Should a mathematics teacher know something about the history of mathematics?*

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Aren't there more important questions to be asked on teacher education? Questions like: should a mathematics teacher know something about mathematics? Or about the mathematics he is teaching? Or about the use of mathematics, about how it is applied (and by that I do not mean a study of so-called Applied Mathematics)?

I just ask because if I did not, other people would wonder — and rightly so — why roam the remote expanses of history as long as problems near at hand have not been solved, nay, not even been tackled? I apologize, it is just my theme: the history of mathematics — what it can mean to the teacher, to instruction, to the student.

Again, isn't it running away from greater responsibilities to cast ourselves upon the mercy of history? Can we instil into "inhuman" mathematics more humanity by convincing the learner that mathematics has been conceived by men, or wouldn't it be a shorter way, a stronger proof, to have some mathematics they are really concerned with recreated by the students themselves?

The argument closest to hand — and the most often heard — is that knowledge of the history of a subject area helps in understanding the subject matter itself. I doubt it — at last as far as mathematics is concerned. Mathematics has a long history, the longest of all sciences. A history of dead ends, in which mankind will not be lost again, and which are only interesting as curiosities. A history of progress where even the present state is not the last judgement. The student, however, learns a mathematics that to him is the *non plus ultra*. No doubt there have been pre-stages, but are they worth remembering? Whoever learns a second modern language learns it in its present state, doesn't he? Well, perhaps at universities one might nurture the belief that a language cannot be taught by disregarding its historical grammar, French should not be detached from medieval French and vulgar Latin, and so on; and indeed this knowledge could be useful to students who aspire to more profound linguistic understanding. But Sanscrit as a precondition for studying modern languages was abolished quite a time ago at European universities, notwithstanding those philologists who taught early in the present century and who now turn in their graves.

History has more and more been eliminated from the university instruction of sciences. Maybe Hippocrates' name will be dropped at least once in courses at medical schools, but no examiner will expect a student to know whether anaesthesia was invented before Christ or later. Up to a few decades ago education as well as philosophy were taught at universities as the history of great educationalists and philosophers. Meanwhile contemporaries have won it from those who had been canonised by history. There are strange exceptions, however. Statistical mechanics, for instance, is still — or at least was until recently — taught as its history: each stage of the development from Maxwell and Boltzmann onwards as a separate theory, and none as the germ of our present knowledge. Isn't teaching a science by or close to its history rather a symptom of a retarded coming of age? What can be the use of the history of mathematics? What do mathematicians themselves know about the history of their science?

Historical notes in textbooks and manuals often make one shudder. An eminent contemporary made Hilbert a student of Felix Klein's and explained Hilbert space as due to Klein's geometric influence. He had Galois prove that the fifth degree equation is not solvable by radicals, had irrationality discovered by Pythagoras and the integral by Riemann (since there is a Riemann integral). Another ascribed common fractions to the Babylonians, the arithmetical laws for fractions to the Egyptians of 2500 B.C., and asserted that Archimedes had been rediscovered as late as our century. As to the history of geometry, there was an author who discovered a theory of regular polygons in the Rhind Papyrus, claimed "our geometry was founded in the pillared halls of the Pythagoreans with their shadows on the sunlit floor tiled with regular polygons", and that Euclid wrote 13 stout volumes on the "Foundations of Geometry". Such are the flowers of blooming historical imagination. The tale that Egyptian *harpedonaptoi* constructed right angles by means of the 3,4,5 triangle has so often been repeated that it cannot any more be conquered. Since a well-known bellelettrist of the history of mathematics called Fermat the Prince of Amateurs, the great man has been ranked among the amateur mathematicians. It is a generally accepted fact that Cauchy once gave a wrong proof of Fermat's theorem — I still remember the astonishment of a young colleague, who had repeated this, when I drew his attention to the fact that almost every day he passed along the stack by Cauchy's Works where he could check at any moment whether the statement were true.

None of these mathematicians would write down a mathematical theorem unless he had convinced himself of its truth. History, however, is simply copied. Or it is invented. One to whom I wrote apologized that he had added the incriminating passage as an adornment. He could not tell the difference between history and fiction and I was unable to explain it to him. Whoever has got a historical background will have big trouble in understanding this mentality.

To what degree are science professionals and students interested in the past of their subject, and if they are at all,

*An edited translation of a lecture I gave at the IREM of Poitiers on June 17, 1977. A German translation of this lecture appeared in: *Zentralbatt der Didaktik der Mathematik* 10 (1978), 75-78.

