DISTANCES, VOLUMES, TIME SCALES IN COSMOLOGY

Including Lambda CDM, but not yet quintessence…

A quick reference guide
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Preface

Many years ago I made a compendium of cosmological formulas, since it was my feeling that I often and repeatedly wasted time in re-deriving distances, volume elements, look-back times, etc. in different model universes, while it was difficult to find a source in the then existing literature that presented such formulas in a satisfactory way. Some of the papers of Sandage in the early sixties came close, but were not complete (see for example Sandage 1961a and Sandage 1961b).

Although the collection of formulas was for personal use only and, to be honest, the very first version was actually more of an exercise for word processors (not yet on PCs but a kind of type-writers with a bit of memory) subsequent technological and software developments (for example LaTex, MSWord) made copying easy. I distributed an early version of the compendium among some colleagues of the Observatory and the Radio Astronomy Institute in Bologna.

A second version was made (in Microsoft Word and html) around 1996, which included now a discussion of flat models with non-zero cosmological constant. I thought that this would be fine for a long time, but new observations and theoretical work have provoked a rapid change in our view of the structure of the Universe. The main causes are certain observations of supernovae, which definitely suggest a re-acceleration of the expansion of the universe, and the incredibly fast evolving field of the CMB. Some other theoretical and observational developments are important as well. All evidence now points to the existence of components other than the "normal" baryonic material: the presence of dark matter is well established, but in particular the power spectrum of the CMB fluctuations can be reproduced by introducing yet another component, the Dark Energy (that is a non-zero cosmological constant, or alternatively a rather mysterious energy component exerting a negative pressure). This latter constituent also goes under the name of Quintessence; a modern, very extensive, discussion can be found in Peebles (2004). The equation of state of such a component is $p=w\rho$, with w negative. This kind of model is much liked by theoreticians, because a cosmological constant (i.e. truly constant) poses some serious problems of interpretation; and in particular there is an enormous discrepancy between value of the vacuum energy (thought to be the source of the cosmological constant) and the value measured for Lambda.
I rearranged this compendium a bit, but did not yet include the Quintessence models (lack of time). I limited the discussion to those models that have a flat space (k=0), although I decided to retain a selection of other models that are of interest for historical reasons only.

It is always a bit annoying -at least to me it is- if you have to use formulas without knowing where they come from. For that reason I first give a discussion of the derivation of the formulas, starting from the Robertson-Walker metric and the Einstein equations. It may indeed be useful to have a quick reference guide, where you can, for example, look up where this or that 1+z factor comes from.

The compendium itself is given in the second part (the appendices), and it is ordered according to the different cosmological models.

Obviously I do not at all pretend to replace a textbook or a lecture course on cosmology; what I do hope is that the compendium can provide a quick and easy way to find the right formula. As such it is intended to be a complement to the usual texts on cosmology, which normally deal with more important and fundamental matters like general relativity.

It would be very surprising if there were no errors, typographical or other, in the following list of equations. Please feel free to send me comments via E-mail (h.deruiter@ira.inaf.it); you can find this compendium on the WEB: see my Home Page, at:

http://www.ira.inaf.it/~deruiter.

You may find this compendium as a file in pdf format, which you can, if you so wish, freely download.

This version was written and completed in June 2005; note that my WEB and E-mail addresses have changed.

1 Introduction

An observational extra-galactic astronomer always needs to have at hand some cosmological tools, for example for the conversion of observed parameters like apparent magnitudes, fluxes, angular diameters to the corresponding intrinsic parameters. Obviously to do that you will have to adopt a particular cosmological model. Often you will see in an astronomical paper some remark like ``we used a model with H_0 = 50 km s^{-1} Mpc^{-1} and q_0=1/2''. In recent years non-standard models (that is with non-zero cosmological constant) have become very popular, in the first place because flat models with positive Λ are a natural consequence of the inflationary-universe theory, and second, because there are now also good observational reasons why such models are feasible. This makes it al the more important to know how to compute distances, look-back times.

This article is organized as follows. First, in this chapter I give some general background information on the various parameters that are relevant for our problem, and use as a starting point (1) the Robertson-Walker metric and (2) the Einstein equations. I decided these are good starting points: no discussion is given of their derivation (I would have to rewrite a textbook on gravitational cosmology, which is far beyond my capacity), nor do I discuss other than homogeneous and isotropic models.
Specific models are then described in Chapters 3 (the standard Friedmann model), 4 (flat models) and 5 (other models).
These chapters make this guide longer than it could have been, but they can be useful for retracing the origin of a formula.
A list of symbols used is given in Appendix A. The compendium of formulas (the ultimate reason for the existence of this guide) is given in Appendices B, C, and D.

1.1 The metric of Space-Time

I only use the metric of a homogeneous and isotropic space-time, often called the Robertson-Walker metric:

$$ds^2 = c^2dt^2 - R^2(t)\left\{\frac{dr^2 + r^2d\theta^2 + r^2 sin^2\theta d\phi^2}{\left(1+\frac{kr^2}{4}\right)^2}\right\}$$

The variables r, θ, ϕ are *co-moving* coordinates; this means that the expansion of the universe is represented by R=R(t). The co-moving coordinates (r, θ, ϕ) do not depend on time and therefore an object will have fixed values of its co-moving coordinates.
The parameter k can be negative, zero, or positive, but without loss of generality we can assign it the possible values -1, 0, or +1. If k=–1 space is negatively curved, while k=+1 corresponds to a positive curvature; for k=0 space is flat.

There is another, more useful, form of the metric, obtained by changing the coordinate r to ω, according to:

$$\sin\omega = \frac{r}{1+\frac{r^2}{4}}$$

For k=+1, ω=r if k=0, while for k=–1, we set

$$\sinh\omega = \frac{r}{1-\frac{r^2}{4}}$$

This can be written more compactly as:

$$\sin\frac{\sqrt{k}\omega}{\sqrt{k}} = \frac{r}{1+\frac{kr^2}{4}}$$

Remembering that by definition sin(ix)/i = sinh x, and cos(ix) = cosh x. Moreover we can take the limit $k \to 0$ in order to find the correct equations for the case k=0. Using this compact form we can now write the equation for the metric as:

$$ds^2 = c^2dt^2 - R^2(t)\{d\omega^2 + \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2)\}$$

This is the form we will always use in the following.

1.2 The redshift

From the metric given in Chapter 2.1 we immediately can find an expression for the redshift as a function of R. Write R_0 for the value of the scale parameter at the present epoch and R_1 for the value it had at the time of emission of a photon. Then the photon will be redshifted by

$$1+z = \frac{R_0}{R_1}$$

Proof: for a photon ds=0, and since ω of the emitting source is fixed, while we can choose θ=ϕ=0, then:

$$\omega = c\int_{t_1}^{t_0}\frac{dt}{R}$$

where t_1 and t_0 are respectively the times of emission and reception of the beginning of the wave packet. For the end of the wave packet we get:

$$\omega = c \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{R}$$

and $\Delta t_0/\Delta t_1 = R_0/R_1$. But Δt is related to the frequency of the photon $(\nu \propto 1/\Delta t)$, and it follows that:

$$1+z = \frac{\nu_{em}}{\nu_{obs}} = \frac{R_0}{R_1}$$

1.3 Distances

It is up to us to define a coordinate distance, and good candidates would be r, or $\sin(\sqrt{k}\omega)/\sqrt{k}$. Of course we would prefer a coordinate distance that has a close relation with the observations. A distance that makes sense is the luminosity distance, which is defined from $S=P/4\pi D^2$ (S is flux, P luminosity), and therefore we search for an expression for D in terms of a coordinate distance. A detailed discussion can be found in McVittie (1965), p. 163--165.
In the usual definition of flux (energy flow per unit area) we take the area of a sphere around the power source and divide the emitted power by this area. In Euclidean space the area of the sphere at distance D, where the observer is on the sphere[1], is found as:

$$A = \int_0^{\pi}\int_0^{2\pi} D^2 \sin(\theta)\, d\theta\, d\phi = 4\pi D^2$$

However, in the metric given in Chapter 2.1 space is not necessarily flat and we therefore must take the area of the pseudo-sphere around the source:

$$A = \int_0^{\pi}\int_0^{2\pi} R_0^2 \left(\frac{\sin\sqrt{k}\,\omega}{\sqrt{k}}\right)^2 \sin\theta \cdot d\theta \cdot d\phi = 4\pi R_0^2 \left(\frac{\sin\sqrt{k}\,\omega}{\sqrt{k}}\right)^2$$

where the ω factor takes account of the curvature of space. The above suggests that a relation between flux and power might be:

$$S = \frac{P}{A} = \frac{P}{4\pi R_0^2 \sin(\sqrt{k}\,\omega)/\sqrt{k}}.$$

Clearly a useful coordinate distance is $\sin\{\sqrt{k}\,\omega\}/\sqrt{k}$, and we call this the geometric distance r_g.

