at least 3 green edges.) Thus, we are guaranteed of obtaining a *monochromatic* triangle, that is, a triangle such that all its edges are of the same color.

In this chapter we extend these ideas further, examining questions such as the following. What is the minimum number of people needed at a party in order to guarantee that there are at least 4 mutual friends or 4 mutual strangers? What is the minimum number needed to guarantee that there are 3 mutual friends or 5 mutual strangers? What is the minimum number needed to guarantee that there are 4 mutual close friends, 7 mutual acquaintances who are not close friends, or 9 mutual strangers?

The minimum numbers in problems such as these are called Ramsey numbers, named after Frank Ramsey*, who in 1930 published a paper [8] on set theory that generalized the pigeonhole principle.

Ramsey Numbers

As we saw earlier, no matter how the edges of K_6 are colored red or green, K_6 contains a subgraph K_3 all edges of which are colored red or a subgraph K_3 all edges of which are colored green. That is, K_6 contains either a red K_3 or a green K_3. We say that the integer 6 has the $(3,3)$-Ramsey property. More generally, we have the following definition.

Definition 1 Let i and j be integers such that $i \geq 2$ and $j \geq 2$. A positive integer m has the *(i,j)-Ramsey property* if K_m contains either a red K_i or a green K_j as a subgraph, no matter how the edges of K_m are colored red or green. □

* See *Discrete Mathematics and Its Applications*, Second Edition, for biographical information on Ramsey.

From the above discussion, we know that 6 has the $(3,3)$-Ramsey property. But is 6 the smallest such number? Suppose we take K_5 and "color" its 10 edges using 1s and 2s, as in Figure 1a.

There are no monochromatic triangles in either Figure 1b or 1c. Therefore this coloring shows that 5 does not have the $(3,3)$-Ramsey property. It is easy to see that no positive integer smaller than 5 has the $(3,3)$-Ramsey property. (See Exercise 3.) Therefore, 6 is the smallest integer with this property, and is called a Ramsey number.

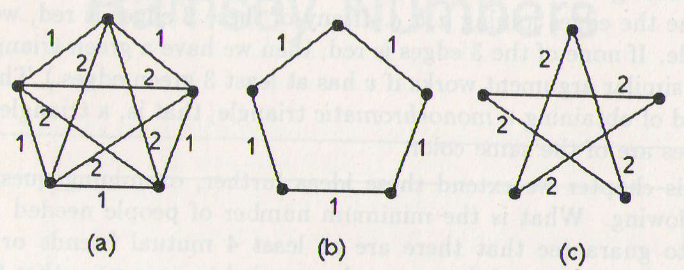


Figure 1. **A coloring of K_5.**

Definition 2 The *Ramsey number* $R(i,j)$ is the smallest positive integer that has the (i,j)-Ramsey property. □

For example, $R(3,3) = 6$, since 6 has the $(3,3)$-Ramsey property but no smaller positive integer has this property.

Example 1 Find $R(2,7)$, the smallest positive integer that has the $(2,7)$-Ramsey property.

Solution: We begin by examining the positive integers n such that every coloring of the edges of K_n with red and green results in either a red K_2 (i.e., a red edge) or a green K_7. The number $R(2,7)$ is the smallest positive integer with this property. Consider any coloring of the edges of K_7. Either there is at least one red edge, or else every edge is colored green. If there is a red edge, then we have the desired red K_2. If every edge was colored green, then we have the desired green K_7. Thus 7 has the $(2,7)$-Ramsey property, which says that $R(2,7) \le 7$.

But $R(2,7) \not< 7$. To see this, consider the coloring of K_6 where each edge is colored red. Then K_6 has neither a red K_2 nor a green K_7, and so 6 does not have the $(2,7)$-Ramsey property.

Therefore $R(2,7) = 7$. □

The argument in the above example can be generalized by replacing 7 with any integer $k \ge 2$ to obtain:

$$R(2,k) = k.$$

(This is left as Exercise 4.) This example also leads us to make four other observations:

1. For all integers $i \ge 2$ and $j \ge 2$, $R(i,j) = R(j,i)$.
2. If m has the (i,j)-Ramsey property, then so does every integer $n > m$.
3. If m does not have the (i,j)-Ramsey property, then neither does any integer $n < m$.
4. If $i_i \ge i_2$, then $R(i_1,j) \ge R(i_2,j)$.

Proofs of these four facts are left to the reader as Exercises 5-8.

Note that when we try to find $R(i,j)$, we are looking for a red K_i or a green K_j; that is, "red" is associated with the variable i and "green" is associated with the variable j. Likewise, when we try to find $R(j,i)$, "red" is associated with j and "green" with i. Since $R(i,j) = R(j,i)$, we need only look for a monochromatic K_i subgraph or a monochromatic K_j subgraph.

The definition of the Ramsey number $R(i,j)$ requires us to find the *smallest* positive integer with the (i,j)-Ramsey property. But suppose that for some i and j there is *no* positive integer with the (i,j)-Ramsey property? In such a case there would be no Ramsey number $R(i,j)$. This leads us to the question: For every choice of positive integers $i \ge 2$ and $j \ge 2$, is there a Ramsey number $R(i,j)$? The following lemma and theorem provide an affirmative answer to this question.