For the Learning of Mathematics **2**, 1 (July 1981)
FLM Publishing Association, Montreal, Quebec, Canada

how well is their historical sense developed? As a young assistant I built into my "Analysis" course historical relations — there are plenty, indeed. This then was to my students the signal to put down their pens and to have a rest. To examination questions as to when logarithms were invented I could expect all periods from fifth century B.C. to the twentieth A.D. In a company of physicists where I once asked the question whether and how Avogadro could have measured the number that bears his name, there was not anybody who knew it nor was interested in it. How many persons were Boyle-Mariotte, Gay-Lussac, Dulong-Petit, Buys-Ballot? What is the use of this kind of knowledge? One has inherited a stock of experience, knowledge, and scientific values, and the only history one is concerned with is what part is scratched during one's lifetime because of obsolescence and what is added because of rejuvenation. Or aren't they things to be concerned with — past and future?

No doubt there are a lot of people interested in history of any kind. If I may be so arrogant as to speak of myself, I can tell that for some time I hesitated whether to study mathematics or history; I have delved profoundly into certain cultural periods (1650-1750); I once dug out the history of the old building that for a certain time served us as a mathematical institute; when I stayed one year in the United States, I learned all about the history of the city and state where I lived. I have forgotten most of the dates of the German kings and emperors and of the margraves and electors of Brandenburg and the kings of Prussia, which I knew as a schoolboy, though the essentials of their history have settled in my mind. But just because I know history, I know better than to impose my interest on others.

Should the one who teaches mathematics at school know something about its history? Let me divide teachers according to the age of their students: 6 — 12 (primary school), 12 — 16 (lower and middle secondary level), above 16 (higher secondary level).

First of all the primary school teachers. In my own country they are trained in — if I am not mistaken — 17 subjects, and though educational theory, taught in difficult educationese, and physical education — as the most dangerous subject — lay a heavy claim on the timetable, mathematics is relatively well endowed with two hours a week during two years. How much history of mathematics could a teacher trainer put into this frame? One might as well ask how much time is available for mathematics along with its didactics. The question is meaningless: it is posed the wrong way. Two hours are too little to be subdivided. Moreover with 17 subjects the student teacher is already more subdivided than becomes somebody who is expected to teach as an undivided person. So much for my own country. I do not know much about other countries, but I am afraid even under other conditions the problems will be the same.

The situation of the lower secondary level is more favourable. At this level teacher training is restricted to two subjects, which on the other hand are to be studied more thoroughly. Future teachers should learn more than they are expected to teach, indeed. This "more" can mean quantity, and then an indeterminate one. It can also mean profundity. Can history contribute to profundity? Yes, provided it means profundity to the trainer. But where do you find this kind of trainer?

At our universities the programmes are more flexible. One of the possible choices of minor subject for the future teacher is the history of mathematics, at least at some universities. What can a restricted study of history mean? Is it worthwhile?

But I have been too rash. Why learn history at all? People who study mathematics, or are at least interested in it, choose it because it represents solid knowledge, dependable knowledge, theorems one can prove, definitions with consequences, proofs one can understand. Sciences are similar, at least up to geography, the truths of which can be checked. These are areas where you can nourish your faith with firm and intelligible rules. Compare this with spelling or grammar. German, for instance: words with a single *a*, with double *a*, with *ah*, all pronounced the same way; in French the terminations *é*, *és*, *ée*, *ées*, to say nothing of English. These things must be learned, the others can be understood. Mathematics, sciences — these are fields of understanding. Should one impose the constraint to memorise upon students who have accepted to go the way of understanding? History, indeed, is again a thing that is memorised, must be memorised. When my eldest boy learned his first history he had one day to memorise certain years — "and tomorrow I will tell you what happened then", the teacher said. And in fact this could even be pleasant. Only it was not history but story telling. It is the way in former times we taught "natural history". Subject: the lion. Or: the cuttlefish. Perhaps history is being taught differently today, but for the average Dutchman the only dates he knows are 100 B.C., "the Batavians come into our country", and 1600 A.D., "the battle of Nieuwpoort", because they are round figures. And then, perhaps, Jan van Schaffelaar's jump from the bell-tower of Barneveld — an unimportant but attractive war story, though undated. Once I overheard a roguish examiner asking whether the Trojan war had taken place before or after the Deluge, which is a tricky question, indeed. But I am sure one can confuse the majority of adults, even people who have graduated from universities, with the question whether Charlemagne lived before or after the discovery of America. (I should, however, add that in Dutch "Karel de Grote" sounds like an ordinary name — there are certainly people with this name, and couldn't they have been bicycle or football champions?)

