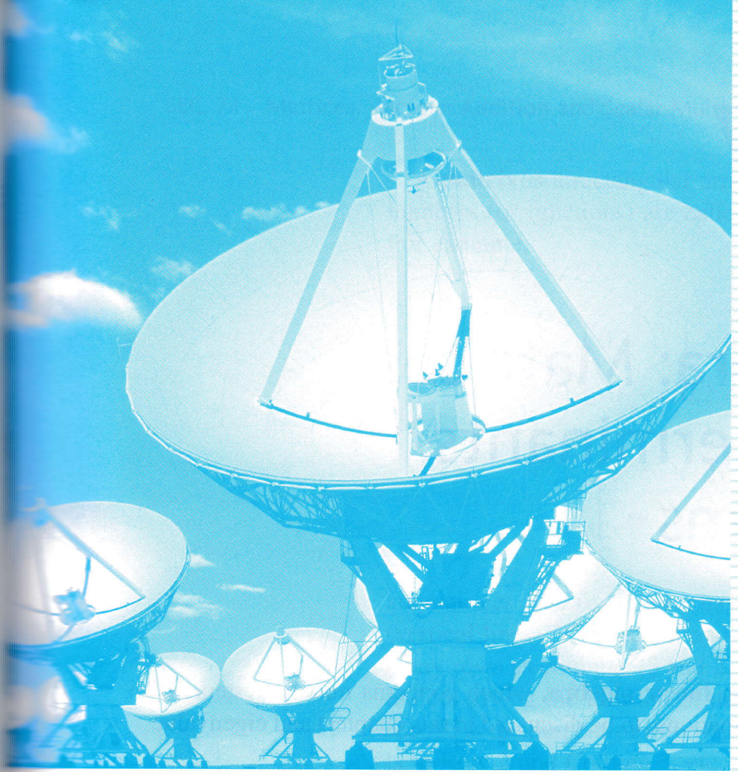


PART B

Linear Algebra. Vector Calculus

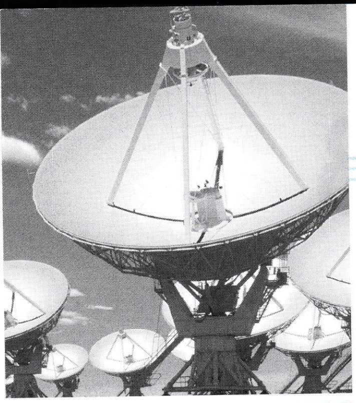
Linear algebra in Chaps. 7 and 8 consists of the theory and application of vectors and matrices, mainly related to linear systems of equations, eigenvalue problems, and linear transformations.

Linear algebra is of growing importance in engineering research and teaching because it forms a foundation of numeric methods (see Chaps. 20–22), and its main instruments, matrices, can hold enormous amounts of data—think of a net of millions of telephone connections—in a form readily accessible by the computer.

Linear analysis in Chaps. 9 and 10, usually called **vector calculus,** extends *differentiation* of functions of one variable to functions of several variables—this includes the vector differential operations grad, div, and curl. And it generalizes *integration* to integrals over curves, surfaces, and solids, with transformations of these integrals into one another, by the basic theorems of Gauss, Green, and Stokes (Chap. 10).

Software suitable for linear algebra (Lapack, Maple, Mathematica, Matlab) can be found in the list at the opening of Part E of the book if needed.

Numeric linear algebra (Chap. 20) ***can be studied directly after Chap. 7 or 8*** because Chap. 20 is independent of the other chapters in Part E on numerics.



CHAPTER 7

Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

This is the first of two chapters on **linear algebra,** which concerns mainly systems of linear equations and linear transformations (to be discussed in this chapter) and eigenvalue problems (to follow in Chap. 8).

Systems of linear equations, briefly called **linear systems,** arise in electrical networks, mechanical frameworks, economic models, optimization problems, numerics for differential equations, as we shall see in Chaps. 21–23, and so on.

As main tools, linear algebra uses **matrices** (rectangular arrays of numbers or functions) and **vectors.** Calculations with matrices handle matrices as single objects, denote them by single letters, and calculate with them in a very compact form, almost as with numbers, so that matrix calculations constitute a powerful "mathematical shorthand".