[1] We put the observer at ω and the source at zero, but in the end put back the observer at zero; this can be done because space-time is homogeneous and isotropic

We have not finished yet however, because there are two more effects we still have to take into account. Call $\varepsilon_{em} = h\nu_{em}$ the energy of an emitted photon. In a time interval Δt_{em} there are n photons emitted, so that the emitted power is $P_{em} = \varepsilon_{em} n / \Delta t_{em}$.
First effect (the energy effect): the wavelength of a photon is redshifted, or $\nu_{obs}=\nu_{em}/(1+z)$, so that $\varepsilon_{obs} = \varepsilon_{em}/(1+z)$.
Second effect (the number effect): photons will arrive at a slower rate. At the source n photons were counted in the interval Δt_{em}; but $\Delta t_{em} = \Delta t_{obs}/(1+z)$, so that the same photons are observed to arrive in an $(1+z)$ times longer interval.
Taking the two effects together and calling the ``observed'' power (from flux and geometric distance) P_{obs}, we have:

$$P_{obs} = \varepsilon_{obs} \times \frac{n}{\Delta t_{obs}} = P_{em} / (1+z)^2 .$$

Finally:

$$S = \frac{P_{em}}{4\pi (1+z)^2 R_o^2 r_g^2} .$$

We can introduce here the luminosity distance D as:

$$D = (1+z) R_0 r_g$$

The derivation given above concerns bolometric fluxes and powers; if we observe in a limited frequency band we also have to take into account that the observed and emitted bandwidths are different by a factor $(1+z)$, and that the flux at frequency ν refers to the power emitted at $\nu(1+z)$.

1.4 Angular Size

The usual formula for angular size $\Delta\theta$ is: $\Delta\theta = L/d_\theta$, where L is the linear size and d_θ the distance, which we now have to specify.
Take two points A and B with co-moving coordinates (ω,θ,ϕ) and $(\omega,\theta+\Delta\theta,\phi)$. A and B could be for example the two components of a double radio source. We assume that a photon is emitted from A, and another one from B at time t_1; A and B are connected with the origin $(0,0,0)$ by null geodetics. The local separation L at time t_1 can be found by putting $dt=d\omega=d\phi=0$, so that $\Delta s^2 = -L^2 = -R_1^2 r_g^2 \Delta\theta^2$ and:

$$\Delta\theta = \frac{(1+z)L}{R_0 r_g}.$$

We can make the identification $d_\theta = R_0 r_g/(1+z)$ and, if we really want, use the standard formulas $S=P/4\pi D^2$ and $\Delta\theta = L/d_\theta$, remembering that:

$$R_0 r_g = (1+z) d_\theta = \frac{D}{(1+z)}$$

$$d_\theta = \frac{D}{(1+z)^2}$$

1.5 The Volume

The co-moving coordinates of an object are, by definition, fixed. Consequently the number of objects (if they are not created or destroyed) per co-moving volume remains constant and therefore this type of volume is the relevant one for computing e.g. a luminosity function. From the metric in Chapter 2.1 we have:

$$dV(\omega,\theta,\varphi) = R_0 d\omega R_0 \frac{\sin\sqrt{k}\omega}{\sqrt{k}} d\theta R_0 \frac{\sin\sqrt{k}\omega}{\sqrt{k}} \sin\theta d\phi$$

Integrating over ω, θ and ϕ we find the volume out to coordinate distance ω:

$$V(\omega) = 4\pi R_0^3 \int_0^{\omega} \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}} \right)^2 d\omega$$

and we can make the change $V(\omega)$ to $V(z)$ if we know $\omega = \omega(z)$; the differential volume in a redshift shell is then simply $(dV/dz)dz$.

1.6 How to compute ω

Let a photon be emitted at time t_1 and be received at t_0 and let it have fixed coordinates θ and ϕ. Since $ds=0$ we find

$$\omega = c\int_{t_1}^{t_0} \frac{dt}{R(t)}$$

where $\omega(t_0) = 0$.

If we can find, at the right-hand side, an expression in terms of H_0, q_0, z then we have solved $\omega = \omega(H_0, q_0, z)$ and distances, volumes, etc. are known in terms of H_0, q_0 and z.

1.7 The scale parameter R

We have already seen that in general $R_0/R_1=(1+z)$. The solution $R=R(t)$ is found from the Einstein equations which I give here without a derivation:

$$8\pi G\rho = \frac{3kc^2}{R^2} + \frac{3\dot{R}^2}{R^2} - \Lambda$$

$$8\pi G\frac{p}{c^2} = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \Lambda$$

It follows directly that:

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{dR^3}{dt} = 0$$

where ρ is the density, p the pressure, and Λ the cosmological constant. The Einstein equations are obtained by equating the curvature tensor to the energy-momentum tensor.

In the most general case the equations are very difficult to solve because we need to know ρ and p (and thus an equation of state), as well as Λ. Although models with non-zero pressure have been calculated (for example assuming a polytropic relation $p \propto \rho^\gamma$), one normally takes $p=0$, so that the Einstein equations become:

$$8\pi G\rho = \frac{3kc^2}{R^2} + \frac{3\dot{R}^2}{R^2} - \Lambda$$

Also:

$$\frac{d}{dt}(\rho R^3) = \frac{d}{dR}(\rho R^3) = 0$$

Important parameters are the Hubble parameter and the deceleration parameter:

$$H = \frac{\dot{R}}{R}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2}$$

1.8 Time scales

Call the present cosmic time t_0; it can be found by measuring H_0, i.e. $t_0=t_0(H_0)$ the functional relation being different for different models. $(H_0)^{-1}$ often called the Hubble time, but it should be kept in mind that in Friedmann models $(\Lambda=0)$ always $t_0 \leq (H_0)^{-1}$.
Another important measure of time is the look-back time τ, defined as

$$\tau = 1 - \frac{t_1}{t_0}$$

We see that $t_1=t_0 \rightarrow \tau=0$, and $t_1=0 \rightarrow \tau=1$. The name look-back time is obvious.

2 The standard (Friedmann) model: $\Lambda=0$

2.1 General characteristics

The Friedmann models are the most important cosmological models and almost exclusively used in observational astronomy. Since $\Lambda=0$, we start from the simplified Einstein equations:

$$\frac{8\pi G\rho}{3} = \frac{kc^2}{R^2} + \frac{\dot{R}^2}{R^2}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = 0$$

As can be seen from above, the first equation states that $\Lambda=0$ implies $q=\sigma$, and therefore the Einstein equations reduce to one (the second). Note that the Friedmann models always have $q\geq0$, because of course or σ or ρ are always ≥0[2].

The second Einstein equation provides a relation between R_0, H_0, q_0:

$$R_0 = \sqrt{\frac{k}{2q_0 - 1}}\frac{c}{H_0}$$

for $k=\pm1$. It follows that

- $k = +1 \rightarrow q_0 > ½$
- $k = 0 \rightarrow q_0 = ½$
- $k = -1 \rightarrow 0 \leq q_0 < ½$

2.2 The solution of R and t in parametric form

The solution of the Friedmann-Einstein equation is well known from equivalent differential equations used in mechanics. Introducing the ``development'' angle ψ we write $R=R(\psi)$ and $t=t(\psi)$.

Remembering that:

$$\dot{R} = \frac{dR}{d\psi}\left(\frac{dt}{d\psi}\right)^{-1}$$

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[2] But negative pressure now has become a definite possibility, since the Quintessence models have been proposed. I should get around discussing these too.

$$\ddot{R} = \left(\frac{dt}{d\psi}\right)^{-2}\left(\frac{d^2R}{d\psi^2} - \frac{dR/d\psi}{dt/d\psi}\frac{d^2t}{d\psi^2}\right)$$

Taking $dt/d\psi = R/c$ we get:

$$-\left(\frac{dR}{d\psi}\right)^2 - 2\frac{d^2R}{d\psi^2}R + kR^2 = 0$$

The solution is:

$$R = \frac{a}{k}(1 - \cos\sqrt{k}\psi).$$

Since $t = \int(R/c)d\psi$:

$$t = \frac{a}{kc}\left(\psi - \frac{\sin\sqrt{k}\psi}{\sqrt{k}}\right).$$

The solutions are valid for all k:

- For $k = -1$ we use $\cos(ix) = \cosh(x)$ and $\sin(ix)/i = \sinh(x)$, then:
  - $R(k{=}-1) = a\{\cosh(\psi)-1\}$,
  - $t(k{=}-1) = (a/c)\{\sinh(\psi)-\psi\}$
- For $k = +1$ we have:
  - $R(k{=}+1) = a\{1-\cos(\psi)\}$
  - $t(k{=}+1) = (a/c)\{\psi-\sin(\psi)\}$
- For $k = 0$ we first write $\cos(\sqrt{k}\psi) = 1 - k\psi^2/2$ and $\sin(\sqrt{k}\psi)/\sqrt{k} = \psi + k\psi^3/6$, since higher terms in k will be zero when taking the limit $k \to 0$. Thus:
  - $R(k{=}0) = a\psi^2/2$
  - $t(k{=}0) = a\psi^3/(6c)$