Lemma 1 If $i \ge 3$ and $j \ge 3$, then

$$R(i,j) \le R(i,j-1) + R(i-1,j). \tag{1}$$

Proof: Let $m = R(i,j-1) + R(i-1,j)$. We will show that m has the (i,j)-Ramsey property. Suppose the edges of K_m have been colored red or green and v is a vertex of K_m. Partition the vertex set V into 2 subsets:

A = all vertices adjacent to v along a red edge

B = all vertices adjacent to v along a green edge.

Since

$$|A| + |B| = |A \cup B| = m - 1 = R(i,j-1) + R(i-1,j) - 1,$$

either $|A| \ge R(i-1,j)$ or $|B| \ge R(i,j-1)$. If this were not the case, we would have $|A| < R(i-1,j)$ and $|B| < R(i,j-1)$, which would imply that

$$|A \cup B| < R(i-1,j) + R(i,j-1) = m - 1.$$

This would contradict the fact that $|A \cup B| = m - 1$.

Consider the case where $|A| \geq R(i-1, j)$. Now consider the complete subgraph on the vertices in A. This is a subgraph of K_m, which we call $K_{|A|}$. We will show that $K_{|A|}$ contains either a red K_i or a green K_j. Since $|A| \geq R(i-1, j)$, $K_{|A|}$ has either a red K_{i-1} or a green K_j. If we have a red K_{i-1}, we add the red edges joining v to the vertices of K_{i-1} to obtain a red K_i. Thus, $K_{|A|}$, and hence K_m, has either a red K_i or a green K_j. This says that m has the (i,j)-Ramsey property. (The case where $|B| \geq R(i, j-1)$ is left as Exercise 9.) Therefore, inequality (1) has been proved. □

This lemma establishes a relationship that Ramsey numbers must satisfy. The following theorem uses mathematical induction together with this relationship to prove the existence of the Ramsey numbers.

Theorem 1 Ramsey's Theorem If i and j are integers ($i \geq 2$ and $j \geq 2$), then there is a positive integer with the (i,j)-Ramsey property (and hence $R(i,j)$ exists).

Proof: We use (1) and the Principle of Mathematical Induction to prove that for all integers $i \geq 2$ and $j \geq 2$ there is a positive integer with the (i,j)-Ramsey property. Let $P(n)$ be the statement:

$P(n)$: If $i + j = n$, then there is an integer with the (i,j)-Ramsey property.

The base case is $P(4)$, since the smallest values of i and j under consideration are $i = j = 2$. We know from the comments following Example 1 that there is an integer with the $(2,2)$-Ramsey property. Therefore $P(4)$ is true.

Now we assume that $P(n)$ is true and show that $P(n+1)$ is also true. Assume that $i + j = n + 1$. Therefore $i + (j-1) = n$ and $(i-1) + j = n$. $P(n)$ states that there are integers with the $(i, j-1)$-Ramsey property and the $(i-1, j)$-Ramsey property. Hence the Ramsey numbers $R(i, j-1)$ and $R(i-1, j)$ exist. Inequality (1) guarantees that $R(i,j)$ must also exist. Therefore $P(n+1)$ is also true. The Principle of Mathematical Induction guarantees that $P(n)$ is true for all $i \geq 2$ and $j \geq 2$. This shows that $R(i,j)$ exists if $i \geq 2$ and $j \geq 2$. ■

So far we know the values of the following Ramsey numbers:

$$R(2,k) = R(k,2) = k$$

$$R(3,3) = 6.$$

If $i \geq 2$ and $j \geq 2$, it rapidly becomes very difficult to find $R(i,j)$ because of the large number of possible edge colorings of the graphs K_n. Hence, it is no

surprise that the list of Ramsey numbers whose exact values are known is very short. We will now evaluate some of these.

Example 2 Find $R(3,4)$.

Solution: From Lemma 1 we know that

$$\begin{aligned} R(3,4) &\leq R(3,3) + R(2,4) \\ &= 6 + 4 \\ &= 10. \end{aligned}$$

To determine if $R(3,4) < 10$, we might consider looking at all possible red/green colorings of the edges of K_9. If one of these colorings has no red K_3 or green K_4, we can conclude that $R(3,4) = 10$. However, if every red/green coloring of K_9 produces either a red K_3 or a green K_4, we must then look at colorings of K_8, etc. Examining all possible red/green colorings of the edges of K_9 is not an easy task—K_9 has 36 edges, so initially there would be $2^{36} \approx 69{,}000{,}000{,}000$ colorings to examine.

Fortunately, we can avoid examining the colorings of K_9, because of the following fact:

$$\text{If } i \geq 3,\ j \geq 3, \text{ and if } R(i, j-1) \text{ and } R(i-1, j) \text{ are even, then } R(i,j) \leq R(i, j-1) + R(i-1, j) - 1. \qquad (2)$$

The proof of this fact follows the proof of Lemma 1, and is left as Exercise 10.

Using (2) with $i = 3$ and $j = 4$ yields

$$\begin{aligned} R(3,4) &\leq R(3,3) + R(2,4) - 1 \\ &= 6 + 4 - 1 \\ &= 9 \end{aligned}$$

and we therefore know that $R(3,4) \leq 9$.

To see that 8 does not have the $(3,4)$-Ramsey property, consider the coloring of the graph K_8, where the red edges are drawn in Figure 2(a) and the green edges in Figure 2(b).