Calculations with matrices and vectors are defined and explained in Secs. 7.1–7.2. Sections 7.3–7.8 center around linear systems, with a thorough discussion of Gauss elimination, the role of rank, the existence and uniqueness problem for solutions (Sec. 7.5), and matrix inversion. This also includes determinants (Cramer's rule) in Sec. 7.6 (for quick reference) and Sec. 7.7. Applications are considered throughout this chapter. The last section (Sec. 7.9) on vector spaces, inner product spaces, and linear transformations is more abstract. Eigenvalue problems follow in Chap. 8.

COMMENT. ***Numeric linear algebra (Secs. 20.1–20.5) can be studied immediately after this chapter.***

Prerequisite: None.
Sections that may be omitted in a short course: 7.5, 7.9.
References and Answers to Problems: App. 1 Part B, and App. 2.

7.1 Matrices, Vectors: Addition and Scalar Multiplication

In this section and the next one we introduce the basic concepts and rules of matrix and vector algebra. The main application to **linear systems** (systems of linear equations) begins in Sec. 7.3.

A **matrix** is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the **entries** (or sometimes the *elements*) of the matrix. For example,

$$(1)\qquad \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$
$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \qquad [a_1 \quad a_2 \quad a_3], \qquad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

are matrices. The first matrix has two **rows** (horizontal lines of entries) and three **columns** (vertical lines). The second and third matrices are **square matrices,** that is, each has as many rows as columns (3 and 2, respectively). The entries of the second matrix have two indices giving the location of the entry. The first index is the number of the row and the second is the number of the column in which the entry stands. Thus, a_{23} (read *a two three*) is in Row 2 and Column 3, etc. This notation is standard, regardless of whether a matrix is square or not.

Matrices having just a single row or column are called **vectors.** Thus the fourth matrix in (1) has just one row and is called a **row vector.** The last matrix in (1) has just one column and is called a **column vector.**

We shall see that matrices are practical in various applications for storing and processing data. As a first illustration let us consider two simple but typical examples.

EXAMPLE 1 Linear Systems, a Major Application of Matrices

In a system of linear equations, briefly called a **linear system,** such as

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\quad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + x_3 &= 10 \end{aligned}$$

the coefficients of the **unknowns** x_1, x_2, x_3 are the entries of the **coefficient matrix,** call it **A,**

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}. \qquad \text{The matrix} \qquad \tilde{\mathbf{A}} = \begin{bmatrix} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{bmatrix}$$

is obtained by augmenting **A** by the right sides of the linear system and is called the **augmented matrix** of the system. In **A** the coefficients of the system are displayed in the pattern of the equations. That is, their position in **A** corresponds to that in the system when written as shown. The same is true for $\tilde{\mathbf{A}}$.

We shall see that the augmented matrix $\tilde{\mathbf{A}}$ contains all the information about the solutions of a system, so that we can solve a system just by calculations on its augmented matrix. We shall discuss this in great detail, beginning in Sec. 7.3. Meanwhile you may verify by substitution that the solution is $x_1 = 3$, $x_2 = \frac{1}{2}$, $x_3 = -1$.

The notation x_1, x_2, x_3 for the unknowns is practical but not essential; we could choose x, y, z or some other letters. ■

EXAMPLE 2 Sales Figures in Matrix Form

Sales figures for three products I, II, III in a store on Monday (M), Tuesday (T), $\cdots$ may for each week be arranged in a matrix

$$\begin{array}{c} \\ \mathbf{A} = \end{array}\begin{array}{cccccc} \text{M} & \text{T} & \text{W} & \text{Th} & \text{F} & \text{S} \end{array}$$

$$\mathbf{A} = \begin{bmatrix} 400 & 330 & 810 & 0 & 210 & 470 \\ 0 & 120 & 780 & 500 & 500 & 960 \\ 100 & 0 & 0 & 270 & 430 & 780 \end{bmatrix} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array}$$

If the company has ten stores, we can set up ten such matrices, one for each store. Then by adding corresponding entries of these matrices we can get a matrix showing the total sales of each product on each day. Can you think of other data for which matrices are feasible? For instance, in transportation or storage problems? Or in recording phone calls, or in listing distances in a network of roads? ■

General Concepts and Notations

We shall denote matrices by capital boldface letters **A, B, C,** $\cdots$ **,** or by writing the general entry in brackets; thus $\mathbf{A} = [a_{jk}]$, and so on. By an $\boldsymbol{m \times n}$ **matrix** (read *m by n matrix*) we mean a matrix with m rows and n columns—rows come always first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{2}$$

The matrices in (1) are of sizes 2×3, 3×3, 2×2, 1×3, and 2×1, respectively.