  In this case, however, the scale parameter R cannot be tied directly to (c/H_0); in fact we might take anything we like. Eliminating ψ we find

$$R(k=0) = \frac{6^{2/3}}{2}a^{1/3}c^{2/3}t^{2/3} \propto t^{2/3}$$

As a next step we eliminate a, by using $H_0 = (2/3)t_0$, and imposing that $R_0 = c/H_0$. We find that $a = (3/4)ct_0$, so we can finally write:

$$R(k=0) = \left(\frac{c}{H_0}\right)\left(\frac{t}{t_0}\right)^{2/3}$$

2.3 The Hubble and deceleration parameters

Using the definitions of H and q given in Chapter 1 the above solutions lead to:

$$H = \frac{c}{R^2}\frac{dR}{d\psi} = \frac{k^2 c}{a}\frac{\sin(\sqrt{k}\psi)/\sqrt{k}}{(1-\cos\sqrt{k}\psi)^2}$$

After some more manipulation:

$$q = \frac{1}{1+\cos\sqrt{k}\psi} \Leftrightarrow \cos\sqrt{k}\psi = \frac{1-q}{q}$$

The explicit relations for $k=-1$ and $k=+1$ are obvious and will not be repeated here. For $k=0$ we can go back to the relation between R and t given in Chapter 2.2, or take the appropriate limits $k\rightarrow 0$ in the formulas given here. Both methods give: $H(k=0)=(2/3)t^{2/3}$ and $q(k=0)\equiv 1/2$.

2.4 Relation between ω and ψ

There is a very simple relation between the radial co-moving coordinate ω and the development angle ψ. Since $c/R=d\psi/dt$ (see Chapter 2.2), we can write:

$$\omega = c\int_{t_1}^{t_0}\frac{dt}{R} = \int_{\psi_1}^{\psi_0} d\psi$$

so that:

$$\omega = \psi_0 - \psi_1$$

Therefore ω and ψ are the same thing, up to a different zero-point: ω is measured with respect to the observer (at $\omega=0$), while ψ gives an absolute scale starting at $R=0$ with $\psi=0$.

2.5 Expressing ψ in the observables q0 and z

Since $\cos(\sqrt{k}\psi)=(1-q)/q$, ψ can be expressed in terms of R; using $R_1=R_0/(1+z)$, we can write:

$$\cos\sqrt{k}\psi_1 = \frac{\cos\sqrt{k}\psi_0 + z}{1+z}$$

or, for $q_0 \neq 1/2$:

$$\cos\sqrt{k}\psi_1 = \frac{1-q_0+q_0 z}{q_0(1+z)}$$

With these formulas we have solved all our problems: ω is a difference between the development angles at t_1 and t_0, and therefore can be expressed in H_0, q_0, z. As a consequence this can also be done with the geometric distance, volume, etc. (see next sections), taking care that for $k = 0$ we take the appropriate limits only at the end of our calculations. The characteristic that all relevant cosmological quantities can be directly expressed in terms of observables is a very pleasant property of Friedmann models (and a few other non-standard models). It is not true in the general case $\Lambda \neq 0$.

2.6 *The geometric distance*

Going back to the original definition for r_g $(=\sin(\sqrt{k}\omega)/\sqrt{k})$ we can write:

$$r_g = \frac{\sin\sqrt{k}(\psi_0-\psi_1)}{\sqrt{k}} = \frac{\sin\sqrt{k}\psi_0\cos\sqrt{k}\psi_1 - \cos\sqrt{k}\psi_0\sin\sqrt{k}\psi_1}{\sqrt{k}}$$

With the knowledge of Chapter 2.5 this can be transformed into:

$$r_g = \left(\frac{2q_0-1}{k}\right)^{1/2}\frac{q_0 z+1-q_0+(q_0-1)(2q_0 z+1)^{1/2}}{q_0^2(1+z)}$$

but since $\sqrt{\{(2q_0-1)/k\}}=c/(R_0H_0)$:

$$r_g = \frac{c}{R_0H_0}\frac{q_0 z+1-q_0+(q_0-1)(2q_0 z+1)^{1/2}}{q_0^2(1+z)}$$

This is the famous Mattig formula (Mattig 1958). Surprisingly enough, Mattig derived it many years after the original work of Friedmann in the early 1920's, and up till then one had to use cumbersome expansions in powers of z. It illustrates how the practical use of cosmology has often lagged behind the theory[3].

Although the limit for $q_0 \rightarrow 0$ exists, Mattig's formula is not very nice in case of very small q_0, because both numerator and denominator go to infinity, as q_0 tends to zero. Terrell (1977) obtained an alternative (and better) form of Mattig's formula (see also Peterson 1997). First we note that the luminosity distance D $(= R_0 r_g(1+z))$ is the difference two numbers:
$A = (1-q_0+q_0 z)/q_0^2$ and $B = (1-q_0)(2q_0 z+1)^{1/2}/q_0^2$.

[3] Mattig's formula was derived without any reference to particular values of k, or q_0, and indeed, it is valid for all k and all $q_0 \geq 0$. For $q_0 = 0$ we can take the limit by expanding the square-root term in powers of $q_0 z$ up to the second order. Zero- and first order terms in q_0 in the numerator cancel exactly, so the limit exists.

We want to write $A-B \propto z(1+C_z)$: since for $q_0=0$ we know that $D= (c/H_0)\, z(1+z/2)$ the latter form should be well behaved. Indeed, solving C_z in terms of A and B and writing $Q=(1+2q_0z)^{1/2}$ we get:

$$D = \frac{cz}{H_0}\left\{1 + \frac{1-q_0}{q_0^2 z}(1+q_0 z - Q)\right\}$$

Multiplying the last term with $(1+q_0z+Q)/(1+q_0z+Q)$ we finally arrive at Terrell's alternative formula:

$$D = \frac{cz}{H_0}\left\{1 + \frac{z(1-q_0)}{1+q_0 z + (1+2q_0 z)^{1/2}}\right\}$$

2.7 The co-moving volume

The general formula for the volume can be integrated immediately:

$$V(\omega) = \frac{2\pi R_0^3}{k}\left(\omega - \frac{\sin 2\sqrt{k}\omega}{2\sqrt{k}}\right)$$

We could leave it at that, since ω is known in terms of (H_0, q_0, z). Although even in the general case we could write the volume explicitly as a function of the observables, I think it is not worth the trouble of doing this (the formulas would be cumbersome): with present day computers (including pocket calculators), it is much easier to calculate ω first (using the formulas in Chapters 2.4 and 2.5) and then apply the above equation. However there are some special values of q_0 where the volume can be easily written in terms of redshift directly. We deal with these cases below.

- $q_0=0$ (the Milne model). Since $\sinh\omega = (z+z^2/2)/(1+z)$ and $\operatorname{arsinh}(x) = \ln\{\sqrt{(x^2+1)}+x\}$:

$$\omega = \ln(1+z)$$

Then, after some more manipulation [using $(1/2)\sinh\omega=\sinh\omega\cosh\omega$]:

$$V(z) = 2\pi\left(\frac{c}{H_0}\right)^3\left\{\frac{1}{4}\frac{(1+z)^4-1}{(1+z)^2} - \ln(1+z)\right\}$$

$$\frac{dV}{dz} = 4\pi\left(\frac{c}{H_0}\right)^3\frac{(1+\frac{1}{2}z^2)^2}{(1+z)^3}$$

- $q_0=1/2$ and $k=0$ (Einstein-de Sitter model). The volume in terms of ω is simply:

$$V(\omega)=\frac{4}{3}\pi R_0^3\omega^3$$

Using the fact that Mattig's formula gives

$$r_g=\omega=2\left\{1-(1+z)^{-1/2}\right\}$$

We have:

$$V(z)=\frac{32}{3}\pi\left(\frac{c}{H_0}\right)^3\left\{1-\frac{1}{(1+z)^{1/2}}\right\}^3$$

$$\frac{dV}{dz}=16\pi\left(\frac{c}{H_0}\right)^3\frac{\left\{(1+z)^{1/2}-1\right\}^2}{(1+z)^{5/2}}$$

- $q_0=1$. Using $\sin(2\omega)=2\sin\omega\,(1-\sin^2\omega)^{1/2}$:

$$V(z)=2\pi\left(\frac{c}{H_0}\right)^3\left\{\arcsin\left(\frac{z}{1+z}\right)-\frac{z(1+2z)^{1/2}}{(1+z)^2}\right\}$$

$$\frac{dV}{dz}=4\pi\left(\frac{c}{H_0}\right)^3\frac{z^2}{(1+z)^3(1+2z)^{1/2}}$$

In all other cases we use ω and $V(\omega)$.