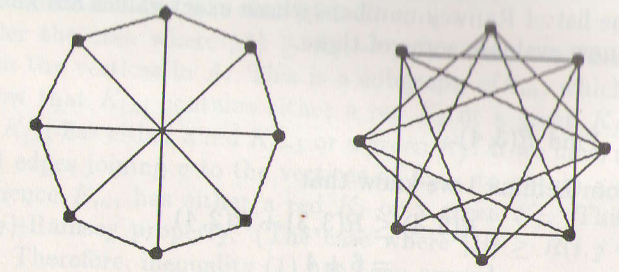
It is easy to see that there is no triangle in 2(a). In Exercise 11 the reader is asked to check that there is no K_4 in 2(b).

Thus, $R(3,4) = 9$. □

Example 3 Find $R(3,5)$.

Solution: From (1) we know that

$$\begin{aligned} R(3,5) &\leq R(3,4) + R(2,5) \\ &= 9 + 5 \\ &= 14 \end{aligned}$$



(a) (b)

Figure 2. A coloring of K_8.

In fact, $R(3,5) = 14$. To see this, we show that 13 does not have the (3,5)-Ramsey property. Draw K_{13} with vertices labeled $1, 2, \ldots, 13$. Color the edge $\{i, j\}$ red if $|i - j| = 1$, 5, 8 or 12, and color the edge green otherwise. We leave it to the reader to verify in Exercise 12 that this graph has no red K_3 or green K_5. □

There are a few other Ramsey numbers whose values are known exactly. Table 1 lists these numbers and ranges for some of the smaller Ramsey numbers. For example, $R(5,4) = R(4,5)$ is known to be 25, 26, or 27.

i \ j	2	3	4	5	6
2	2				
3	3	6			
4	4	9	18		
5	5	14	25–27	43–52	
6	6	18	34–43	57-94	102-169
7	7	23	≥ 49	≥ 76	
8	8	28	≥ 53	≥ 94	
9	9	36	≥ 69		

Table 1. Ramsey Numbers $R(i,j)$.

From the remark following Example 1, we know exact values of all Ramsey numbers $R(i,j)$ when $i = 2$ or $j = 2$. Aside from these, only eight other Ramsey numbers have known values. The Ramsey numbers $R(3,3)$, $R(4,3)$, $R(5,3)$, and $R(4,4)$ were found in 1955 by A. M. Gleason and R. E. Greenwood; $R(6,3)$ was found by J. G. Kalbfleisch in 1966; $R(7,3)$ was found by J. E. Graver and J. Yackel in 1968; $R(8,3)$ was found recently by B. McKay and Z. Ke Min; $R(9,3)$ was found by C. M. Grinstead and S. M. Roberts in 1982.

Many upper and lower bounds for Ramsey numbers are known. The following theorem gives one of these.

Theorem 2 If $i \geq 2$ and $j \geq 2$, then $R(i,j) \leq C(i+j-2, i-1)$.

Proof: A proof by induction can be given here, following closely the induction proof given for Theorem 1. This is left as Exercise 14. ■

It is also known that if $k \geq 2$, then $2^{k/2} \leq R(k,k) \leq 2^{2k-3}$, and if $j \geq 13$, then $R(3,j) \leq C(j,2) - 5$. Proofs of these facts can be found in Tomescu [10].

Generalizations

The Ramsey numbers discussed in the last section represent only one family of Ramsey numbers. In this section we will briefly study some other families of these numbers.

For example, suppose we take K_n and color its edges red, yellow, or green. What is the smallest positive integer n that guarantees that K_n has a subgraph that is a red K_3, a yellow K_3, or a green K_3? We might also think of this problem in terms of friendships rather than colors. A group of n countries each send a diplomat to a meeting. Each pair of countries are friendly toward each other, neutral toward each other, or enemies of each other. What is the minimum number of countries that need to be represented at the meeting in order to guarantee that among the countries represented there will be 3 mutual friends, 3 mutually neutral ones, or 3 mutual enemies? The minimum such number is written $R(3,3,3;2)$. The "2" is written as part of $R(3,3,3;2)$ because edges (i.e., the objects being colored) are determined by 2 vertices. (As we shall see, the 2 can be replaced by a larger positive integer.)

The three 3s can be replaced by any number of positive integers to obtain other families of Ramsey numbers. For example, using the 3 colors red, yellow, and green, $R(5,4,7;2)$ is the smallest positive integer n with the property that if the edges of K_n are colored with the 3 colors, then K_n contains a red K_5, a yellow K_4, or a green K_7.

Definition 3 Suppose $i_1, i_2, \ldots, i_n$ are positive integers, where each $i_j \geq 2$. A positive integer m has the $(i_1, \ldots, i_n; 2)$-*Ramsey property* if, given n colors $1, 2, \ldots, n$, K_m has a subgraph K_{i_j} of color j, for some j, no matter how the edges of K_m are colored with the n colors.

The smallest positive integer with the $(i_i, \ldots, i_n; 2)$-Ramsey property is called the **Ramsey number** $R(i_i, \ldots, i_n; 2)$. □

Note that if $n = 2$, the Ramsey numbers $R(i_1, i_2; 2)$ are the Ramsey numbers $R(i_1, i_2)$ studied in the previous section.

Again we are faced with the question: Do these numbers always exist? That is, for a given list $i_i, \ldots, i_n$, are there positive integers with the $(i_i, \ldots, i_n; 2)$-Ramsey property so that there is a smallest one (i.e., the Ramsey number $R(i_i, \ldots, i_n; 2)$)? The answer is yes, and a proof using the Principle of Mathematical Induction can be found in Graham, Rothschild, and Spencer [6].

