supposed square] minus 4, of which the square will therefore be 4 times the square $+16 - 16$ times the root, this being equal to the units of 16 minus the square. Adding the same defect to both sides, and taking like from like, it comes out that 5 times the square equals 16 times the root, and so the root is $\frac{16}{5}$. Therefore one square will be $\frac{256}{25}$ and the other $\frac{144}{25}$, and the sum of both is $\frac{400}{16}$ or 16, and both are squares.

1.4 BEGINNINGS OF ALGEBRA

Algebra has been left until last in this chapter because while it is relatively easy to understand what is meant by 'arithmetic' or 'geometry' it is much more difficult to define 'algebra'. The popular understanding is that it is a kind of calculation with 'letters for numbers' and this is indeed one aspect of it, but those who have studied mathematics far enough know that the word 'algebra' is also used in a quite different sense for the study of abstract structures such as groups or vector spaces; indeed there are now entire systems that are themselves called 'algebras'. The different meanings are not unrelated, and in Chapters 12 and 13 we shall try to untangle some of the threads that are woven into modern algebra, but here we look only at the earliest precursors of the subject. The Arabic word *al-jabr* first entered mathematics in connection with equation-solving, and so that is where we shall begin.

1.4.1 Completing the square, *c.* 1800 BC

Problems involving unknown numbers that must satisfy certain conditions have been posed in many cultures; we have already seen some in the *Arithmetica* of Diophantus, and many others are to be found in Indian texts from about 500 AD onwards. Of particular interest with regard to the later development of algebra, however, are some much earlier examples, from Babylonian tablets of around 1800 BC, in which it was required to find two numbers whose sum (or difference) and product were known. Transcribed into modern notation, such questions lead to quadratic equations, but in 1800 BC they were solved by a kind of cut-and-paste geometry. In the problem below, for example, one is asked to find two numbers that are reciprocal with respect to 60 (that is, whose product is 60) and whose difference is 7. This can be solved by representing the two unknowns as sides of a rectangle of area 60, with one side 7 units longer than the other, and by then supplementing and rearranging the pieces to form a square whose side can be calculated.

With practice, the geometric manoeuvres can be replaced by a set of instructions that can be carried out without drawing figures, but in the example below many traces of practical activity linger on in the language, and the process as a whole is still described as 'completing the square'.[2]

2. For further details see Robson 2002, 114–116.

Completing the square

from an unkown scribe, *c.* 1800 BC, Yale Babylonian Collection 6967, reproduced from Neugebauer and Sachs, 1945, as translated by Eleanor Robson, 2002

Notation
Square brackets show restorations of missing text.

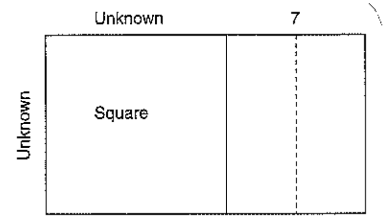
TRANSLATION

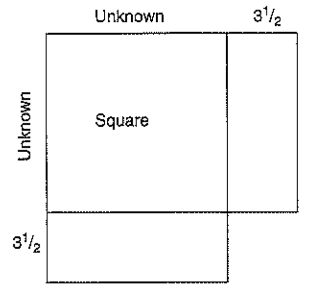
[A reciprocal] exceeds its reciprocal by 7. What are [the reciprocal] and its reciprocal?

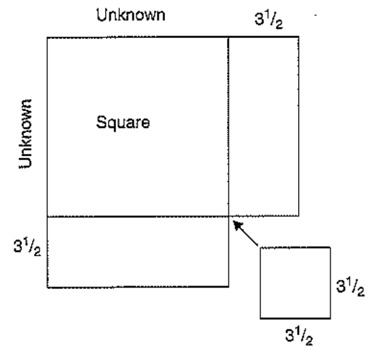
You: break in half the 7 by which the reciprocal exceeds its reciprocal, and 3; 30 (will come up). Multiply 3; 30 by 3; 30 and 12; 15 (will come up).

Append [1 00, the area,] to the 12; 15 which came up for you and 1 12; 15 (will come up). What is [the square-side of 1] 12; 15? 8; 30.

Put down [8; 30 and] 8; 30, its equivalent, and subtract 3; 30, the takiltum-square, from one (of them); append (3; 30) to one (of them). One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.







1.4.2 Al-Khwārizmī's *Al-jabr*, *c.* 825 AD

Babylonian two-number problems recur (though with long gaps in the record) until about 200 BC, but then disappear. One thousand years later, however, a seminal treatise on equation-solving, the inspiration for all subsequent work on the subject, came from the same part of the world: al-Khwārizmī's *Al-kitāb almukhtasar fi hisāb al-jabr w'al-muqābala* (*The book on restoration and balancing*), written in Baghdad around 825 AD. *Al-jabr* means 'restoration', especially of broken bones,[3] and presumably referred originally to the geometric process of putting together a square. Later it came to have the more abstract but equivalent meaning of supplying a term in an equation, in particular a positive term to counteract a negative. Of course, such a quantity must be added to *both* sides of an equation, and hence the concept of *al-muqābala*, or balancing, another practical word that came to acquire a more abstract meaning.