2.8 Time Scales

2.8.1 The look-back time τ

By definition

$$\tau=\frac{t_0-t_1}{t_0}$$

It is normalized such that $t_1=t_0\rightarrow\tau=0$ and $t_1=0\rightarrow\tau=1$. From the solution of t in terms of ψ (Chapter 2.2) τ can be written as

Commento [HDR1]:

$$\tau=1-\frac{\psi_1-\sin\sqrt{k}\psi_1/\sqrt{k}}{\psi_0-\sin\sqrt{k}\psi_0/\sqrt{k}}$$

As for the volume the easiest way to proceed is to find ψ_0 and ψ_1 in terms of the observables H_0, q_0, z. Below I give the explicit expressions for τ as a function of the observables, for the cases $q_0=0, 1/2, 1$.

- $q_0 = 0$. Taking the limit $q_0 \to 0$ the above equation yields:

$$\tau = \frac{z}{1+z}$$

A much easier method, which obviously gives the same result: $q=0$ implies $d^2R/dt^2=0$, so that $R=ct$ and consequently:

$$\frac{R_0}{R_1} = \frac{t_0}{t_1} = 1+z$$

This leads immediately to the expression for τ.

- $q_0 = ½$. From the equation given in Chapter 2.2:

$$\tau = 1 - \left(\frac{\psi_1}{\psi_0}\right)^3$$

and therefore:

$$\tau = 1 - \frac{1}{(1+z)^{3/2}}$$

- $q_0 = +1$. Then $\cos\psi_0 = (1-q_0)/q_0 = 0$, or $\psi_0 = \pi/2$. Also $\cos\psi_1 = (\cos\psi_0+z)/(1+z)=z/(1+z)$, so that:

$$\tau = 1 - \frac{\arccos\left(\frac{z}{1+z}\right) - \left\{1 - \left(\frac{z}{1+z}\right)^2\right\}^{1/2}}{\frac{\pi}{2} - 1}$$

2.8.2 The age of the Universe (t_0) expressed in H_0 and q_0.

Taking together the expressions for t and H in terms of ψ (Chapters 2.2 and 2.3) and we get:

$$t_0 H_0 = \left(\frac{q_0}{k}\right)\left(\frac{k}{2q_0 - 1}\right)^{3/2} (\psi_0 - \sin\sqrt{k}\psi_0 / \sqrt{k})$$

- $q_0=0$. Then clearly, since $R_0=ct_0$, and $R_0=c/H_0$

$$t_0 H_0 = 1$$

This can, as before, also be derived by taking the limit $q_0 \rightarrow 0$ in the general formula.

- $q_0=1/2$. Going back to Chapter 2.2 we directly have:

$$t_0 H_0 = 2/3$$

- $q_0=1$. Applying the general formula with $k=1$ and $q_0=1$:

$$t_0 H_0 = \frac{\pi}{2} - 1$$

With these formulas the description of the Friedmann models is complete.

3 Flat Models ($k=0$, $\Lambda\neq 0$)

3.1 Introductory Remarks

With the exception of the Einstein-de Sitter model, flat models ($k=0$) have never attracted much attention. The reasons are that a zero cosmological constant is appealing from a philosophical point of view, and anyway there has never been any compelling observational reason why we should introduce still another parameter.
Nevertheless every now and then non-standard models become popular with cosmologists. Usually this popularity quickly fades away, but in one occasion (the most recent one) a non-Friedmann model appears to have a good chance to survive for a long time. The introduction of the ``inflationary universe'' is the cause of renewed interest for a model with $\Lambda \neq 0$: according to the theory of inflation the cosmological constant might very well be positive: it is related to the properties of the vacuum, see Börner (1988). Also, the metric of space would be flat.

The matter is very complicated, because it is also claimed that very stringent observational limits can be set upon Λ, but a discussion of these points is obviously far beyond the scope of the present work. What makes the flat models with a positive cosmological constant so important is that there is now (from about the year 2000) serious observational evidence that this type of model may actually the correct description of the universe. Therefore I give the general equations for flat models, with particular emphasis on the ones with $\Lambda>0$; also I write the equations for a model with $\Omega_M = 1/3$ and $\Omega_\Lambda = 2/3$ (see for the definition below), since this is the model that is presently (2002) favoured by many cosmologists.

It is worth wile to look a bit closer at the Einstein equations, by writing them in terms of present day Hubble parameter, deceleration parameter and density parameter (defined as $\sigma_0 = 4\pi G\rho_0/(3H_0^2)$); then:

$$\Lambda = 3H_0^2(\sigma_0 - q_0)$$

$$k\left(\frac{c}{R_0 H_0}\right)^2 = 3\sigma_0 - q_0 - 1$$

It is appropriate to introduce at this point two often used parameters that replace in the modern literature σ_0 and Λ; they are $\Omega_M = 2\sigma_0$ and $\Omega_\Lambda = \Lambda/3H_0^2$. It is easy to see from the above equations that $\Omega_M + \Omega_\Lambda = 1$, for $k = 0$. The ratio of these two parameters will be used extensively in this chapter, and is denoted by $A = \Omega_M/\Omega_\Lambda$.

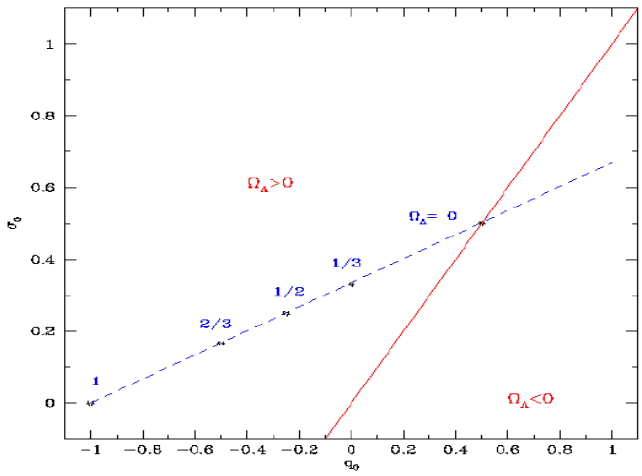

Figure 1 The q0-σ0 plane. The lines representing Friedmann models and flat models are drawn.

For a positive cosmological constant the value of *A* will be between zero (a pure deSitter model, see Chapter 4.2) and infinity (a pure Einstein-deSitter model, see Chapter 2).

In fig. 1 we show the q_0-σ_0 plane. The parts of the plane with $\Omega_M>0$ and $\Omega_\Lambda<0$ are indicated. We can recognize various distinct regions: the two lines $\sigma_0=q_0$ ($\Lambda=0$) and $3\sigma_0=q_0+1$ ($k=0$) give two special families of (relatively simple) solutions, discussed in detail in Chapter 2 and in this Chapter. More general solutions will only be described briefly (and limited to a few special cases), (see Chapter 4) not only because they can be complicated, but also because they are, with few exceptions, not very interesting. I looked a bit in more detail at models with $k=0$ and $\Lambda\neq 0$. I will first show that a general solution exists for all flat models, regardless of the value of Λ, somewhat similar to the general solution for Friedmann models.

3.2 The general solution

As a starting point we use the Einstein equations with $k=0$:

$$\frac{8\pi G\rho_0 R_0^3}{3R^3} = \frac{\dot{R}^2}{R^2} - \frac{\Lambda}{3}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}=\Lambda$$

It can be easily verified that the solution of this system of two equations is:

$$R=(\varepsilon A^*)^{1/3}R_0\left(\frac{\sin\gamma^* t}{\sqrt{\varepsilon}}\right)^{2/3}$$

If $\Lambda>0 \rightarrow \varepsilon=-1$, if $\Lambda<0 \rightarrow \varepsilon=+1$; also:

$$A^*=\frac{8\pi G\rho_0}{-\Lambda}$$

$$\gamma^*=\frac{1}{2}\sqrt{-3\Lambda}$$

Going back to Chapter 3.1 we can see that:

- if $\Lambda < 0$: $\Lambda/3H^2=1-2\sigma$, so that $\sigma>1/2$, while $A^*=2\sigma/(2\sigma-1)>1$.
- if $\Lambda>0$: $\sigma<1/2$, and consequently $-A^*=2\sigma/(1-2\sigma)>0$.

