2.2 IMPROVEMENTS IN NOTATION

Good mathematical notation serves more than one purpose. First, it displays mathematics in a clear and concise form, often encapsulating in a few symbols ideas that cannot be at all easily or briefly expressed in words; once one know the rules, mathematics written symbolically is more easily visualized, communicated, and understood. This in turn helps to generate new ideas, sometimes because the symbols themselves can be manipulated to show new connections, but at a deeper level because clear exposition almost invariably opens the way to further and faster progress in the mind of both reader and writer.

New symbols usually lag some way behind the concepts they are required to express, and can take some time to settle into a standard form. Mathematical notation continues to evolve today, but widespread publication and rapid communication have made the process very much faster than it was four hundred years ago. Modern notation did not begin to appear until the late fifteenth century, when $+$ and $-$ were first used in Germany. They were followed some time later by the $=$ sign, which first appeared in Robert Recorde's *Whetstone of witte* in 1557 (and was described by Recorde as 'a paire of paralleles, or Gemowe [twin] lines of one lengthe, thus: ====== bicause noe .2. thynges, can be moare equalle'). It was many years, however, before these signs or any others became standard; many sixteenth-century writers continued to use *p.* for plus and *m.* for minus (or other inventions of their own), and continental writers followed René Descartes in using a version of ∞ as an equality sign well into the seventeenth century.

Girolamo Cardano's great book, the *Ars magna*, of 1545 (see 12.1.1), was written entirely verbally, with some useful abbreviations but no genuine symbolic notation. Rafael Bombelli, trying to present Cardano's work more clearly, devised the notation $\overset{2}{\smile}$, $\overset{3}{\smile}$, and so on, for squares, cubes, and higher powers, an idea that was taken up with slight modification by Stevin, who greatly admired Bombelli. François Viète, writing in the 1590s, retained the verbal forms *quadratus* and *cubus* for 'squared' and 'cubed', but contributed the idea of using vowels A, E, I, ... for unknowns, and consonants B, C, D, ... for known or given quantities, which meant that equations could be expressed entirely in letters or, as he called them, 'species'. Viète still used words, however, for the linking operations ('multiplied by', 'equals', and so on), so that his text remained primarily verbal rather than symbolic.

2.2.1 Harriot's notation, *c*. 1600

The earliest mathematical notation that appears to a modern reader both familiar and easy to read is that of Thomas Harriot, whose gift for devising lucid symbolism is apparent in all aspects of his mathematical and scientific work from the early 1590s onwards. Harriot's mathematical notation was based on Viète's insofar as it used vowels a, e, ... for unknown quantities, and consonants b, c, d, ... for known quantities or coefficients, but now in lower case rather than as capitals. He also introduced the convention of writing ab for $a \times b$, with aa, aaa, and so on, for squares, cubes, and higher powers. His equals sign incorporated two short cross strokes to distinguish it from Recorde's simple parallels (since the latter were sometimes used by Viète to indicate subtraction) but these were abandoned in the printed versions. He also invented the now standard inequality signs $<$ and $>$; in manuscript they, like his equals sign, included two vertical cross strokes, but these too were dropped as soon as the signs went into print. Also to be found in Harriot's manuscripts is the three-dot 'therefore' sign, designed to suggest that two propositions imply a third.

Below is a single page from Harriot's posthumously published *Artis analyticae praxis* (*The practice of the analytic art*) of 1631, one of the few extracts in this present book that needs almost no translation.

Harriot's notation

from Harriot, *Artis analyticae praxis*, 1631, 10

SECTIO PRIMA.

Comparationis signa in sequentibus vsurpanda.