Each entry in (2) has two subscripts. The first is the *row number* and the second is the *column number*. Thus a_{21} is the entry in Row 2 and Column 1.

If $m = n$, we call **A** an $n \times n$ **square matrix.** Then its diagonal containing the entries $a_{11}, a_{22}, \cdots, a_{nn}$ is called the **main diagonal** of **A.** Thus the main diagonals of the two square matrices in (1) are a_{11}, a_{22}, a_{33} and e^{-x}, $4x$, respectively.

Square matrices are particularly important, as we shall see. A matrix that is not square is called a **rectangular matrix.**

Vectors

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by ***lowercase*** boldface letters **a, b,** $\cdots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots, \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Matrix Addition and Scalar Multiplication

What makes matrices and vectors really useful and particularly suitable for computers is the fact that we can calculate with them almost as easily as with numbers. Indeed, we now introduce rules for addition and for scalar multiplication (multiplication by numbers) that were suggested by practical applications. (Multiplication of matrices by matrices follows in the next section.) We first need the concept of equality.

DEFINITION

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal,** written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different.** Thus, matrices of different sizes are always different.

EXAMPLE 3 **Equality of Matrices**

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \qquad \text{if and only if} \qquad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

The following matrices are all different. Explain!

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$$ ■

DEFINITION

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

As a special case, the **sum $\mathbf{a} + \mathbf{b}$** of two row vectors or two column vectors, which must have the same number of components, is obtained by adding the corresponding components.

EXAMPLE 4 **Addition of Matrices and Vectors**

If $\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$.

$\mathbf{A}$ in Example 3 and our present $\mathbf{A}$ cannot be added. If $\mathbf{a} = [5 \quad 7 \quad 2]$ and $\mathbf{b} = [-6 \quad 2 \quad 0]$, then $\mathbf{a} + \mathbf{b} = [-1 \quad 9 \quad 2]$.

An application of matrix addition was suggested in Example 2. Many others will follow. ■

DEFINITION

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

Here $(-1)\mathbf{A}$ is simply written $-\mathbf{A}$ and is called the **negative** of $\mathbf{A}$. Similarly, $(-k)\mathbf{A}$ is written $-k\mathbf{A}$. Also, $\mathbf{A} + (-\mathbf{B})$ is written $\mathbf{A} - \mathbf{B}$ and is called the **difference** of $\mathbf{A}$ and $\mathbf{B}$ (which must have the same size!).

EXAMPLE 5 **Scalar Multiplication**

If $\mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$, then $-\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$, $\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$, $0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

If a matrix $\mathbf{B}$ shows the distances between some cities in miles, $1.609\mathbf{B}$ gives these distances in kilometers. ■

Rules for Matrix Addition and Scalar Multiplication. From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size $m \times n$, namely,

$$\textbf{(3)} \qquad \begin{aligned} &\text{(a)} & \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} \\ &\text{(b)} & (\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \mathbf{A} + (\mathbf{B} + \mathbf{C}) \qquad (\text{written } \mathbf{A} + \mathbf{B} + \mathbf{C}) \\ &\text{(c)} & \mathbf{A} + \mathbf{0} &= \mathbf{A} \\ &\text{(d)} & \mathbf{A} + (-\mathbf{A}) &= \mathbf{0}. \end{aligned}$$

Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero. (The last matrix in Example 5 is a zero matrix.)

Hence matrix addition is *commutative* and *associative* [by (3a) and (3b)].

Similarly, for scalar multiplication we obtain the rules

$$\textbf{(4)} \qquad \begin{aligned} &\text{(a)} & c(\mathbf{A} + \mathbf{B}) &= c\mathbf{A} + c\mathbf{B} \\ &\text{(b)} & (c + k)\mathbf{A} &= c\mathbf{A} + k\mathbf{A} \\ &\text{(c)} & c(k\mathbf{A}) &= (ck)\mathbf{A} \qquad (\text{written } ck\mathbf{A}) \\ &\text{(d)} & 1\mathbf{A} &= \mathbf{A}. \end{aligned}$$

PROBLEM SET 7.1

1–8 ADDITION AND SCALAR MULTIPLICATION OF MATRICES AND VECTORS

Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 4 \\ -1 & 2 & 2 \\ 6 & 5 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & -5 & -3 \\ -5 & 2 & 4 \\ -3 & 4 & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 6 & 1 \\ -4 & 7 \\ -8 & 3 \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4.5 \\ 0.8 \\ 1.2 \end{bmatrix}.$$

Find the following expressions or give reasons why they are undefined.