We can in principle write general solutions for H, q, τ, etc., however, only the case $\Lambda>0$ is of interest and will be discussed in full below: for σ, q $>1/2$ the cosmological constant is an extra attractive force, and its effect is only to keep space flat, compared to the corresponding Friedmann model, for which $q_0>1/2$ implies a positive space curvature. Therefore from now on we will only consider $\Lambda>0$. We can now write:

$$R=A^{1/3}R_0\sinh^{2/3}(\gamma t)$$

Here we have used $A=-A^*$:

$$A=\frac{8\pi G\rho_0}{\Lambda}=\frac{\Omega_M}{\Omega_\Lambda}$$

and

$$\gamma=\frac{1}{2}\sqrt{3\Lambda}=\frac{3}{2}H_0\sqrt{\Omega_\Lambda}$$

So the general flat models with non-negative cosmological constant allow a direct solution of R in terms of t; but it can also be suspected that the rather complicated dependence (a hyperbolic sine to the two-third power) may create some problems in finding manageable expressions for geometric distance, volumes, etc. It will be

shown in the following that this is true only to some extent. Nevertheless the general properties of the models are quite simple.
A special case is the model that is nowadays favoured by many, with (roughly, the precise values may change as new observations become available): $\Omega_M=1/3$ and $\Omega_\Lambda=2/3$, so that $A = 1/2$. This model goes under the name of Concordance model. Then:

$$R=\left(\frac{1}{2}\right)^{1/3} R_0 \sinh^{2/3}(\sqrt{\frac{3}{2}}H_0 t)$$

3.3 H_0, q_0 and t_0 in terms of the parameter A

From the general solution we immediately find for $\Lambda>0$, by using $\sinh(\gamma t_0)=A^{-1/2}$ and $\cosh(\gamma t_0) = \{(1+A)/A\}^{1/2}$:

$$H_0=\sqrt{\frac{\Lambda}{3}}(1+A)^{1/2}=\sqrt{\frac{\Lambda}{3\Omega_\Lambda}}$$

and

$$q_0=\frac{1}{2}\left(\frac{A-2}{A+1}\right)=\frac{1}{2}(1-3\Omega_\Lambda)\Leftrightarrow A=2\frac{1+q_0}{1-2q_0}$$

and

$$t_0 H_0=\frac{2}{3}(1+A)^{1/2}\ln\{A^{-1/2}+(1+A^{-1})^{1/2}\}=\frac{2}{3}\Omega_\Lambda^{-1/2}\ln\left(\frac{1+\Omega_\Lambda^{1/2}}{\Omega_M^{1/2}}\right)$$

For $A=0.5$:

$$H_0=\sqrt{\frac{\Lambda}{2}}$$

$$q_0=-0.5$$

$$t_0 H_0=1.40\times 2/3=0.933$$

It can be easily seen that for $A \to \infty$, that is the usual Einstein-deSitter universe, we find (as we should) $H^2 \to 8\pi G\rho_0/3$, $q0 \to 1/2$,and $t_0H_0 \to 2/3$. For $A \to 0$ we have $H_0 \to \sqrt{(\Lambda/3)}$, $q_0 \to -1$ and $t_0H_0 \to \infty$, which indeed are the equations for a deSitter universe (see Chapter 4.2). Similar formulas can be obtained for the case $\Lambda < 0$.

3.4 H, q and τ in terms of A and z

Again from the solution we can find some general expressions for H(z), q(z) and the look-back time τ(z); for $\Lambda > 0$:

$$(1-2q)H^2 = \Lambda$$

$$H(z) = \sqrt{\frac{\Lambda}{3}}\{1 + A(1+z)^3\}^{1/2} = H_0\left(\frac{1 + A(1+z)^3}{1+A}\right)^{1/2}$$

$$1 - 2q(z) = \frac{3}{1 + A(1+z)^3}$$

Note that there may have been a moment in the past in which q was zero; this represents a transition point in which the early stage of deceleration was changing into one of acceleration. In the model with A=0.5 this happened at a redshift z(q=0)=0.587.
For the look-back time we get:

$$\tau = 1 - \frac{\ln[\{1 + A(1+z)^3\}^{1/2}] - \frac{3}{2}\ln(1+z) - \frac{1}{2}\ln A}{\ln\{1 + (1+A)^{1/2}\} - \frac{1}{2}\ln A}$$

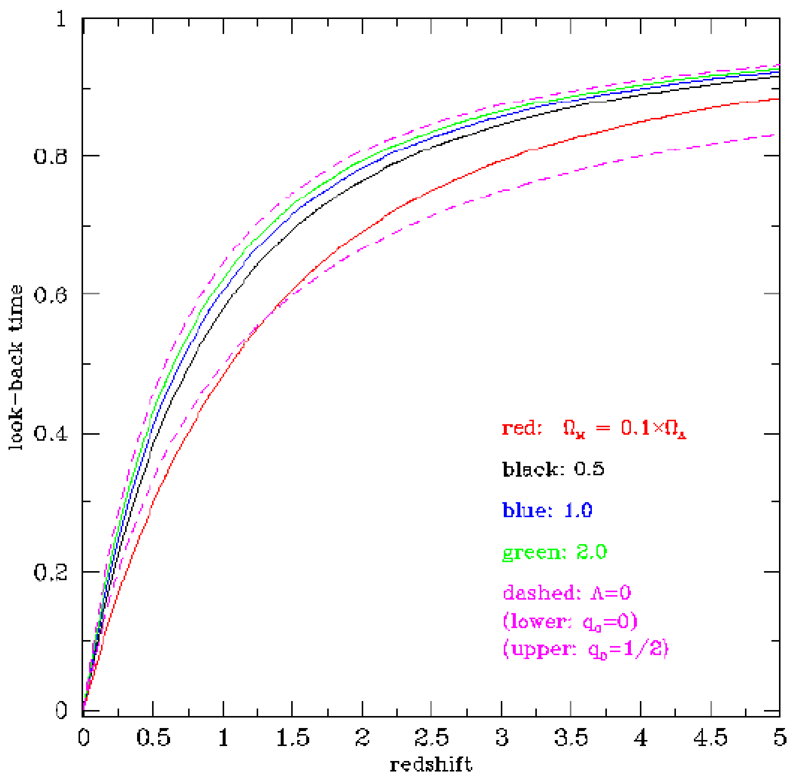

Figure 2 The look-back time for some flat models with positive cosmological constant. The full lines represent models with: A=0.1, 0.5, 1.0. The A=0.5 model is close to the Concordance model. Also shown are standard $q_0 = 0$ and $q_0 = 1/2$ models (lower and upper full lines respectively).

For A=0.5:

$$H(z) = H_0 \left(\frac{1 + \frac{1}{2}(1+z)^3}{\frac{3}{2}} \right)^{1/2}$$

$$q(z) = q_0 \left(\frac{1 - \frac{1}{4}(1+z)^3}{1 + \frac{1}{2}(1+z)^3} \right)$$

Therefore $q(z=\infty) \to 0.5$, because $q_0 = -0.5$.
We give no separate equation for $\tau(z, A=0.5)$, as this does not clarify much. The dependence of τ on redshift for different values of A is given in fig. 2.
Similar formulas are found for the case $\Lambda < 0$. The appearance of the term $A(1+z)^3$ suggests that it is a fundamental term for flat universes. This will be clear from the following section, where we discuss the geometric distance.

3.5 The geometric distance and volume elements

By definition the geometric distance is the integral in time of the inverse of the scale parameter. In our case we find therefore:

$$r_g = c \int_{t_1}^{t_0} \frac{dt}{R} = c \int_{t_1}^{t_0} \frac{dt}{A^{1/3} R_0 \sinh^{2/3} \gamma t}$$

Changing from t to z (using the fact that $(1+z) = R_0/R$), we find, after some manipulation:

$$r_g = \frac{c}{R_0 H_0} (1+A)^{1/2} \int_0^z \frac{d\zeta}{\{1 + A(1+\zeta)^3\}^{1/2}}$$

It is quickly verified that for $A \to \infty$ (Einstein-De Sitter), we have

$$r_g \Rightarrow \frac{c}{R_0 H_0} 2\{1 - \frac{1}{(1+z)^{1/2}}\}$$

and for $A \to 0$:

$$r_g \Rightarrow \frac{c}{R_0 H_0} z$$

These are indeed the correct formulas. Although the general formula for the geometric distance has no analytical solution, it is simple enough that a numerical integration is straightforward. From that we arrive immediately at the solution for the volume ($V=4\pi\ R_0^3 r_g^3/3$) and the differential volume $dV/dz = 4\pi R_0^3 r_g^2 dr_g dz$. The dependence of r_g on z for different values of A is given in fig. 3.