Very little is also known about the numbers $R(i_1, \ldots, i_n; 2)$ if $n \geq 3$. However, if $i_j = 2$ for all j, then

$$R(2, \ldots, 2; 2) = 2.$$

This is left as Exercise 15. If each $i_j \geq 3$, the only Ramsey number whose value is known is $R(3, 3, 3; 2)$.

Example 4 Show that $R(3, 3, 3; 2) = 17$.

Solution: Consider any coloring of the edges of K_{17}, using the colors red, yellow, and green. Choose any vertex v. Of the 16 edges incident on v, there is a single color that appears on at least 6 of them, by the Pigeonhole Principle. Suppose this color is red, and that there are 6 red edges joining v to vertices labeled 1, 2, 3, 4, 5, 6. If one of the edges joining i and j $(1 \leq i \leq 6,\ 1 \leq j \leq 6)$ is also red, then we have a red K_3. Otherwise, every one of the edges joining i and j is yellow or green. These edges give a graph K_6 colored with 2 colors, and we know that such a graph has a monochromatic K_3 (yellow or green) since $R(3, 3) = 6$. This says that K_{17} must have a K_3 that is red, yellow, or green. Therefore $R(3, 3, 3; 2) \leq 17$.

To see that $R(3, 3, 3; 2) \geq 17$, consult Berge [1, opposite p.420] for a coloring of K_{16} with 3 colors that has no monochromatic triangle. □

So far we have dealt with Ramsey numbers of the form $R(i_1, \ldots, i_n; 2)$ by looking at them from the standpoint of coloring the edges of the graph K_m with n colors and looking for monochromatic subgraphs K_{i_j}. But there is another way to develop these numbers and further generalize ideas.

Consider the problem in the Introduction—the problem of finding a red or green subgraph K_3 in K_6. Start with K_6 and its vertex set V. Take all 2-element subsets of V (i.e., the edges of K_6) and divide these subsets into 2 collections C_1 and C_2. The number 6 has the $(3, 3)$-Ramsey property if and only if:

(i) There is a 3-element subset of V with all its 2-element subsets in C_1, or
(ii) There is a 3-element subset of V with all its 2-element subsets in C_2.

Thinking of C_1 as the set of edges colored red and C_2 as the set of edges colored green, we see that we have a red triangle if and only if condition (i) is satisfied, and we have a green triangle if and only if condition (ii) is satisfied.

But note that this statement of the Ramsey property does not require us to use a graph. The property is phrased only in terms of a set and properties of a certain collection of its subsets. This notion can be extended to include dividing subsets of size r (rather than 2) into any number of collections (rather than only the two collections C_1 and C_2).

This leads us to the following general definition of the classical Ramsey numbers.

Definition 4 Suppose that $i_1, i_2, \ldots, i_n, r$ are positive integers where $n \geq 2$ and each $i_j \geq r$. A positive integer m has the $(i_1, \ldots, i_n; r)$-*Ramsey property* if the following statement is true:

> If S is a set of size m and the r-element subsets of S are partitioned into n collections $C_1, \ldots, C_n$, then for some j there is a subset of S of size i_j such that each of its r-element subsets belong to C_j.

The **Ramsey number** $R(i_1, \ldots, i_n; r)$ is the smallest positive integer that has the $(i_1, \ldots, i_n; r)$-Ramsey property. □

In particular, when $r = 2$, we can think of this definition in terms of coloring the edges of K_m with n colors and taking C_j as the set of edges of color j. The numbers i_j give the number of vertices in the monochromatic K_{i_j} subgraphs.

We mention without proof Ramsey's Theorem regarding the existence of these Ramsey numbers. A proof can be found in Graham, Rothschild, and Spencer [6].

Theorem 3 Ramsey's Theorem If $i_1, \ldots, i_n, r$ are positive integers where $n \geq 2$ and each $i_j \geq r$, the Ramsey number $R(i_1, \ldots, i_n; r)$ exists. ■

Earlier in this chapter we displayed the known Ramsey numbers of the form $R(i_1, i_2; 2)$. We also proved that $R(3, 3, 3; 2) = 17$. If $r \geq 3$, very little is known about exact values of the Ramsey numbers. See Exercise 17 for an example of one type of Ramsey number whose exact value is known.

If $r = 1$, the Ramsey numbers $R(i_1, \ldots, i_n; 1)$ are easy to find, since in this case we need only consider the 1-element subsets of S. The following theorem gives a formula for these numbers.

Theorem 4 $R(i_1,\ldots i_n;1) = i_1 + \ldots i_n - (n-1)$.

Proof: Let $i_1 + \ldots + i_n - (n-1) = m$. We first show that the integer m has the $(i_1,\ldots,i_n;1)$-Ramsey property.

Take a set S of size m and divide its 1-element subsets into n classes $C_1,\ldots,C_n$. Observe that there must be a subscript j_0 such that $|C_{j_0}| \geq i_{j_0}$. (If $|C_j| < i_j$ for all j, then $|C_j| \leq i_j - 1$. Therefore $m = |C_1| + \ldots + |C_n| \leq (i_1 - 1) + \ldots + (i_n - 1) = i_1 + \cdots + i_n - n = m - 1$, and hence $m \leq m-1$, which is a contradiction.) If we take any i_{j_0} elements of C_{j_0}, we have a subset of S of size i_{j_0} that has all its 1-element subsets belonging to C_{j_0}. This shows that $R(i_1,\ldots,i_n;1) \leq i_1 + \cdots + i_n - (n-1)$.