1. $\mathbf{C} + \mathbf{D}$, $\mathbf{D} + \mathbf{C}$, $6(\mathbf{D} - \mathbf{C})$, $6\mathbf{C} - 6\mathbf{D}$
2. $4\mathbf{C}$, $2\mathbf{D}$, $4\mathbf{C} + 2\mathbf{D}$, $8\mathbf{C} - 0\mathbf{D}$
3. $\mathbf{A} + \mathbf{C} - \mathbf{D}$, $\mathbf{C} - \mathbf{D}$, $\mathbf{D} - \mathbf{C}$, $\mathbf{B} + 2\mathbf{C} + 4\mathbf{D}$
4. $2(\mathbf{A} + \mathbf{B})$, $2\mathbf{A} + 2\mathbf{B}$, $5\mathbf{A} - \frac{1}{2}\mathbf{B}$, $\mathbf{A} + \mathbf{B} + \mathbf{C}$
5. $3\mathbf{C} - 8\mathbf{D}$, $4(3\mathbf{A})$, $(4 \cdot 3)\mathbf{A}$, $\mathbf{B} - \frac{1}{10}\mathbf{A}$
6. $5\mathbf{A} - 3\mathbf{C}$, $\mathbf{A} - \mathbf{B} + \mathbf{D}$, $4(\mathbf{B} - 6\mathbf{A})$, $4\mathbf{B} - 24\mathbf{A}$
7. $33\mathbf{u}$, $4\mathbf{v} + 9\mathbf{u}$, $4(\mathbf{v} + 2.25\mathbf{u})$, $\mathbf{u} - \mathbf{v}$
8. $\mathbf{A} + \mathbf{u}$, $12\mathbf{u} + 10\mathbf{v}$, $0(\mathbf{B} - \mathbf{v})$, $0\mathbf{B} + \mathbf{u}$

9. **(Linear system)** Write down a linear system (as in Example 1) whose augmented matrix is the matrix **B** in this problem set.

10. **(Scalar multiplication)** The matrix **A** in Example 2 shows the numbers of items sold. Find the matrix showing the number of units sold if a unit consists of (a) 5 items, (b) 10 items?

11. **(Double subscript notation)** Write the entries of **A** in Example 2 in the general notation shown in (2).

12. **(Sizes, diagonal)** What sizes do **A, B, C, D, u, v** in this problem set have? What are the main diagonals of **A** and **B,** and what about **C?**

13. **(Equality)** Give reasons why the five matrices in Example 3 are different.

14. **(Addition of vectors)** Can you add (a) row vectors whose numbers of components are different, (b) a row and a column vector with the same number of components, (c) a vector and a scalar?

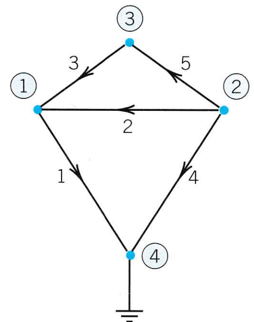
15. **(General rules)** Prove (3) and (4) for general 3×2 matrices and scalars c and k.

16. **TEAM PROJECT. Matrices in Modeling Networks.** Matrices have various applications, as we shall see, in a form that these problems can be efficiently handled on the computer. For instance, they can be used to characterize connections in electrical networks, in nets of roads, in production processes, etc., as follows.

(a) Nodal incidence matrix. The network in Fig. 152 consists of 5 *branches* or *edges* (connections, numbered 1, 2, · · ·, 5) and 4 *nodes* (points where two or more branches come together), with one node being grounded. We number the nodes and branches and give each branch a direction shown by an arrow. This we do arbitrarily. The network can now be described by a *"nodal incidence matrix"* $\mathbf{A} = [a_{jk}]$, where

$$a_{jk} = \begin{cases} +1 & \text{if branch } k \text{ leaves node } (j) \\ -1 & \text{if branch } k \text{ enters node } (j) \\ \;\;\,0 & \text{if branch } k \text{ does not touch } (j). \end{cases}$$

Show that for the network in Fig. 152 the matrix **A** has the given form



$$\begin{array}{l} \text{Branch} \\ \text{Node } (1) \\ \text{Node } (2) \\ \text{Node } (3) \\ \text{Node } (4) \end{array} \begin{array}{c} \begin{array}{ccccc} \;1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 152. Network and nodal incidence matrix in Team Project 16(a)

(b) Find the nodal incidence matrices of the networks in Fig. 153.