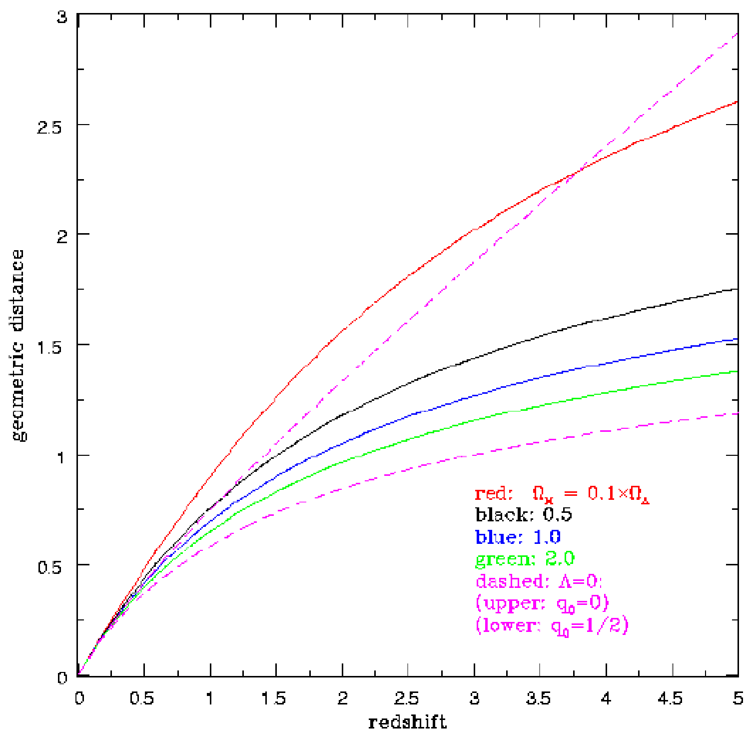

Figure 3: The geometric distance as a function of redshift, for A=0.1,0.5, 1.0 and 2.0. For comparison also the Friedmann models with q_0=0 and 0.5 are shown

4 A selection of Models with $k \neq 0$, $\Lambda \neq 0$

4.1 Zero-density model with q0 > 0

Although this model is unrealistic, because it presupposes $\sigma \equiv 0$, it is still of some interest: it allows q_0 to be of order one, while the matter density may be much smaller, such that $\sigma_0 = 0$ is a good first order approximation. As is immediately clear from the discussion in Chapter 3.1, a similar combination of σ and q, with $\sigma < q$, implies that $\Lambda < 0$. Therefore Λ provides a universal attractive force, which causes the deceleration parameter to be >0, even though no matter is present.

First we go back to the Einstein equations, putting $\rho = 0$:

$$\frac{3kc^2}{R^2} + \frac{3\dot{R}^2}{R^2} - \Lambda = 0$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} - \Lambda = 0$$

Since we require that $q_0 > 0$, it follows from the second Einstein equation as written in Chapter 3.1, that $3\sigma_0 - q_0 - 1 = -q_0 - 1 < 0$, so that $k = -1$.
Eliminating Λ from the two Einstein equations given above we get one equation:

$$\ddot{R}R - \dot{R}^2 + c^2 = 0$$

This can be written at present epoch in terms of the usual variables as:

$$R_0 = \frac{1}{(q_0+1)^{1/2}} \frac{c}{H_0}$$

The Einstein equation of this model allows an explicit solution $R = R(t)$, as can be easily verified by trying a form like $R = a \sin(bt)$. We find:

$$R = \frac{c}{H_0 q_0^{1/2}} \sin(H_0 q_0^{1/2} t)$$

with the boundary condition:

$$\sin(H_0 q_0^{1/2} t) = \left(\frac{q_0}{q_0+1}\right)^{1/2}$$

The Hubble and deceleration parameters as a function of t are:

$$H = H_0 q_0^{1/2} \cot(H_0 q_0^{1/2} t)$$

and:

$$q = \tan^2(H_0 q_0^{1/2} t)$$

Some comments:

- The model is closed, although it has a negatively curved metric (k=–1): R increases up to a maximum $c/(H_0 q_0^{1/2})$ at $t = \pi/(2H_0 q_0^{1/2})$, and goes to zero again at $t = \pi/(H_0 q_0^{1/2})$.
- The particular moment $q_0=1$ corresponds to $\sin(H_0 t_0)=(1/2)\sqrt{2}$, or $t_0 H_0 = \pi/4$.
- If one wishes to do so one can combine the solution and its boundary condition into one equation, to read:

$$R = \frac{1}{(q_0+1)^{1/2}} \frac{c}{H_0} [\cos\{H_0 q_0^{1/2}(t-t_0)\} + q_0^{-1/2} \sin\{H_0 q_0^{1/2}(t-t_0)\}]$$

Making use of the formulas $\int dx/\sin x = \ln \tan x/2$, $\tan x/2 = \sin x/ (\cos x + 1)$ and $\text{arcosh } x = \ln\{\sqrt{(x2\text{-}1)}+x\}$, and doing a bit of calculation we find:

$$r_g = \sinh \omega = \frac{c}{R_0 H_0} \frac{\{(1+q_0)(1+z)^2 - q_0\}^{1/2} - (1+z)}{q_0}$$

Therefore an equivalent of Mattig's formula does exist in this model. Note that for $q_0 \to 0$ we should recover the standard $q_0=0$ model of Milne, because $\sigma \equiv 0$, and thus $\Lambda \propto (\sigma_0 - q_0) \to 0$. It can be easily verified that taking the limit $q_0 \to 0$ in the above formula gives the correct expression for r_g.
The look-back time can be written as:

$$\tau = 1 - \frac{\arcsin\left\{\frac{q_0^{1/2}}{(q_0+1)^{1/2}(1+z)}\right\}}{\arcsin\left(\frac{q_0}{q_0+1}\right)^{1/2}}$$

and t_0 as:

$$t_0 H_0 = \frac{\arctan q_0^{1/2}}{q_0^{1/2}}$$

This completes our discussion of this model.

4.2 The Lemaître model: Λ>0 and k=+1

Before turning to the Lemaître model proper, two other very simple models should be briefly discussed, because they are closely related.

- The Einstein model. This is in fact the first cosmological model, given by Einstein. He tried to get a static solution (the recession of galaxies was only being discovered at that time by Hubble), and had to introduce the cosmological constant Λ to obtain this goal. Requiring R=constant we have $dR/dt=d^2R/dt^2=0$. The Einstein equations become:

$$8\pi G\rho + \Lambda = \frac{3kc^2}{R^2}$$

$$\frac{kc^2}{R^2} = \Lambda$$

It follows that $8\pi G\rho = 2kc^2/R^2$, or $k=1$ (since of course $\rho>0$) and $\Lambda>0$. We find

$$R_E = \frac{c}{\sqrt{\Lambda}}$$

and

$$\frac{kc^2}{R^2} = \Lambda$$

- The De Sitter model, which was already discussed in Chapter 3, as a limiting case of the flat models. Its relevance in this section comes from the fact that it is also a limiting case for the Lemaître models. We summarize some results:

$$q \equiv -1$$

$$R = R_0 e^{H_0(t-t_0)}$$

$$H = \sqrt{\frac{\Lambda}{3}}$$

The Lemaître model was one of the most famous models in the 1930's, and was revived for a short time after about 1965, as attempts were made to explain the

(apparently significant) redshift peak of quasars near $z = 2$. The reason for this is made clear below. For more information see for example Petrosian (1967).
In the Lemaître model we start with $k = +1$ and $\Lambda > 0$, just like in the Einstein model. However, we put slightly more matter in it: as a consequence the Lemaître model is unstable (as is, incidentally, also the Einstein model) against perturbations. The dynamics is determined by the competing forces of the density, which tries to slow down the expansion, and the cosmological constant which, being positive, acts as a universal repulsive force.
I follow the discussion of Weinberg (1972). It is convenient to write the scale parameter in terms of the Einstein radius R_E, and define $x \equiv R/R_E$. We put more matter in the universe than the Einstein value ρ_E, so that, since $\rho R^3 = \text{constant}$, we can write $R^3 = \rho x^3 R_E^3 = \alpha\rho_E R_E^3$, with $\alpha > 1$. Then:

$$\rho = \frac{\alpha\rho_E}{x^3} = \frac{\alpha\Lambda}{4\pi G x^3}$$

The Einstein equations expressed in x and α become:

$$\dot{x}^2 = \frac{\Lambda}{3x}(x^3 - 3x + 2\alpha)$$

$$\ddot{x} = \frac{\Lambda}{3x^2}(x^3 - \alpha)$$

A lot can be learned by studying the asymptotic behaviour of the equations:

- *For* $x \ll 1$ we find:

$$x = (3\alpha\Lambda)^{1/3} t^{2/3}$$

and thus:

$$H = \frac{2}{3t}$$

$$q = 1/2$$

This is of course nothing but the Einstein-De Sitter model: For $0 < x < 1$ the Lemaître model behaves as the Einstein-De Sitter model.}

- For $x \gg 1$ we find

$$\dot{x}^2 = \frac{1}{3}\Lambda x^2$$

or:

$$H = \sqrt{\frac{\Lambda}{3}}$$

and

$$q = -1$$

For $x >> 1$ the Lemaître model behaves as the De Sitter model. Note that it has these two features in common with the flat models with $\Lambda > 0$, discussed in Chapter 3. The characteristic properties of the Lemaître model are seen in the intermediate range of x:

- x of order unity. From the Einstein equations we see that dx/dt has a minimum:

$$\dot{x}^2 = \Lambda(\alpha^{2/3} - 1)$$

for

$$x_{\min} = \alpha^{2/3}$$

Then $q_{min} = 0$. So this is the phase where the deceleration (with $q = 1/2$), has stopped; after that q will eventually become -1 and the expansion will accelerate. This can be understood as follows: in the early phase the density is so high that it wins against the cosmological constant, until a radius $x_{min} = \alpha^{1/3}$ is reached. For some time the radius will stay close to this value. However, the matter density does not succeed in stopping the expansion definitively (dx/dt remains >0). After some time the density, which goes as R^{-3}, has dropped enough that the cosmological constant can take over. In the end $\rho \to 0$ and we have reached the De Sitter model stage.
We can force dx/dt (min) to become arbitrarily close to zero, by letting $\alpha \to 1$. For α close to unity we have:

$$\dot{x}^2 - \dot{x}_{\min}^2 = \Lambda(\frac{x^2}{3} + \frac{2\alpha}{3x} - \alpha^{2/3}) \approx \frac{\Lambda}{3}(x - \alpha^{1/3})^2$$

Then:

$$\dot{x}^2 \cong \Lambda(\alpha^{2/3} - 1) + \frac{\Lambda}{3}(x - \alpha^{1/3})^2$$

and by trying a solution of the kind $x=\alpha^{1/3}+A\sinh B(t-t_m)$, we find that

$$A = \sqrt{3}(\alpha^{2/3} - 1)^{1/2}$$

and

$$B = \left(\frac{\Lambda}{3}\right)^{1/2}$$

Since the redshift in this period (the ``coasting'' phase) can be written as:

$$\Delta z = \frac{R_0 - R_1}{R_1} = \frac{x - \alpha^{1/3}}{\alpha^{1/3}}$$

we finally have

$$\dot{x}^2 = \frac{\Lambda}{3}\left(\frac{x^3}{3} - x + \frac{2\alpha}{3}\right)$$

$$\sinh\left\{\sqrt{\frac{\Lambda}{3}}(t - t_m)\right\} = \frac{\Delta z}{\sqrt{3}}(1 - \alpha^{-2/3})^{-1/2}$$

Suppose we make $t-t_m = \Delta t$ very large. Then we still can make for small Δz the right hand side as large as we want, by letting $\alpha \to 1$. In other words during a very long period we can keep Δz very small.

It now becomes clear why it has been attempted to explain the redshift peak near $z=2$ with this model: one devises a model in which a coasting period occurs around $z=2$. Later observations have shown that such a redshift peak is in reality non-existent, so the need for introducing the Lemaître model has disappeared. One more comment: it will be obvious that in this model one has to take recourse to numerical integration in order to find the relevant parameters like geometric distance, time scales, etc.
This completes our discussion of the Lemaître model.

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Appendix A: Parameters and symbols used

General Symbols

- z: the redshift
- r_g: the geometric distance
- D: the luminosity distance
- $\Delta\theta$: the angular size (L is used for linear size)
- V: the volume in co-moving coordinates
- τ: the look-back time $(t_0 - t_1)/t_0$, t_0 present epoch, t_1 epoch at redshift z
- ψ: the development angle, used in Friedmann models ($\Lambda=0$)

A.1 The Metric

$$ds^2 = c^2 dt^2 - R^2(t)\left\{d\omega^2 + \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2)\right\}$$

- ω: the radial co-moving coordinate
- θ: an angular co-moving coordinate
- ϕ: a second angular co-moving coordinate
- R(t): the scale parameter

N.B. the co-moving coordinates are fixed once and for all for any galaxy.

A.2 The Einstein Equations

$$8\pi G\rho = 3\frac{kc^2}{R^2} + 3\frac{\dot{R}^2}{R^2} - \Lambda$$

$$8\pi G\frac{p}{c^2} = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \Lambda$$

- ρ: the matter density; also often used $\sigma = 4\pi G\rho/(3H^2)$
- p: the pressure; we only consider $p = 0$
- Λ: the cosmological constant
- H: the Hubble parameter, $H = (dR/dt)(1/R)$
- q: the deceleration parameter, $q = -\{(d^2R/dt^2)R\}/(dR/dt)^2$

A.3 General Relations

- $1+z = R0/R1$: redshift
- $\omega = c\int(dt/R)$, from t_1 to t_0: radial coordinate
- $r_g = \sin\sqrt{(k)}\omega/\sqrt{(k)}$: geometric distance
- $D = (1 + z)\ R_0 r_g$: luminosity distance
- $\Delta\theta = (1+z)L/(R_0 r_g)$: angular size
- $V(\omega) = 4\pi R03\int[\sin\{\sqrt{(k)}\omega\}/\sqrt{(k)}]^2 d\omega$, from 0 to ω volume

Appendix B: The Friedmann model (Λ=0)

See Chapter 2 for information on the derivation of the formulas.

B.1 General Relations ($q_0 \geq 0$)

$$R = \frac{a}{k}(1 - \cos\sqrt{k}\psi)$$

$$t = \frac{a}{kc}(\psi - \frac{\sin\sqrt{k}\psi}{\sqrt{k}})$$

$$\cos\sqrt{k}\psi_0 = \frac{1-q_0}{q_0}$$

$$\cos\sqrt{k}\psi_1 = \frac{\cos\sqrt{k}\psi_0 + z}{1+z}$$

$$\omega = \psi_0 - \psi_1$$

$$R_0 = \left(\frac{k}{2q_0 - 1}\right)^{1/2} \frac{c}{H_0}$$

$$H = \frac{k^2 c}{a} \frac{\sin\sqrt{k}\psi / \sqrt{k}}{(1-\cos\sqrt{k}\psi)^2} \Leftrightarrow H = H_0(1+z)(1+2q_0 z)^{1/2}$$

$$q = \frac{1}{1+\cos\sqrt{k}\psi} \Leftrightarrow q = q_0 \frac{1+z}{1+2q_0 z}$$

$$r_g = \frac{c}{R_0 H_0} \frac{q_0 z + 1 - q_0 + (q_0 - 1)\sqrt{(2q_0+1)}}{q_0^2(1+z)}$$

$$\Delta\theta = 57.3 \frac{(1+z)L}{R_0 r_g} (\text{deg})$$

$$V(\omega) = \frac{2\pi R^3}{k}\left(\omega - \frac{\sin 2\sqrt{k}\omega}{2\sqrt{k}}\right)$$

$$\frac{dV}{dz} = \frac{dV}{d\omega}\frac{d\omega}{dz}$$

$$\tau = 1 - \frac{\psi_1 - \sin\sqrt{k}\psi_1}{\psi_0 - \sin\sqrt{k}\psi_0}$$

$$t_0 H_0 = \frac{k^{3/2} q_0}{k(2q_0 - 1)^{3/2}}\left(\psi_0 - \frac{\sin\sqrt{k}\psi_0}{\sqrt{k}}\right)$$

B.2 k = –1; 0 ≤ q0 < 1/2

$$R = a(\cosh\psi - 1)$$

$$t = \frac{a}{c}(\sinh\psi - \psi)$$

$$\cosh\psi_0 = \frac{1 - q_0}{q_0}$$

$$\cosh\psi_1 = \frac{\cosh\psi_0 + z}{1 + z}$$

$$\omega = \psi_0 - \psi_1$$

$$R_0 = (1 - 2q_0)^{-1/2}\frac{c}{H_0}$$

$$H = \frac{c}{a}\frac{\sinh\psi}{(1 - \cosh\psi)^2}$$

$$q = \frac{1}{1 + \cosh\psi}$$

$$V(\omega) = 2\pi R_0^3\left(\frac{1}{2}\sinh 2\omega - \omega\right)$$

$$\frac{dV}{dz} = 2\pi R_0^3(\cosh 2\omega - 1)\frac{d\omega}{dz}$$

$$\tau = 1 - \frac{\sinh\psi_1 - \psi_1}{\sinh\psi_0 - \psi_0}$$

$$t_0 H_0 = \frac{q_0}{(1-2q_0)^{3/2}}(\sinh\psi_0 - \psi_0)$$

B.3 k = +1; q0 > 1/2

$$R = a(1-\cos\psi)$$

$$t = \frac{a}{c}(\psi - \sin\psi)$$

$$\cos\psi_0 = \frac{1-q_0}{q_0}$$

$$\cos\psi_1 = \frac{\cos\psi_0 + z}{1+z}$$

$$\omega = \psi_0 - \psi_1$$

$$R_0 = (2q_0 - 1)^{-1/2}\frac{c}{H_0}$$

$$H = \frac{c}{a}\frac{\sin\psi}{(1-\cos\psi)^2}$$

$$q = \frac{1}{1+\cos\psi}$$

$$v(\omega) = 2\pi R_0^3\left(\omega - \frac{1}{2}\sin 2\omega\right)$$

$$\frac{dV}{dz} = 2\pi R_0^3(1-\cos 2\omega)\frac{d\omega}{dz}$$

$$\tau = 1 - \frac{\psi_1 - \sin\psi_1}{\psi_0 - \sin\psi_0}$$

$$t_0 H_0 = \frac{q_0}{(2q_0-1)^{3/2}}(\psi_0 - \sin\psi_0)$$

B.4 k = –1; q_0 = 0

$$R = ct$$

$$\omega = \ln(1+z)$$

$$H = t^{-1}$$

$$r_g = \frac{z + \frac{1}{2}z^2}{1+z}$$

$$V(z) = 2\pi\left(\frac{c}{H_0}\right)^3\left(\frac{1}{4}\frac{(1+z)^4 - 1}{(1+z)^2} - \ln(1+z)\right)$$

