Text 8: The Method of Archimedes. From T. L. Heath, ed. (1953). *The Works of Archimedes with the Method of Archimedes*. New York: Dover Publications, pp. 12–21. Adopted from J. Lützen and K. Ramskov, eds. (1999). *Kilder til matematikkens historie*. 2nd ed. København: Matematisk Afdeling, Københavns Universitet, pp. 22–26.

THE METHOD OF ARCHIMEDES TREATING OF MECHANICAL PROBLEMS — TO ERATOSTHENES

Archimedes to Eratosthenes greeting.

I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. [. . .]

Archimedes then describes some theorems that he has found and mentions that he has included the proofs. He continues

[. . .] Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure* though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it† and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

This is followed by some theorems about centers of gravity and the argument for the above mentioned theorem. This argument concludes with the following remark:

Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published.‡

Proposition 2

We can investigate by the same method the proposition that

(1) *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*

(2) *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is* $1\frac{1}{2}$ *times the sphere.*

(1) Let $ABCD$ be a great circle of a sphere, and AC, BD diameters at right angles to one another.

Let a circle be drawn about BD as diameter and in a plane perpendicular to AC, and on this circle as base let a cone be described with A as vertex. Let the

*περὶ τοῦ εἰρημένου σχήματος, in the singular. Possibly Archimedes may have thought of the case of the pyramid as being the more fundamental and as really involving that of the cone. Or perhaps "figure" may be intended for "type of figure."

†Cf. Preface to *Quadrature of Parabola.*

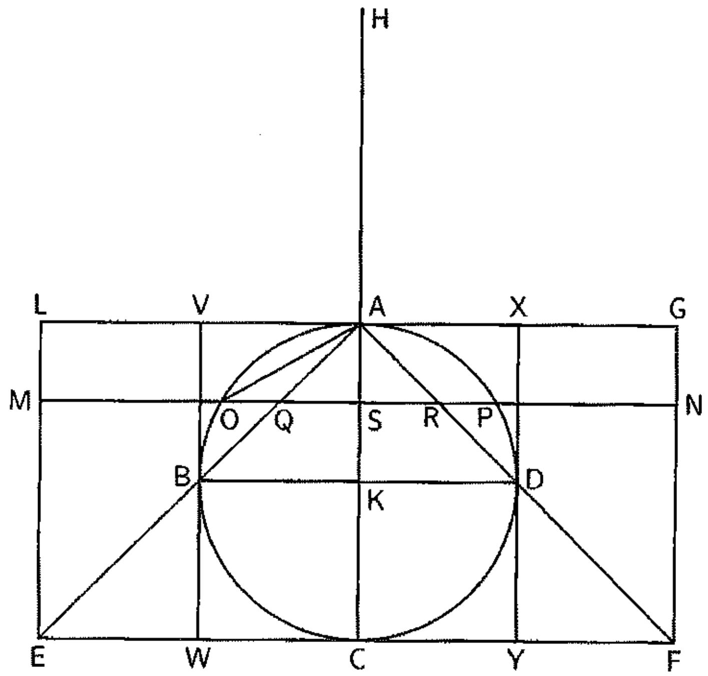
‡The word governing *τὴν γεωμετρουμένην ἀπόδειξιν* in the Greek text is *τάξομεν*, a reading which seems to be doutbful and is certainly difficult to translate. Heiberg translates as if *τάξομεν* meant "we shall give lower down" or "later on", but I agree with Th. Reinach (*Revue générale des sciences pures et appliquées*, 30 November 1907, p. 918) that it is questionable whether Archimedes would really have written out in full once more, as an appendix a proof which, as he says, had already been published (i.e. presumably in the *Quadrature of a Parabola*). *τάξομεν*, if correct, should apparently mean "we shall appoint", "prescribe" or "assign."

surface of this cone be produced and then cut by a plane through C parallel to its base; the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC, and produce CA to H, making AH equal to CA.

Let CH be regarded as the bar of a balance, A being its middle point.

Draw any straight line MN in the plane of the circle $ABCD$ and parallel to BD. Let MN meet the circle in O, P, the diameter AC in S, and the straight lines AE, AF in Q, R respectively. Join AO.

Through MN draw a plane at right angles to AC; this plane will cut the cylinder in a circle with diameter MN, the sphere in a circle with diameter OP, and the cone in a circle with diameter QR.



Now, since $MS = AC$, and $QS = AS$,

$$\begin{aligned} MS.SQ &= CA.AS \\ &= AO^2 \\ &= OS^2 + SQ^2. \end{aligned}$$

$$\begin{aligned} HA : AS &= CA : AS \\ &= MS : SQ \\ &= MS^2 : MS.SQ \\ &= MS^2 : (OS^2 + SQ^2) \quad \text{from above} \\ &= MN^2 : (OP^2 + QR^2) \\ &= (\text{circle, diam. } MN) : (\text{circle, diam. } OP + \text{circle, diam. } QR). \end{aligned}$$

That is, $HA : AS =$ (circle in cylinder):(circle in sphere + circle in cone).

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both the latter circles are placed with their centres of gravity at H.

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF.

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder, the sphere and the cone, and which make up those solids respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about A with the sphere and the cone together, when both are placed with their centres of gravity at H.

Therefore, since K is the centre of gravity of the cylinder,

$$HA : AK = (\text{cylinder}) : (\text{sphere} + \text{cone } AEF).$$

But $HA = 2AK$; therefore

$$\text{cylinder} = 2\ (\text{sphere} + \text{cone } AEF).$$

Now

$$\text{cylinder} = 3\,(\text{cone } AEF); \qquad \text{[Eucl. XII.10]}$$

therefore

$$\text{cone } AEF = 2\,(\text{sphere}).$$

But, since $EF = 2BD$,

$$\text{cone } AEF = 8\,(\text{cone } ABD);$$

therefore

$$\text{sphere} = 4\,(\text{cone } ABD).$$

(2) Through B, D draw VBW, XDY parallel to AC; and imagine a cylinder which has AC for axis and the circles on VX, WY as diameters for bases.

Then

$$\begin{aligned} \text{cylinder } VY &= 2\,(\text{cylinder } VD) \\ &= 6\,(\text{cone } ABD) \quad \text{[Eucl. XII.10]} \\ &= \tfrac{3}{2}\,(\text{sphere}), \quad \text{from above.} \end{aligned}$$

Q.E.D.

From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle, with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.§