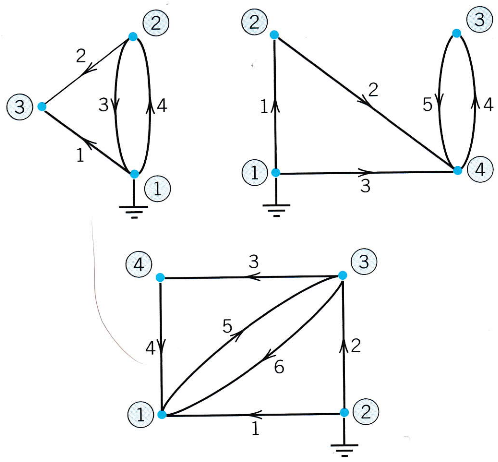


Fig. 153. Networks in Team Project 16(b)

(c) Graph the three networks corresponding to the nodal incidence matrices

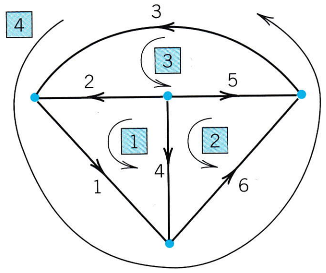
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix}.$$

(d) Mesh incidence matrix. A network can also be characterized by the *mesh incidence matrix* $\mathbf{M} = [m_{jk}]$, where

$$m_{jk} = \begin{cases} +1 \text{ if branch } k \text{ is in mesh } \boxed{j} \\ \text{and has the same orientation} \\ -1 \text{ if branch } k \text{ is in mesh } \boxed{j} \\ \text{and has the opposite orientation} \\ \ \ 0 \text{ if branch } k \text{ is not in mesh } \boxed{j} \end{cases}$$

and a mesh is a loop with no branch in its interior (or in its exterior). Here, the meshes are numbered and directed (oriented) in an arbitrary fashion. Show that in Fig. 154 the matrix $\mathbf{M}$ corresponds to the given figure, where Row 1 corresponds to mesh 1, etc.



$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 154. Network and matrix $\mathbf{M}$ in Team Project 16(d)

(e) Number the nodes in Fig. 154 from left to right 1, 2, 3 and the low node by 4. Find the corresponding nodal incidence matrix.

7.2 Matrix Multiplication

Matrix multiplication means multiplication of matrices by matrices. This is the last algebraic operation to be defined (except for *transposition,* which is of lesser importance). Now matrices are *added* by *adding* corresponding entries. In *multiplication,* do we *multiply* corresponding entries? The answer is no. Why not? Such an operation would not be of much use in applications. The standard definition of multiplication looks artificial, but will be fully motivated later in this section by the use of matrices in "linear transformations," by which this multiplication is suggested.

DEFINITION

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array} \tag{1}$$

The condition $r = n$ means that the second factor, **B,** must have as many rows as the first factor has columns, namely n. As a diagram of sizes (denoted as shown):

$$\begin{array}{ccccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times r] & = & [m \times r]. \end{array}$$

c_{jk} in (1) is obtained by multiplying each entry in the jth row of **A** by the corresponding entry in the kth column of **B** and then adding these n products. For instance, $c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1}$, and so on. One calls this briefly a ***"multiplication of rows into columns."*** See the illustration in Fig. 155, where $n = 3$.

$$m = 4 \left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3} \right. \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} = \left. \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2} \right\} m = 4$$

Fig. 155. Notations in a product $\mathbf{AB} = \mathbf{C}$

EXAMPLE 1

Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$. The product **BA** is not defined. ■

EXAMPLE 2

Multiplication of a Matrix and a Vector

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 3 + 8 \cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix} \quad \text{whereas} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \quad \text{is undefined.}$$ ■

EXAMPLE 3

Products of Row and Column Vectors

$$[3 \quad 6 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19], \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} [3 \quad 6 \quad 1] = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$ ■