$$\frac{dV}{dz} = 4\pi\left(\frac{c}{H_0}\right)^3 \frac{(z + z^2/2)^2}{(1+z)^3}$$

$$\tau = \frac{z}{1+z}$$

$$t_0 H_0 = 1$$

B.5 k = 0 ; q_0 = 1/2

$$R = \frac{c}{H_0}\left(\frac{t}{t_0}\right)^{2/3}$$

$$\omega = 2\left(1 - \frac{1}{(1+z)^{1/2}}\right)$$

$$H = \frac{2}{3}t^{-1}$$

$$r_g = \omega$$

$$V(z) = \frac{32}{3}\pi\left(\frac{c}{H_0}\right)^3\left(1 - \frac{1}{(1+z)^{1/2}}\right)^3$$

$$\frac{dV}{dz} = 16\pi\left(\frac{c}{H_0}\right)^3 \frac{\{(1+z)^{1/2} - 1\}^2}{(1+z)^{5/2}}$$

$$\tau = 1 - \frac{1}{(1+z)^{3/2}}$$

$$t_0 H_0 = 2/3$$

B.6 k = +1; q_0 = 1

$$\omega = \frac{\pi}{2} - \arccos\left(\frac{z}{1+z}\right)$$

$$r_g = \frac{z}{1+z}$$

$$V(z) = 2\pi\left(\frac{c}{H_0}\right)^3 \left\{\arcsin\left(\frac{z}{1+z}\right) - \frac{z(1+2z)^{1/2}}{(1+z)^2}\right\}$$

$$\frac{dV}{dz} = 4\pi\left(\frac{c}{H_0}\right)^3 \frac{z^2}{(1+z)^3(1+2z)^{1/2}}$$

$$\tau = 1 - \frac{\arccos\left(\frac{z}{1+z}\right) - \left\{1 - \left(\frac{z}{1+z}\right)^2\right\}^{1/2}}{\frac{\pi}{2} - 1}$$

$$t_0 H_0 = \frac{\pi}{2} - 1$$

Appendix C: Flat Models (k = 0; Λ > 0)

See Chapter 3 for information on the derivation of the formulas.

$$R = A^{1/3} R_0 \sinh^{2/3} \gamma t$$

$$A = \frac{8\pi G \rho_0}{\Lambda}$$

$$\gamma = \frac{1}{2}\sqrt{3\Lambda}$$

$$H_0 = \sqrt{\frac{\Lambda}{3}(1+A)}$$

$$q_0 = \frac{1}{2}\left(\frac{A-2}{A+1}\right) \Leftrightarrow A = 2\left(\frac{1+q_0}{1-2q_0}\right)$$

$$t_0 H_0 = \frac{2}{3}(1+A)^{1/2} \ln\{A^{-1/2} + (1+A^{-1})^{1/2}\}$$

$$(1-2q)H^2 = \Lambda$$

$$H(z) = \sqrt{\frac{\Lambda}{3}\{1 + A(1+z)^3\}}$$

$$1 - 2q = \frac{3}{1 + A(1+z)^3}$$

$$r_g = \frac{c}{R_0 H_0}(1+A)^{1/2} \int_0^z \frac{d\zeta}{\{1 + A(1+\zeta)^3\}^{1/2}}$$

$$\tau = 1 - \frac{\ln[\{1 + A(1+z)^3\}^{1/2} + 1] - \frac{3}{2}\ln(1+z) - \frac{1}{2}\ln A}{\ln\{1 + (1+A)^{1/2}\} - \frac{1}{2}\ln A}$$

Appendix D: A Selection of Other Models

See Chapter 4 for information on the derivation of the formulas.

D.1 Zero-Density Model with $q_0 > 0$

$\rho \equiv 0$ $\Lambda < 0$

$$R = \frac{c}{H_0 q_0^{1/2}} \sin(H_0 q_0^{1/2} t)$$

$$\sin(H_0 q_0^{1/2} t_0) = \left(\frac{q_0}{q_0 + 1}\right)^{1/2}$$

These two equations can be written together as:

$$R = \frac{c}{H_0 (q_0 + 1)^{1/2}} [\cos\{H_0 q_0^{1/2}(t - t_0)\} + q_0^{-1/2} \sin\{H_0 q_0^{1/2}(t - t_0)\}]$$

$$\omega = \operatorname{arcosh}\left\{\left(\frac{q_0 + 1}{q_0}\right)^{1/2} (1 + z)\right\} - \operatorname{arcosh}\left(\frac{q_0 + 1}{q_0}\right)^{1/2}$$

$$H = H_0 q_0^{1/2} \cot(H_0 q_0^{1/2} t)$$

$$q = \tan^2(H_0 q_0^{1/2} t)$$

$$r_g = \left(\frac{c}{R_0 H_0}\right) \frac{\{(1 + q_0)(1 + z)^2 - q_0\}^{1/2} - (1 + z)}{q_0}$$

$$V(\omega) = 2\pi R_0^3 (\frac{1}{2} \sinh 2\omega - \omega)$$

$$\tau = 1 - \frac{\arcsin\left\{\frac{q_0^{1/2}}{(q_0 + 1)^{1/2}(1 + z)}\right\}}{\arcsin\left(\frac{q_0}{q_0 + 1}\right)^{1/2}}$$

$$t_0 H_0 = q_0^{-1/2} \arctan q_0^{1/2}$$

D.2 The Einstein model

$k = +1$

$\Lambda > 0$

$R_E = \frac{c}{\sqrt{\Lambda}}$

$\rho_E = \frac{\Lambda}{4\pi G}$

$\dot{R} = \ddot{R} = 0$

$H = q \equiv 0$

D.3 The De Sitter model

$k = 0$

$\Lambda > 0$

$\rho \equiv 0$

$R = R_0 \exp\{H_0(t - t_0)\}$

$\omega = \left(\frac{c}{R_0 H_0}\right) z$

$H = \sqrt{\frac{\Lambda}{3}}$

$q \equiv -1$

$r_g = \left(\frac{c}{R_0 H_0}\right) z$

$V(z) = \frac{4\pi}{3}\left(\frac{c}{H_0}\right)^3 z^3$

D.4 The Lemaître model

$k = +1$

$\Lambda > 0$

$x = \frac{R}{R_E}$ (definition)

$\alpha = x^3 \frac{\rho}{\rho_E}$ (definition)

The Einstein equations are then:

$$\dot{x}^2 = \frac{\Lambda}{3x}(x^3 - 3x + 2\alpha)$$

$$\ddot{x} = \frac{\Lambda}{3x^2}(x^3 - \alpha)$$

Three different phases can be distinguished:

- for x << $\alpha^{1/3}$, the behaviour is as in the Einstein-De Sitter model.

$$x = \left(\frac{3\alpha}{2}\right)^{1/3} \Lambda^{1/2} t^{2/3}$$

$$H = \frac{2}{3} t^{-1}$$

$$q = \frac{1}{2}$$

- for x >> $\alpha^{1/3}$, the behaviour is as in the De Sitter model.

$$x = x_0 \exp\{H_0(t - t_0)\}$$

$$H_0 = \sqrt{\frac{\Lambda}{3}}$$

$$q \equiv -1$$

- $x \approx \alpha^{1/3}$, the *coasting phase*:

$$x \approx \alpha^{1/3} + (\alpha^{2/3} - 1)^{1/2} \sinh \sqrt{\frac{\Lambda}{3}}(t - t_0)$$

The duration of the coasting phase is:

$$\Delta t \approx \Lambda^{-1/2} \,|\{\ln(1 - \alpha^{-2/3})\,|$$

We can make this period arbitrarily long, because $\Delta t \to \infty$ for $\alpha \to 1$.

The End