We now show that $m-1 = i_1 + \cdots + i_n - n$ does not have the $(i_1,\ldots,i_n;1)$-Ramsey property. Take a set S where $|S| = i_1 + \cdots + i_n - n$. Partition its 1 - element subsets into n classes $C_1,\ldots,C_n$ where $|C_j| = i_j - 1$. With this partition there is no subset of S of size i_j that has all its 1-element subsets belonging to C_j. ∎

Note the particular case of this theorem when $i_1 = \cdots = i_n = 2$:

$$R(2,\ldots,2;1) = n+1.$$

This fact shows how Ramsey theory can be thought of as a generalization of the pigeonhole principle. In the terminology of Ramsey numbers, the fact that $R(2,\ldots,2;1) = n+1$ means that $n+1$ is the smallest positive integer with the property that if S has size $n+1$ and the subsets of S are partitioned into n sets $C_1,\ldots,C_n$, then for some j there is a subset of S of size 2 such that each of its elements belong to C_j. Hence, some C_j has at least 2 elements. If we think of a set S of $n+1$ pigeons and the subset C_j, $(j = 1,\ldots,n)$ as the set of pigeons roosting in pigeonhole j, then some pigeonhole must have at least 2 pigeons in it. Thus, the Ramsey numbers $R(2,\ldots,2;1)$ give the smallest number of pigeons that force at least 2 to roost in the same pigeonhole.

Schur's Theorem

Suppose the integers $1,2,3,4,5$ are each colored red or green. That is, suppose $S = \{1,2,3,4,5\}$ is partitioned into 2 subsets, R and G, where the integers in R are colored red and the integers in G are colored green. Then it is known that the equation

$$x + y = z$$

has a monochromatic solution. (See Exercise 18.) That is, the equation $x+y=z$ is satisfied for some $x,y,z \in R$ or some $x,y,z \in G$. For example, if $R = \{2,4,5\}$ and $G = \{1,3\}$, we have a red solution: $2+2=4$. Also, if $R = \{2,5\}$ and $G = \{1,3,4\}$, we have a green solution: $1+3=4$.

However, if $S = \{1,2,3,4\}$, we are not guaranteed a monochromatic solution. To see this, take $R = \{1,4\}$ and $G = \{2,3\}$. The equation $x+y=z$ is not satisfied for any choices of $x,y,z \in R$ or for any choices of $x,y,z \in G$.

The number 5 can be looked on as a "dividing point" here. If $n \geq 5$, then partitioning $\{1,\ldots,n\}$ into red and green sets will always result in a monochromatic solution to $x+y=z$, whereas if $n < 5$, we may fail to have a monochromatic solution. We write

$$S(2) = 5,$$

where the number 2 indicates the fact that we are using 2 colors. The letter "S" is used in honor of I. Schur*, who in 1916 developed this material while studying a problem related to Fermat's Last Theorem ($x^n + y^n = z^n$ has no integer solutions if $n > 2$).

Rather than work with only 2 colors, we can color $\{1,2,\ldots,n\}$ with k colors and look for the minimum value of n, written $S(k)$, that guarantees a monochromatic solution to $x+y=z$.

If $k = 1$, it is easy to see that

$$S(1) = 2.$$

(If $\{1,2\}$ is colored with 1 color, then we have the solution $1+1=2$; the smaller set $\{1\}$ yields no solution to $x+y=z$.) If $k=3$, it is known that

$$S(3) = 14.$$

That is, no matter how the integers in the set $\{1,\ldots,14\}$ are colored with 3 colors, $x+y=z$ will have a monochromatic solution.

It is natural to ask the question: For each positive integer k, is there a number $S(k)$? For example if we wish to use 4 colors, is there a positive integer $S(4)$ such that any coloring of $\{1,\ldots,S(4)\}$ with 4 colors will give a monochromatic solution? The following theorem shows that the numbers $S(k)$ always exist and are bounded above by the family of Ramsey numbers $R(3,\ldots,3;2)$.

Theorem 5 If k is a positive integer, then

$$S(k) \leq R(3,\ldots,3;2)$$

(where there are k 3s in the notation for the Ramsey number).

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* Issai Schur (1875–1941) was a Russian mathematician who attended schools in Latvia and Germany. Most of his teaching career was spent at the University of Berlin. Forced into retirement by the Nazis in 1935, he eventually emigrated to Palestine where he died two years later. He contributed to many areas of mathematics, and is best known for his work in the area of abstract algebra called the representation theory of groups.

Proof: We will show that if the integers $1, 2, \ldots, R(3, \ldots, 3; 2)$ are colored with k colors, then the equation $x + y = z$ will have a monochromatic solution. (This implies that $S(k) \leq R(3, \ldots, 3; 2)$ since $S(k)$ is the smallest integer with the monochromatic solution property.)

Let $n = R(3, \ldots, 3; 2)$ and color the integers $1, 2, \ldots, n$ with k colors. This gives a partition of $1, 2, \ldots, n$ into k sets, $S_1, S_2, \ldots, S_k$, where integers in the same set have the same color.