For most people the past is a vast pulp without any structure at all, and instruction in history does not seem the way to structure it. Charlemagne was crowned emperor in 800, and Columbus discovered America in 1492. But even if one does not know these dates, there may be global structures: the Holy Roman Empire, the Moors in Spain, the age of the great discoveries, which overarch such details. In a well-known popular book on the history of mathematics I saw Sargon of Babylon confused with Sargon of Assur — a difference of two millennia — and the whole book was pulp with little — and then wrong — structure.

An experiment in a third grade astonished me, when it appeared that none of these 8-9 year olds had any articulated representation of the past — the past was to them an amorphous mass. But it also appeared that all of them could easily learn to articulate the past — to articulate it by means of a sequence of generational pictures: mother, grandmother, greatgrandmother, up to great-great-great, dressed according to fashion, with furniture and carriages in the corresponding appropriate style. This happened in a mathematics lesson while introducing the time axis on which third graders had to mark events — true or imagined — from their own life and the lives of their ancestors, and I think it is a better contribution to history than traditional history instruction. Of course there might be earlier initiatives in a child's life. A 4-5 year old, interested in cars: "How did they look when Mom was as old as I am now, when Grandaddy was as old, and Grandaddy's father?" There weren't any. "And T.V., when was it invented (radio, phone)? The steam engine in the museum, which among my ancestors used it?"

Well, this is the most recent past, which can be paced by steps of grandfather of grandfather of grandfather. There are other scales, that of the history of states, of mankind, of Earth, of Universe. A seven year old cannot yet accommodate them to each other. (Can adults?) It requires scale transformations beyond his faculties. Logarithmic scales would be appropriate. Yet how many adults can fathom millions and billions?

What *is* history? Telling stories — yes, this too. But stories can also be invented, and maybe this is the reason why some people believe they are allowed to invent history. In fact there are many other, more pleasant, more logical courses, mathematics and science could have taken to grow towards that what they are nowadays. Why wasn't affine geometry discovered before Euclidean geometry, function fields before elliptic functions, why was the quantum derived from the cumbersome statistics of black radiation rather than in the easy way from the photo effect, why the detour of the phlogiston theory?

About all this one can tell true stories, and *this* then is history. When I was a schoolby, I was taught history from Solon up to the French revolution or Napoleon's downfall. Today exams require history from the French revolution onwards, or even only from World War I or II. Why? Because this is what a citizen has to know? And if this is the reason, is it appreciated this way? Is history really useful as a preparation for joining the polity?

Should history mean this for other people too? I cannot urge it and certainly not impose it. Among all species man is the only one that cares about his past and future. Mankind's heritage is not only biological, it is also tradition. What you inherited from your forefathers, acquire it in order to possess it, as Goethe said; but acquisition includes getting to know how it came about. In an old tale of Ahasuerus it happens again and again after centuries that people assert that what stands there had stood so for eternity. But already the Babylonian kings who believed they had uncovered the foundations of antediluvian temples knew better.

To the majority the past is an amorphous pulp where school instruction has scattered a few glass marbles. I believe it becomes man to understand the past of his race, of the Earth, of the Universe in a structured way, and I will try to contribute to this goal. This to my view is the use of the history of mathematics and adjacent areas: serving history rather than mathematics; rather than the comprehension of mathematics promoting that of history. As a bonus it can aid mathematics too.

Let us give examples:

Numbers — where do they come from, what do they point to, what do they mean?
The numerals and their shapes — could they have been different and why are there ten?
Why does the day have 24 hours?
The hour sixty minutes?
The minute sixty seconds?
The year 365 days, and sometimes 366?
The week seven days?
Snowwhite seven dwarfs?
The right angle 90 degrees?
A dozen twelve pieces?
Why is a meter 100 cm long?
Why does water boil at 100 centigrades?
And freeze at zero?
Why is $-273°$ the absolute zero?
Why has the sky four quarters?
The year four seasons?
Why is the equator so nicely 40,000 km, and yet a little more?
The velocity of light so nicely 300,000 km/sec?
The velocity of sound so nicely 333 m/sec?
The nautical mile a crazy 1852 m?
And the statute mile 1609 m?
Why does a stamp for a domestic letter cost 65 (Dutch) cents?
Why is π about $3\frac{1}{7}$?
What is the natural feature of the basis of natural logarithms?
Why does a man have 32 teeth?
And a deck of cards 32 or 52 pieces?
Why are there 9 men in skittles and 10 in bowling?
Why does February have 28 days?

Well, I could continue this way quite a while. It looks higgledy-piggledy: mathematics, conventions, tradition, history, and old lace. But in order to understand this, or only to think about which queries are the same kind and which are different, one is required to show initiative, comprehension and historical feeling.