Æqualitatis $=$ *vt* $a = b$. *significet a æqualem ipsi b.*
Maioritatis $>$ *vt* $a > b$. *significet a maiorem quam b.*
Minoritatis $<$ *vt* $a < b$ *significet a minorem quam b.*

Fractiones reducibiles reductitijs suis æquatæ.

$$\frac{ba}{b} = a. \quad \Big| \quad \frac{bca}{b} = ca. \quad \Big| \quad \frac{bca}{c} = ba \quad \Big| \quad \frac{bcda}{ca} = bd$$

$$\frac{ba}{c} + d = \frac{ba}{c} + \frac{dc}{c} = \frac{ba+dc}{c} \quad \Big| \quad \frac{ac}{b} + d = \frac{ac+db}{b}$$

$$\frac{ac}{b} + \frac{dd}{g} = \frac{acg}{bg} + \frac{bdd}{bg} = \frac{acg+bdd}{bg}.$$

$$\frac{ac}{b} - d = \frac{ac}{b} - \frac{db}{b} = \frac{ac-db}{b}$$

$$\frac{ac}{b} - \frac{dd}{g} = \frac{acg}{bg} - \frac{ddb}{bg} = \frac{acg-ddb}{bg}$$

$$\left.\begin{array}{c} ac \\ \hline b \\ \hline b \end{array}\right| = \frac{acb}{b} = ac. \quad \Big| \quad \left.\begin{array}{c} ac \\ \hline b \\ \hline d \end{array}\right| = \frac{acd}{b} \quad \Big| \quad \left.\begin{array}{c} ac \\ \hline b \\ dd \\ \hline g \end{array}\right| = \frac{acdd}{bg}$$

$$\begin{array}{c} \frac{aaa}{b} \\ \hline\hline d \end{array} = \frac{aaa}{bd} \quad \Big| \quad \begin{array}{c} bg \\ \hline\hline \frac{ac}{d} \end{array} = \frac{bgd}{ac} \quad \Big| \quad \begin{array}{c} \frac{bbb}{c} \\ \hline\hline \frac{aaa}{dg} \end{array} = \frac{bbbdg}{caaa}$$

Æquatio-

TRANSLATION

Signs of comparison used in what follows.

Equality $=$ *as* $a = b$ *signifies a is equal to b.*
Greater $>$ *as* $a > b$ *signifies a is greater than b.*
Less $<$ as $a < b$ signifies a is less than b.

Reducible fractions reduced to their equivalents

[...]

2.2.2 Descartes' notation, 1637

Because of its enormous influence, Descartes' *La géométrie* (1637) was the book that did more than any other to standardize modern algebraic notation. Like Harriot, Descartes used lower case letters, and he began by using a, b, ..., from the beginning of the alphabet, but later adopted the convention of using x, y, and z as unknown quantities. Also like Harriot, he wrote xy for x times y, and xx for x times x, but introduced superscript notation x^3, x^4, ... for higher powers (though for some reason the convention of writing xx for x-squared lingered on well into the eighteenth century).

Continental mathematicians took up Descartes' notation wholeheartedly. In England, Harriot's a rather than Descartes' x survived until at least the end of the seventeenth century, but then fell out of use. On the other hand, Descartes' ∞ for equality was eventually ousted by Recorde's $=$, and Descartes' $\sqrt{C.}$ by the more adaptable $\sqrt[3]{\ }$.

The extract below is a very short section from Descartes' *La géométrie*, illustrating not only his notation but also some of the difficulties of contemporary typesetting.

Descartes' notation

from Descartes, *La géométrie*, 1637, 326

Mais ſuppoſons la icy eſtre poſſible, & pour en abreger les termes, au lieu des quantités $\frac{cfglz - dekzz}{ez^3 - cgzz}$ eſcriuons $2m$, & au lieu de $\frac{dezz + cfgz - bcgz}{ez^3 - cgzz}$ eſcriuons $\frac{2n}{z}$; & ainſi nous aurons

$$yy \infty 2my - \frac{2n}{z}xy \frac{+ bcfglx - bcfgxx}{ez^3 - cgzz},$$

dont la racine eſt

$$y \infty m - \frac{nx}{z} + \sqrt{mm - \frac{2mnx}{z} + \frac{nnxx}{zz} \frac{+ bcfglx - bcfgxx}{ez^3 - cgzz}}.$$