Now take the graph K_n, label its vertices $1, 2, \ldots, n$, and label its edges according to the following rule:

$$\text{edge } \{i, j\} \text{ has label } |i - j|$$

This labeling has a special feature: every triangle has two sides with labels whose sum is equal to the label on the third side. (To see this, suppose we have a triangle determined by i_1, i_2, i_3, and assume $i_1 > i_2 > i_3$. The 3 edge labels are $i_1 - i_2$, $i_2 - i_3$, and $i_1 - i_3$, and we have $(i_1 - i_2) + (i_2 - i_3) = (i_1 - i_3)$.

We will use these labels to color the edges of K_n: use color m on edge $\{i, j\}$ if its label $|i - j| \in S_m$. This gives a coloring of K_n with the k colors used for the elements in the sets $S_1, \ldots, S_k$. Since $n = R(3, \ldots, 3; 2)$, n has the $(3, \ldots, 3; 2)$-Ramsey property, and so there must be a monochromatic triangle in K_n. If its 3 vertices are i_1, i_2, i_3, and if we let $x = i_1 - i_2$, $y = i_2 - i_3$, and $z = i_1 - i_3$, then we have $x + y = z$ where all 3 values have the same color. ■

Note that the theorem shows only the existence of $S(k)$, by showing that these numbers are bounded above by certain Ramsey numbers. For example, if $k = 3$, the theorem states that

$$S(3) \leq R(3, 3, 3; 2) = 17.$$

Therefore, we know that $x + y = z$ has a monochromatic solution, no matter how the integers $1, \ldots, 17$ are colored with 3 colors. But the theorem does not give the value of the minimum number with this property, $S(3)$, which is known to be 14.

It is natural to ask what happens if we have more than 3 variables in the equation. For an extension to equations of the form $x_1 + \ldots + x_{n-1} = x_n$, see Beutelspacher and Brestovansky [2].

Convex Sets

In this section we will show how Ramsey numbers play a role in constructing convex polygons. A **convex polygon** is a polygon P such that if x and y are points in the interior of P, then the line segment joining x and y lies completely inside P. In Figure 3 the hexagon (6-gon) is convex, whereas the quadrilateral (4-gon) is *concave* (i.e., not convex) since the line segment joining x and y intersects the exterior of the 4-gon.

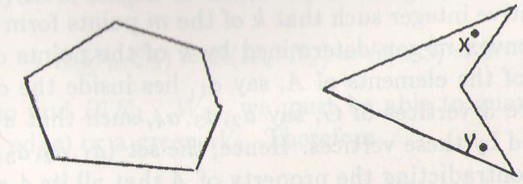

Figure 3. Two polygons.

If we take $n (\geq 3)$ points in the plane, no 3 of which are collinear, we can always connect then to form an n-gon. But can we guarantee that the n-gon will be convex? (The answer is no, even for 4 points, as shown by the 4-gon in Figure 3.) But suppose we do not select all n points? For example, given 5 points, no 3 of which are collinear, can we always find 4 of the 5 points so that when they are connected we obtain a 4-gon that is convex? Then answer is yes, and is left to the reader as Exercise 19.

In general, suppose we want to guarantee that we can obtain a convex m-gon by connecting m of n given points, no 3 of which are collinear. Is there always an integer n that guarantees a convex m-gon, and, if so, how large must n be? For example, if we want a convex 5-gon, how many points, n, must we start with in order to guarantee that we can obtain a convex 5-gon? The following theorem, proved by Erdös and Szekeres [3] in 1935, shows that Ramsey numbers can be used to find a solution. In fact, it was this paper that provided the impetus for the study of Ramsey numbers and suggested the possibility of its wide applicability in mathematics.

Theorem 6 Suppose m is a positive integer and there are n given points, no 3 of which are collinear. If $n \geq R(m, 5; 4)$, then a convex m-gon can be obtained from m of the n points.

Proof: Suppose $n \geq R(m, 5; 4)$ and S is a set of n points in the plane, with no 3 points collinear. Since n has the $(m, 5; 4)$-Ramsey property, no matter how we divide the 4-element subsets of S into 2 collections C_1 and C_2, either

(i) There is a subset of S of size m with all its 4-element subsets in C_1, or
(ii) There is a subset of S of size 5 with all its 4-element subsets in C_2.

We will take C_1 to be the collection of all subsets of S of size 4 where the 4-gon determined by the points is convex and take C_2 to be the collection of all subsets of S of size 4 where the 4-gons are concave.

But note that alternative (ii) cannot happen. That is, it is impossible to have 5 points in the plane that give rise only to concave 4-gons. (This is Exercise 19.) Therefore, we are guaranteed of having a subset $A \subseteq S$ where $|A| = m$ and all 4-gons determined by A are convex.

We will now show that the m points of A determine a convex m-gon. Let k be the largest positive integer such that k of the m points form a convex k-gon. Suppose G is a convex m-gon determined by k of the points of A. If $k < m$, then at least one of the elements of A, say a_1, lies inside the convex k-gon G. Therefore there are 3 vertices of G, say a_2, a_3, a_4, such that a_1 lies inside the triangle determined by these vertices. Hence, the set $\{a_1, a_2, a_3, a_4\}$ determines a concave 4-gon, contradicting the property of A that all its 4-gons are convex. Therefore $k \not< m$, and we must have $k = m$. This says that the m points of A determine the desired convex m-gon. ■

The Ramsey number $R(m, 5; 4)$ in Theorem 7 gives a set of values for n that guarantee the existence of a convex m-gon. But it remains an unsolved problem to find the smallest integer x (which depends on m) such that if $n \geq x$, then a convex m-gon can be obtained from m of the n points.