Al-Khwārizmī dealt entirely with equations between 'squares', 'roots', and 'numbers' and classified them into six types, for each of which he gave rules for solution. These were:

squares equal to roots
squares equal to numbers
roots equal to numbers
squares and roots equal to numbers
squares and numbers equal to roots
roots and numbers equal to squares

Although they are separated by over 2500 years, there are striking resemblances between al-Khwārizmī's text and the Mesopotamian example given earlier: the 'recipes' for solution are essentially the same, and so is the way the instructions are addressed directly to the reader. Although the concrete instructions of the Mesopotamian text ('break in half', 'append', 'put down') have been replaced by arithmetic operations such as 'divide into two', 'add', or 'subtract', traces of older geometric thinking are still discernible. Indeed, al-Khwārizmī provided geometric demonstrations for several of his examples by drawing and completing squares. The list of instructions could be applied by rote, but geometry was still regarded as the justification and foundation of the method. At the same time, as one might expect, the later text shows an increased sophistication of understanding. It states more explicitly that we are dealing with entire classes of equations and that the rules will apply to all others of the same type. It is understood that the type 'squares plus numbers equal roots' gives rise to two (positive) solutions, and the text also points out some possible pitfalls: cases where the method either breaks down or becomes trivial.

3. In the Spanish of Cervantes' *Don Quixote*, for example, an *algebrista* is a bone-setter.

Al-Khwārizmī's treatise ends with a lengthy collection of problems arising, for example, from Islamic inheritance laws, but with no further application to quadratic equations.

The rules for solving equations were taken up by other Islamic writers, particularly abū-Kāmil (*c.* 850–930) and al-Karajī (*c.* 1010), but it was not until the twelfth century that they reached western Europe, initially through Latin translations of al-Khwārizmī's *Al-jabr* by Robert of Chester (1145) and Gerard of Cremona (*c.* 1175).

Given below is Chapter V of al-Khwārizmī's treatise, here translated into English from the twelfth-century Latin version made by Robert of Chester. Observe that the idea of a square as a geometric shape still persists in the phrase '49 [units] fill (*adimplent*) the square'.

Al-Khwārizmī's treatment of a quadratic equation

Al-Khwārizmī, *Al-jabr wa'l-muqābala*, Chapter V, translated into English from the Latin version given in Karpinski, 1915, 74 and 76

TRANSLATION

On squares and numbers equalling roots Chapter V.

The proposition is of this kind, as you say:

A square and 21 units are equal to 10 roots.

For this investigation a rule of this kind is given, as you say: What is the square, to which if you adjoin 21 units the whole sum at the same time is worth 10 roots of the same square. The solution to questions of this kind is conceived in this way, that first you divide the [number of] roots into two, and they come in this case to 5, this multiplied with itself produces 25. From this subtract the 21 units, which a little earlier we mentioned together with the square, and there remain 4, of which you take the square root, which is 2, which you subtract from half the roots, 5, and there is left 3, constituting one root of this square, and clearly the number nine gives the square. If you so wish, you may of course add this 2, which you have already subtracted from half the roots, to half the roots, 5, and there comes 7; which gives one root of the square, and 49 [units] fill the square. Therefore when any problem of this kind is proposed to you, investigate it by this method of addition, as we have said, but when you cannot find it by addition, without doubt you will find it by subtraction. For this kind alone requires both addition and subtraction which in other previous kinds you do not find at all.

It must also be understood, when according to this case you take half the roots, and then you multiply the half with itself, if what arises from the multiplication turns out to be less than the number of units announced with the square: the question proposed to you is void. But if it is equal to the units, or if the root of the square turns out to be the same as half the roots that are with the square, it is solved, without either addition or subtraction.

Robert of Chester's translation does not appear to have become widely known, but in 1202 Leonardo Pisano, towards the end of his *Liber abaci*, gave rules for al-Khwārizmī's six types of equation, drawing extensively on problems from abū-Kāmil. Through the *Liber abaci* and later texts derived from it, the technique of *al-jabr*, or algebra, gradually spread during the thirteenth to fifteenth centuries into France and Germany, and became known as the 'cossick art', from the Italian *cosa* (thing) for an unknown quantity. By the sixteenth century al-Khwārizmī's rules, by now attributed in a vague way to someone called Mohammed, or sometimes to an astronomer named al-Geber, were available in printed textbooks in Spain, France, Germany, and England.