& derechef pour abreger, au lieu de $-\frac{2mn}{z} + \frac{bcfgl}{ez^3 - cgzz}$ eſcriuons o, & au lieu de $\frac{nn}{zz} \frac{- bcfg}{e^3 - cgzz}$ eſcriuons $\frac{p}{m}$. car ces quantités eſtant toutes données, nous les pouuons nommer comme il nous plaiſt. & ainſi nous auons

$$y \infty m - \frac{n}{z}x + \sqrt{mm + ox - \frac{p}{m}xx},$$

qui doit eſtre la longeur de la ligne B C, en laiſſant A B, ou x indeterminée.

TRANSLATION

But let us suppose here that it is possible, and to shorten the terms, in place of the quantities $\frac{cflgz - dekzz}{ez^3 - egzz}$ let us write $2m$, and in place of $\frac{dezz + cfgz - bcgz}{ez^3 - cgzz}$ let us write $\frac{2n}{z}$; and thus we will have $yy = 2my - \frac{2n}{z}xy + \frac{bcfglx - bcfgxx}{ez^3 - cgzz}$ of which the root is

$$y = m - \frac{nx}{z} + \sqrt{mm - \frac{2mnx}{z} + \frac{nnxx}{zz} \frac{+bcfglx - bcfgxx}{ez^3 - egzz}}.$$

And again to shorten it, in place of $-\frac{2mn}{z} + \frac{bcfgl}{ez^3 - egzz}$ let us write o, and in place of $\frac{nn}{zz}[+]\frac{-bcfg}{ez^3 - cgzz}$ let us write $\frac{p}{m}$. For these quantities all being given, we can name them as we please, and thus we have $y = m - \frac{n}{z}x + \sqrt{mm + ox - \frac{p}{m}xx}$, which must be the length of the line BC, leaving AB, or x undetermined.

2.3 ANALYTIC GEOMETRY

The invention of analytic geometry is usually attributed to Descartes and Fermat, but the foundations were laid before either was born, by the French lawyer and mathematician François Viète in the early 1590s. In this section we look at some of the difficult but important ideas put forward by Viète, and how they were used, adapted, and eventually superseded by Fermat and Descartes.

2.3.1 Viète's introduction to the analytic art, 1591

Viète's most important contribution to mathematics was his recognition that geometric relationships could be expressed and explored through algebraic equations, leading to a powerful fusion of two previously distinct legacies: classical Greek geometry and Islamic algebra. The central technique of algebra, taught in many sixteenth-century texts as the 'Rule of Algebra', was that one should assign a symbol or letter to an unknown quantity and then, bearing in mind the requirements of the problem, manipulate it alongside known quantities to produce an equation. For Viète, never content with a simple idea unless he could clothe it in a Greek term, this was the classical method of 'analysis' in which, he claimed, one assumes that what one is seeking is somehow known and then sets up the relationships or equations it must satisfy. Thus Viète saw the application of algebra to geometric quantities as the restored art of analysis.

From this there followed some important consequences. One was that, in solving a problem or proving a theorem, all the relevant geometric magnitudes, given or sought, had to be represented by letters or 'species'. Viète set up a scale of dimensions: length, square, cube, square-square, square-cube, and so on, and frequently introduced artificially contrived 'species' such as A_{planum} or $Z_{solidum}$ to keep his equations homogeneous. This made his notation almost impossible to generalize beyond three or four dimensions, but it led for the first time to equations in which all the quantities, known or unknown, were represented by letters rather than numbers.