Graph Ramsey Numbers

So far, we have examined colorings of K_m and looked for monochromatic K_i subgraphs. But suppose we don't look for complete subgraphs K_i, but rather try to find other subgraphs, such as cycle graphs (C_i), wheels (W_i), complete bipartite graphs ($K_{i,j}$), or trees? Problems such as these give rise to other families of Ramsey numbers, called *graph Ramsey numbers*.

Definition 5 Suppose $G_1, \ldots, G_n$ are graphs, each with at least one edge. An integer m has the *$(G_1, \ldots, G_n)$-Ramsey property* if every coloring of the edges of K_m with the n colors $1, 2, \ldots n$ yields a subgraph G_j of color j, for some j.

The *graph Ramsey number* $R(G_1, \ldots, G_n)$ is the smallest positive integer with the $(G_1, \ldots, G_n)$-Ramsey property. □

We note that, just as in the case of the classical Ramsey numbers discussed earlier, the "order" of particular colors does not matter. That is, if every coloring of K_m has a red G_1 or a green G_2, then every coloring of K_m has a green G_1 or a red G_2, and vice versa. Therefore, $R(G_1, G_2) = R(G_2, G_1)$, and the problem can be phrased as one of finding a monochromatic G_1 or G_2 (rather than a G_1 or G_2 of a specified color).

Example 5 Some graph Ramsey numbers are easy to determine because they are the classical Ramsey numbers in disguise.

Suppose we want to find $R(W_3, C_3)$, where W_3 is the wheel with 3 spokes and C_3 is the cycle of length 3. Since W_3 is the graph K_4 and C_3 is K_3, we have

$$R(W_3, C_3) = R(K_4, K_3) = R(4, 3) = 9.$$

Similarly, to find $R(K_{1,1}, W_3)$, we must be able to guarantee either a red $K_{1,1}$ (i.e., a red edge) or a green W_3. Therefore,

$$R(K_{1,1}, W_3) = R(K_2, K_4) = 4.$$

The graph Ramsey numbers always exist, and are bounded above by related classical Ramsey numbers. To see this, suppose we wish to prove the existence of $R(G_1, \ldots, G_n)$ where each graph G_j has i_j vertices. If we let $m = R(i_1, \ldots, i_n; 2)$, then Theorem 4 guarantees that every coloring of K_m with the n colors $1, 2, \ldots, n$ yields a subgraph K_{i_j} of color j, for some j. But G_j is a subgraph of K_{i_j}, and hence we obtain a subgraph G_j of K_m that has color j. Therefore $R(G_1, \ldots, G_n)$ is bounded above by $R(i_1, \ldots, i_n; 2)$. We state this as the following theorem.

Theorem 7 For $j = 1, \ldots, n$, suppose that G_j is a graph with i_j vertices (where $i_j \geq 2$). Then the graph Ramsey number $R(G_1, \ldots, G_n)$ exists, and

$$R(G_1, \ldots, G_n) \leq R(i_1, \ldots, i_n; 2).$$ ■

This area of Ramsey theory has been a focus of much activity by many mathematicians, including Erdös and Graham*, during the last twenty years. See, for example Gardner [5] and Graham, Rothschild, and Spencer [6].

The case where a monochromatic K_i subgraph is "almost" obtained has also been studied. That is, rather than looking for the monochromatic K_i subgraphs (which is done when we try to find the classical Ramsey numbers $R(i, j)$), we look for monochromatic subgraphs $K_i - e$, where $K_i - e$ is the graph K_i with any one edge removed. The following example gives one of these numbers.

* Ronald L. Graham (1935–) was born in Taft, California, and received his bachelor's degree from the University of Alaska while in the Air Force. He received his Ph.D. from Berkeley (at which time he was also a member of a trampoline group) and then went to work at Bell Laboratories. At Bell Labs he has headed the Mathematical Studies Center and is currently Adjunct Director of the Research Information Sciences Division. His research work has been in the area of combinatorial mathematics, with many contributions in the field of Ramsey Theory. For his work in this area, in 1972 he was named a corecipient of the Polya Prize, given by the Society for Industrial and Applied Mathematics. He has also earned acclaim for his skill as a juggler, and has been past president of the International Jugglers Association.

Example 6 Prove that $R(K_3 - e, K_3 - e) = 3$.

Solution: First note that $K_3 - e$ is the graph on 3 vertices consisting of 2 edges. No matter how the 3 edges of K_3 are colored red or green, there are at least 2 of the same color, thereby giving the monochromatic subgraph $K_3 - e$. Therefore $R(K_3 - e, K_3 - e) \leq 3$. It is easy to see that $R(K_3 - e, K_3 - e) > 2$, since we can take K_2 and color one edge red and the other green. Therefore

$$R(K_3 - e, K_3 - e) = 3.$$

□

It is also known that

$$R(K_4 - e, K_4 - e) = 10$$
$$R(K_5 - e, K_5 - e) = 22$$
$$42 \leq R(K_6 - e, K_6 - e) \leq 86$$
$$R(K_4 - e, K_5 - e) = 13$$
$$R(K_4 - e, K_6 - e) = 17$$
$$R(K_4 - e, K_7 - e) = 28.$$

For more information on these numbers, the reader is referred to Faudree, Rousseau, and Schelp [4] and Radziszowski [7]. The last two numbers in this list were found by J. McNamara (personal communication, August 14, 1990).