Is this a ridiculous enumeration? I think that this list can be a useful guide for the sort of didacticians who have picked up the word "information" without knowing there are various kinds of it, and who parrot so-called philosophers, who should have stated that any cognitive system is nothing but language, in order to claim that cognitive instruction is mere transfer of information in a suitable language.

And what about the larger public? Ask them why the day has 24 hours and the hour 60 minutes. Did they ever care about it? Isn't it as selfevident as their having ten fingers and ten toes? It is on the dial, isn't it? Sure it is. But what is behind it can be worth being consciously experienced. Much of it can mean a start on long excursions, some of it as early as primary school, and most of it would be within a teacher's reach who has been taught how and where to look up data.

The migration of the Indian numerals via the Arabian speaking countries to the West — how many ethnic and cultural movements have not played a part in it? The "60" brings us back to the Sumerians, the 12 signs of the zodiac — what kind of zoo is this? — the 7 planets and the days of the week with the names of pagan deities, what is the matter with it? Who was the first to have measured the Earth, and why did this Greek polyhistor live in Egypt, in a city named after the Greek king Alexander? How did he measure the Earth, and how did his successors do it — a Dutchman, French *sansculottes* — and where did the 40,000 km come from, and why is it actually a bit more? All this is history, the past structured by today's features and by development. How could the Greek Meton on the 5th century B.C. have figured out the length of the year up to a few minutes? *How long have our clocks indicated minutes, indeed?* No, he did not use clocks: between two eclipses at the same spot of the sky there is a lapse of 6940 days and 19 years (and since it is the same spot they must be complete years), and indeed Meton's data clearly shows its origin in a division of 6940 by 19. Meton lived in Athens. *What kind of city was Athens at that time, and where did Meton's knowledge come from?* From Babylon. *What was happening in Babylon at that time?* Anaxagoras, also from Athens, is said to have been exiled because he had called the Sun an incandescent stone bigger than the Peloponnese. Aristarchus of Samos put the sun at the centre. *Which other man came from Samos?* Aristarchus calculated the distances and sizes of the sun and Moon. How did he do it? How and when did the first seafarers venture onto the ocean, and what did they take their bearings from? *What did they look for in foreign countries and what did they discover?* What are longitude and latitude on the globe, and how do we find them? Telescopes, sextants, satellites of Jupiter, Olaf Romer, the velocity of light — it is a long road one can pursue, if one likes, even up to Einstein.

This is how I understand history of mathematics and sciences in the classroom, and the teacher's intellectual baggage required: integrated knowledge. Integrated because familiar to the teacher and a cornucopia available for instruction, not hidden in drawers that are opened at pre-established moments.

One can continue this way in the higher secondary grades and at the university. Not long ago mathematicians wrote in a different style, still witnessed to by physicists and textbooks on physics. Understanding why they did so can help one understanding why Bourbaki wrote his. A bit of historical sense can help the teacher and the student to distinguish horses and hobbies: the genuine horse that has once been a foal, and the hobby that was born wood. Why did people in former times write functions as $f(x)$ and why is it now f? What has happened meanwhile? Why was it by turns allowed and forbidden, and again allowed and forbidden to use differentials? Why did it take so long for logical symbols to be admitted to mathematical texts and why aren't they used more frequently than they are? Did Riemann spaces fall out of the blue sky, and where did they fall? Why in Poincaré's *Analysis Situs* was the change from set theoretic to algebraic addition so important and nevertheless so hidden that both he himself and his contemporaries hardly noticed it? Why did Brouwer and Lebesgue not apply Poincaré's algebraic methods?

Again at this level I stress the history of science as integrated knowledge rather than items stored in well-stocked drawers, each of them labelled and opened when the timetable announces the history of the subject matter. I do not exclude this latter kind. At the university students who like it should be given the opportunity to be active in the history of their subject. Being active does not necessarily mean attending courses in this field. History is worth being studied at the source rather than by reading and copying what others have read and copied before. Sources are nowadays easily accessible, though astonishingly few know this fact. Whoever is interested in the history of mathematics should study the processes rather than the products of mathematical creativity. The most appropriate way to learn and teach the history of mathematics is through seminars rather than courses. At Utrecht University this has successfully been tried, thanks to Dr. H.J.M. Bos.

KNOWING

Unfortunately we find systems of education today which have departed so far from plain truth that they now teach us to be proud of what we know and ashamed of ignorance... To any person prepared to enter with respect into the realm of his great and universal ignorance, the secrets of being will eventually unfold, and they will do so in a measure according to his freedom from natural and indoctrinated shame in his respect of their revelation.

G. Spencer Brown, *Laws of form*