A second consequence of Viète's method was that the art of creating equations (which he called *zetetics*) and solving them (which he called *exegetics*) came to be seen as essential tools for analysing and solving geometric problems, and Viète wrote separate treatises on each. In particular he introduced a numerical method for solving equations that could not be handled algebraically, the first European mathematician to do so.[4] Viète believed that his methods of analysis and equation-solving could not only help to restore the lost or incomplete work of the Greek geometers, but could also enable mathematicians to handle previously intractable problems, in particular the classical problems of doubling the cube and trisecting an angle. So inspired was he by these new possibilities that he ended his first treatise, *Ad artem analyticem isagoge* (*Introduction to the analytic art*) of 1591, with his hopes for the future written in capital letters: NULLUM NON PROBLEMA SOLVERE (To leave no problem unsolved).

Viète's idiosyncratic blend of Greek terminology and awkward notation make him a difficult author to read. Understanding of his ideas grows only with time and repeated reading of his various tracts, both singly and in relation to each other. Nevertheless, the opening chapter of the *Isagoge* is given below because it contains the seeds of some vitally important ideas. It was here, for example, that the word 'analysis' first entered modern European mathematics; it has since evolved through several changes of meaning but has never disappeared from the mathematical lexicon. It was here too that Viète first claimed that geometric magnitudes could be discovered through setting up and solving equations, and argued that all of this could be done in symbols. These were profound ideas that were to lead eventually to the development of powerful general techniques.

4. The relevant treatises are *Zeteticorum libri quinque* (1593), *De numerosa potestatum ad exegesin resolutione* (1600), and *De recognitione aequationum* (1615).

Viète's vision of the 'analytic art'

from Viète, *Isagoge*, 1591, 4

IN ARTEM ANALYTICEM

ISAGOGE.

De definitione & partitione Analyseos, & de ijs quæ iuuant Zeteticem. CAPVT I.

EST veritatis inquirendæ via quædam in Mathematicis, quam Plato primus inuenisse dicitur, à Theone nominata Analysis, & ab eodem definita, Adsumptio quæsiti tanquam concessi per consequentia ad verum concessum. Vt contrà Synthesis, Adsumptio concessi per consequentia ad quæsiti finem & comprehensionem. Et quanquam veteres duplicem tantùm proposuerunt Analysim ζητητικὴν ϗ ποριστικὴν, ad quas definitio Theonis maximè pertinet, constitui tamen etiam tertiam speciem, quæ dicatur ῥητικὴ ἢ ἐξηγητικὴ, consentaneum est, vt sit Zetetice quâ inuenitur æqualitas proportioue magniiudinis de quâ quæritur cum ijs quæ data sunt. Poristice, quâ de æqualitate vel proportione ordinati Theorematis veritas examinatur. Exegetice, quâ ex ordinata æqualitate vel proportione ipsa de qua quæritur exibetur magnitudo. Atque adeò tota ars Analytice triplex illud sibi vendicans officium definiatur, doctrina bene inueniendi in Mathematicis. Ac quod ad Zeteticem quidem attinet, instituitur arte logicâ per syllogismos & enthymemata, quorum firmamẽta sunt ea ipsa quibus æqualitates & proportiones concluduntur symbola, tam ex communibus deriuanda notionibus, quàm ordinandis vi ipsius Analyseos theorematis. Formã autẽ Zetesim ineundi ex arte propriâ est, non iam in numeris suam logicam exercẽte, quæ fuit oscitantia veterum Analystarum, sed per logisticem sub specie nouiter inducendam, feliciorem multò & potiorem numerosâ ad comparandum inter se magnitudines, propositâ primùm homogeneorum lege, & inde constitutâ, vt fit, solemni magnitudinum ex genere ad genus vi suâ proportionaliter adscendentium vel descendentium serie seu scalâ, quâ gradus earundem & genera in comparationibus designentur ac distinguantur.

De Symbolis æqualitatum & proportionum. CAPVT II.

SYmbola æqualitatum & proportionum notiora quæ habentur in Elementis adsumit Analytice vt demonstrata, qualia sunt ferè, .

1 Totum suis partibus æquari.

2 Quæ eidem æquantur, inter se esse æqualia.

3 Si æqualia æqualibus addantur, tota esse æqualia.

TRANSLATION

INTRODUCTION
TO THE ANALYTIC ART

On the definition and parts of Analysis, and thence what is useful to Zetetics. CHAPTER I.