Suggested Readings

1. C. Berge, *Graphs and Hypergraphs*, American Elsevier, New York, 1973.
2. A. Beutelspacher and W. Brestovansky, "Combinatorial Theory," Proceedings, *Lecture Notes in Mathematics*, Vol. 969, 1982, Springer-Verlag, Berlin, pp. 30–38.
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4. R. Faudree, C. Rousseau, and R. Schelp, "Studies Related to the Ramsey Number $R(K_5 - e)$," in *Graph Theory and Its Application to Algorithms and Computer Science*, ed. Y. Alavi, 1985.
5. M. Gardner, "Mathematical Games," *Scientific American*, Vol. 235, No. 5, 1977, pp. 18–28.
6. R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, Wiley, New York, 1980.
7. S. Radziszowski, "Small Ramsey Numbers", unpublished.
8. F. Ramsey, "On a Problem of Formal Logic," *Proceedings of the London Mathematical Society*, Vol. 30, 1930, pp. 264–286.
9. F. S. Roberts, *Applied Combinatorics*, Prentice-Hall, Englewood Cliffs, N. J., 1984.
10. I. Tomescu, *Problems in Combinatorics and Graph Theory*, translated by R. Melter, Wiley, New York, 1985.

Exercises

1. Prove that $R(2,3) = 3$, by showing that every coloring of the edges of K_3 with red and green gives either a red K_2 or a green K_3, and then showing that K_2 does not have this property.
2. Prove that $R(3,4) > 6$ by finding a coloring of the edges of K_6 that has no red K_3 or green K_4.
3. Prove that if $0 < n < 5$, then n does not have the $(3,3)$-Ramsey property.
4. Prove that $R(2,k) = k$ for all integers $k \geq 2$.
5. Prove that $R(i,j) = R(j,i)$ for all integers $i \geq 2$, $j \geq 2$.
6. Suppose that m has the (i,j)-Ramsey property and $n > m$. Prove that n has the (i,j)-Ramsey property.
7. Suppose that m does not have the (i,j)-Ramsey property and $n < m$. Prove that n does not have the (i,j)-Ramsey property.
8. Prove that $R(i_1, j) \geq R(i_2, j)$ if $i_1 \geq i_2$.
9. In the proof of Lemma 1, prove that if $|B| \geq R(i, j-1)$, then m has the (i,j)-Ramsey property.

⋆10. Suppose $i \geq 3$, $j \geq 3$, and $R(i, j-1)$ and $R(i-1, j)$ are even integers. Prove that $R(i,j) \leq R(i,j-1) + R(i-1,j) - 1$. (*Hint*: Follow the proof of Lemma 1, choosing the vertex v so that it has even degree.)

11. Prove that the graph in Figure 2(b) does not contain K_4 as a subgraph.

⋆12. Draw K_{13} and color its edges red and green according to the rule in Example 3. Then prove that the graph has no red K_3 or green K_5.

13. Use inequality (1) of Lemma 1 to prove that $R(4,4) \leq 18$. (*Note*: Roberts [9, p.330] contains a coloring of K_{17} that contains no monochromatic K_4. This proves that $R(4,4) = 18$.)

14. Prove that if G has nine vertices, then either G has a K_4 subgraph or $\overline{G}$ has a K_3 subgraph.

15. Prove Theorem 3.

$\star$16. Suppose $i_1 = \cdots = i_n = 2$. Prove that $R(i_1, \ldots, i_n; 2) = 2$, for all $n \geq 2$.

$\star$17. Prove that $R(7, 3, 3, 3, 3; 3) = 7$. (*Note*: This example can be generalized to $R(m, r, r, \ldots, r; r) = m$ if $m \geq r$).

18. Prove that $S(2) = 5$ by showing that $x + y = z$ has a monochromatic solution no matter how $1, 2, 3, 4, 5$ are colored with two colors.

$\star$19. Prove that $S(3) > 13$ by showing that there is a coloring of $1, \ldots, 13$ with three colors such that the equation $x+y = z$ has no monochromatic solution.

$\star$20. Prove that for every five points in the plane (where no three are collinear), four of the five points can be connected to form a convex 4-gon.

21. Consider the following game for two players, A and B. The game begins by drawing six dots in a circle. Player A connects two of the dots with a red line. Player B then connects two of the dots with a green line. The player who completes a triangle of one color wins.
   a) Prove that this game cannot end in a draw.
   b) Prove that this game can end in a draw if only five dots are used.
   c) Suppose the game is played with six dots, but the player who completes a triangle of one color loses. Prove that the game cannot end in a draw.

22. Suppose the game in Exercise 20 is modified for three players, where the third player uses the color blue.
   a) Prove that the game can end in a draw if 16 points are used.
   b) Prove that there must be a winner if 17 points are used.

23. Suppose the game of Exercise 20 is played by two players with 54 dots, and the winner is the first player to complete a K_5 of one color. If the game is played and happens to end in a tie, what conclusion about a Ramsey number can be drawn?

24. Verify that $R(K_{1,1}, K_{1,3}) = 4$.

Computer Projects

1. Write a computer program that proves that $R(3, 3) \leq 6$, by examining all colorings of the edges of K_6 and showing that a red K_3 or a green K_3 is always obtained.

2. Write a computer program that checks all edge-colorings of K_5 and determines the number of colorings that contain neither a red K_3 nor a green K_3.

3. Write a computer program that finds all colorings of $\{1, \ldots, 13\}$ with three colors such the the equation $x + y = z$ has no monochromatic solution.