EXAMPLE 4 **CAUTION! Matrix Multiplication Is Not Commutative, $\mathbf{AB} \neq \mathbf{BA}$ in General**

This is illustrated by Examples 1 and 2, where one of the two products is not even defined, and by Example 3, where the two products have different sizes. But it also holds for square matrices. For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

It is interesting that this also shows that $\mathbf{AB} = \mathbf{0}$ does ***not*** necessarily imply $\mathbf{BA} = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. We shall discuss this further in Sec. 7.8, along with reasons when this happens. ■

Our examples show that the ***order of factors*** in matrix products ***must always be observed very carefully.*** Otherwise matrix multiplication satisfies rules similar to those for numbers, namely.

$$\textbf{(2)} \quad \begin{aligned} &\text{(a)} && (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) && \textit{written } k\mathbf{AB} \textit{ or } \mathbf{A}k\mathbf{B} \\ &\text{(b)} && \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} && \textit{written } \mathbf{ABC} \\ &\text{(c)} && (\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \\ &\text{(d)} && \mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB} \end{aligned}$$

provided **A, B,** and **C** are such that the expressions on the left are defined; here, k is any scalar. (2b) is called the **associative law.** (2c) and (2d) are called the **distributive laws.**

Since matrix multiplication is a multiplication of rows into columns, we can write the defining formula (1) more compactly as

$$\textbf{(3)} \qquad c_{jk} = \mathbf{a}_j\mathbf{b}_k, \qquad j = 1, \cdots, m; \quad k = 1, \cdots, p,$$

where $\mathbf{a}_j$ is the jth row vector of **A** and $\mathbf{b}_k$ is the kth column vector of **B,** so that in agreement with (1),

$$\mathbf{a}_j\mathbf{b}_k = [a_{j1} \quad a_{j2} \quad \cdots \quad a_{jn}] \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}.$$

EXAMPLE 5 **Product in Terms of Row and Column Vectors**

If $\mathbf{A} = [a_{jk}]$ is of size 3×3 and $\mathbf{B} = [b_{jk}]$ is of size 3×4, then

$$\textbf{(4)} \qquad \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4 \\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4 \end{bmatrix}.$$

Taking $\mathbf{a}_1 = [3 \quad 5 \quad -1]$, $\mathbf{a}_2 = [4 \quad 0 \quad 2]$, etc., verify (4) for the product in Example 1. ■

Parallel processing of products on the computer is facilitated by a variant of (3) for computing $\mathbf{C} = \mathbf{AB}$, which is used by standard algorithms (such as in Lapack). In this method, **A** is used as given, **B** is taken in terms of its column vectors, and the product is computed columnwise; thus,

$$\textbf{(5)} \qquad \mathbf{AB} = \mathbf{A}[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_p].$$

Columns of **B** are then assigned to different processors (individually or several to each processor), which simultaneously compute the columns of the product matrix $\mathbf{Ab}_1$, $\mathbf{Ab}_2$, etc.

EXAMPLE 6 Computing Products Columnwise by (5)

To obtain

$$\mathbf{AB} = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

from (5), calculate the columns

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

of **AB** and then write them as a single matrix, as shown in the first formula on the right. ■

Motivation of Multiplication by Linear Transformations

Let us now motivate the "unnatural" matrix multiplication by its use in **linear transformations.** For $n = 2$ variables these transformations are of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 \\ y_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{6*}$$

and suffice to explain the idea. (For general n they will be discussed in Sec. 7.9.) For instance, (6*) may relate an x_1x_2-coordinate system to a y_1y_2-coordinate system in the plane. In vectorial form we can write (6*) as

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}. \tag{6}$$

Now suppose further that the x_1x_2-system is related to a w_1w_2-system by another linear transformation, say,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Bw} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_{11}w_1 + b_{12}w_2 \\ b_{21}w_1 + b_{22}w_2 \end{bmatrix}. \tag{7}$$

Then the y_1y_2-system is related to the w_1w_2-system indirectly via the x_1x_2-system, and we wish to express this relation directly. Substitution will show that this direct relation is a linear transformation, too, say,

$$\mathbf{y} = \mathbf{Cw} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_{11}w_1 + c_{12}w_2 \\ c_{21}w_1 + c_{22}w_2 \end{bmatrix}. \tag{8}$$