There is a certain way of seeking the truth in mathematics, which Plato is said to have first discovered, called Analysis by Theon, and defined by him as: assuming what is sought as though given, from the consequences the truth is given. As opposed to Synthesis: assuming what is given, from the consequences one arrives at and understands what is sought. And although the ancients proposed only two kinds of analysis, zetetics and poristics, to which the definition of Theon wholly applies, I have nevertheless also added a third kind, which is called rhetics or exegetics, agreeing that it is zetetic by which are found equalities in the ratios of magnitudes, between that which is sought and those that are given. Poristic is that by which the truth of a stated theorem is examined from the equality or ratio. Exegetic is that by which the magnitude one seeks is discovered from that stated equality or ratio. And therefore the whole analytic art taking these three tasks to itself, may be defined as the doctrine of proper discovery in mathematics. And what particularly pertains to zetetics is founded on the art of reasoning with syllogisms and enthymemes, of which the main features are those laws from which equalities and ratios may be deduced, to be derived as much from common notions as from the theorems stated on the strength of Analysis itself. Moreover the way of discovery by zetetics is by its own art, exercising its reasoning not now in numbers, which was the shortcoming of the ancient analysts; but by a newly introduced reasoning with symbols, much more fruitful and powerful than the numerical kind for comparing magnitudes, first by the proposed law of homogeneity, and then by setting up, as it were, of an ordered series or scale of magnitudes, ascending or descending proportionally from one kind to another by power, from which the degree of each and comparisons between them are denoted and distinguished.

2.3.2 Fermat and analytic geometry, 1636

Pierre de Fermat was introduced to the mathematics of Viète by his friend Etienne d'Espagnet while he was studying law in Bordeaux between 1626 and 1630; d'Espagnet's father had known Viète personally and owned copies of his books, which were otherwise not easy to obtain. Fermat was profoundly influenced by what he read, and Viète's style of thinking and writing are clearly discernible in Fermat's early work.

Viète had firmly established the connection between algebra and geometry, but his constructions remained essentially fixed and static, whereas Fermat began to investigate the *locus*, or range of possible positions, of a point constrained by certain conditions. In 1636 he attempted a restoration of two lost books of Apollonius on plane loci, and shortly afterwards wrote a short treatise entitled 'Ad locos planos et solidos isagoge'

('An introduction to plane and solid loci'), where he showed that all second degree equations correspond to one or other of the conic sections. (Fermat referred to general second-degree equations as 'affected equations', that is to say, the square term is 'affected' by the addition or subtraction of terms of lower degree.)

Fermat's treatment was based directly on that of Apollonius, where each conic section is distinguished by a particular relationship between the ordinates and the diameter (see 1.2.4 for the parabola). That is to say, each section has a natural co-ordinate representation in which the diameter is taken as one axis and any line parallel to the ordinates as the other. The axes are not external, but embedded into the curve itself, and need not be at right angles, and Fermat was able to demonstrate that simple variations in the equation correspond only to changes in the choice of axes.

Fermat sent 'Ad locos' to Marin Mersenne in Paris in 1637, but by the end of the year it was overtaken by the publication of Descartes' *La géométrie*, and remained unpublished in Fermat's lifetime. After his death, his son Samuel published some of his papers and letters in a compilation entitled *Varia opera mathematica* (1679), and the extract below is taken from that edition. While editors of mathematical texts generally try to remain true to the words of an original, they sometimes have less compunction about 'improving' the notation or diagrams, and Samuel was no exception. Fermat had used Viète's notation *A quadratus* or *Aq*, but Samuel upgraded this to the more modern A^2. It is likely that Fermat's diagrams were also redrawn for the printed edition, and we do not know how close they are to the originals. Later editions and translations have introduced yet further changes to notation and diagrams (compare, for example, the various extracts from Fermat in this book with those in Fauvel and Gray or other source books).