Indeed, substituting (7) into (6), we obtain

$$\begin{aligned} y_1 &= a_{11}(b_{11}w_1 + b_{12}w_2) + a_{12}(b_{21}w_1 + b_{22}w_2) \\ &= (a_{11}b_{11} + a_{12}b_{21})w_1 + (a_{11}b_{12} + a_{12}b_{22})w_2 \\ y_2 &= a_{21}(b_{11}w_1 + b_{12}w_2) + a_{22}(b_{21}w_1 + b_{22}w_2) \\ &= (a_{21}b_{11} + a_{22}b_{21})w_1 + (a_{21}b_{12} + a_{22}b_{22})w_2. \end{aligned}$$

Comparing this with (8), we see that

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} \qquad c_{22} = a_{21}b_{12} + a_{22}b_{22}.$$

This proves that $\mathbf{C} = \mathbf{AB}$ with the product defined as in (1). For larger matrix sizes the idea and result are exactly the same. Only the number of variables changes. We then have m variables y and n variables x and p variables w. The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C} = \mathbf{AB}$ then have sizes $m \times n$, $n \times p$, and $m \times p$, respectively. And the requirement that $\mathbf{C}$ be the product $\mathbf{AB}$ leads to formula (1) in its general form. *This motivates matrix multiplication completely.*

Transposition

Transposition provides a transition from row vectors to column vectors and conversely. More generally, it gives us a choice to work either with a matrix or with its transpose, whatever will be more practical in a specific situation.

DEFINITION

Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^{\mathsf{T}}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^{\mathsf{T}} = [a_{kj}]$, written out

$$\mathbf{A}^{\mathsf{T}} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}. \tag{9}$$

As a special case, transposition converts row vectors to column vectors and conversely.

EXAMPLE 7 **Transposition of Matrices and Vectors**

If
$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}.$$

A little more compactly, we can write

$$\begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 3 & 8 \\ 0 & -1 \end{bmatrix}$$

$$[6 \quad 2 \quad 3]^{\mathsf{T}} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^{\mathsf{T}} = [6 \quad 2 \quad 3].$$

Note that for a square matrix, the transpose is obtained by interchanging entries that are symmetrically positioned with respect to the main diagonal, e.g., a_{12} and a_{21}, and so on. ■

Rules for transposition are

$$
\textbf{(10)}\qquad
\begin{aligned}
&\text{(a)} && (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}\\
&\text{(b)} && (\mathbf{A} + \mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}\\
&\text{(c)} && (c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}}\\
&\text{(d)} && (\mathbf{A}\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}.
\end{aligned}
$$

CAUTION! Note that in (10d) the transposed matrices are ***in reversed order.*** We leave the proofs to the student. (See Prob. 22.)

Special Matrices

Certain kinds of matrices will occur quite frequently in our work, and we now list the most important ones of them.

Symmetric and Skew-Symmetric Matrices. Transposition gives rise to two useful classes of matrices, as follows. *Symmetric matrices* and *skew-symmetric matrices* are ***square*** matrices whose transpose equals the matrix itself or minus the matrix, respectively:

$$
\textbf{(11)}\qquad \underset{\text{Symmetric Matrix}}{\mathbf{A}^{\mathsf{T}} = \mathbf{A} \quad (\text{thus } a_{kj} = a_{jk}),} \qquad \underset{\text{Skew-Symmetric Matrix}}{\mathbf{A}^{\mathsf{T}} = -\mathbf{A} \quad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0).}
$$

EXAMPLE 8 Symmetric and Skew-Symmetric Matrices

$$
\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \quad \text{is symmetric, and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{is skew-symmetric.}
$$

For instance, if a company has three building supply centers C_1, C_2, C_3, then $\mathbf{A}$ could show costs, say, a_{jj} for handling 1000 bags of cement on center C_j, and a_{jk} $(j \neq k)$ the cost of shipping 1000 bags from C_j to C_k. Clearly, $a_{jk} = a_{kj}$ because shipping in the opposite direction will usually cost the same.

Symmetric matrices have several general properties which make them important. This will be seen as we proceed. ■

Triangular Matrices. Upper triangular matrices are square matrices that can have nonzero entries only on and ***above*** the main diagonal, whereas any entry below the diagonal must be zero. Similarly, **lower triangular matrices** can have nonzero entries only on and ***below*** the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

EXAMPLE 9 Upper and Lower Triangular Matrices

$$
\underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix},}_{\text{Upper triangular}} \qquad \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}.}_{\text{Lower triangular}} \qquad ■
$$

Diagonal Matrices. These are square matrices that can have nonzero entries only on the main diagonal. Any entry above or below the main diagonal must be zero.

If all the diagonal entries of a diagonal matrix **S** are equal, say, c, we call **S** a **scalar matrix** because multiplication of any square matrix **A** of the same size by **S** has the same effect as the multiplication by a scalar, that is,

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A}. \tag{12}$$

In particular, a scalar matrix whose entries on the main diagonal are all 1 is called a **unit matrix** (or **identity matrix**) and is denoted by $\mathbf{I}_n$ or simply by **I**. For **I**, formula (12) becomes

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}. \tag{13}$$

EXAMPLE 10 **Diagonal Matrix D. Scalar Matrix S. Unit Matrix I**

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applications of Matrix Multiplication

Matrix multiplication will play a crucial role in connection with linear systems of equations, beginning in the next section. For the time being we mention some other simple applications that need no lengthy explanations.

EXAMPLE 11 **Computer Production. Matrix Times Matrix**

Supercomp Ltd produces two computer models PC1086 and PC1186. The matrix **A** shows the cost per computer (in thousands of dollars) and **B** the production figures for the year 2005 (in multiples of 10000 units.) Find a matrix **C** that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\begin{array}{c} \\ \\ \mathbf{A} = \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} \text{PC1086} & \text{PC1186} \end{array} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \qquad \mathbf{B} = \begin{array}{c} \text{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{PC1086} \\ \text{PC1186} \end{array}$$

Solution.

$$\mathbf{C} = \mathbf{AB} = \begin{array}{c} \text{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array}$$

Since cost is given in multiples of \$1000 and production in multiples of 10 000 units, the entries of **C** are multiples of \$10 millions; thus $c_{11} = 13.2$ means \$132 million, etc.

EXAMPLE 12 **Weight Watching. Matrix Times Vector**

Suppose that in a weight-watching program, a person of 185 lb burns 350 cal/hr in walking (3 mph), 500 in bycycling (13 mph) and 950 in jogging (5.5 mph). Bill, weighing 185 lb, plans to exercise according to the matrix shown. Verify the calculations (W = Walking, B = Bicycling, J = Jogging).

$$\begin{array}{l} \\ \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}
\begin{array}{c} \begin{array}{ccc} \text{W} & \text{B} & \text{J} \end{array} \\ \begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix} \end{array}
\begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix} = \begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix}
\begin{array}{l} \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}$$

EXAMPLE 13 **Markov Process. Powers of a Matrix. Stochastic Matrix**

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is

C: Commercially Used 25% I: Industrially Used 20% R: Residentially Used 55%.

Find the states in 2009, 2014, and 2019, assuming that the transition probabilities for 5-year intervals are given by the matrix **A** and remain practically the same over the time considered.

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} \text{From C} & \text{From I} & \text{From R} \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{To C} \\ \text{To I} \\ \text{To R} \end{array}$$

A is a **stochastic matrix,** that is, a square matrix with all entries nonnegative and all column sums equal to 1. Our example concerns a **Markov process**[1], that is, a process for which the probability of entering a certain state depends only on the last state occupied (and the matrix **A**), not on any earlier state.

Solution. From the matrix **A** and the 2004 state we can compute the 2009 state,

$$\begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7\cdot 25 + 0.1\cdot 20 + 0\cdot 55 \\ 0.2\cdot 25 + 0.9\cdot 20 + 0.2\cdot 55 \\ 0.1\cdot 25 + 0\cdot 20 + 0.8\cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

To explain: The 2009 figure for C equals 25% times the probability 0.7 that C goes into C, plus 20% times the probability 0.1 that I goes into C, plus 55% times the probability 0 that R goes into C. Together,

$$25\cdot 0.7 + 20\cdot 0.1 + 55\cdot 0 = 19.5\ [\%]. \qquad \text{Also} \qquad 25\cdot 0.2 + 20\cdot 0.9 + 55\cdot 0.2 = 34\ [\%].$$

Similarly, the new R is 46.5%. We see that the 2009 state vector is the column vector

$$\mathbf{y} = [19.5 \quad 34.0 \quad 46.5]^{\mathsf{T}} = \mathbf{Ax} = \mathbf{A}\,[25 \quad 20 \quad 55]^{\mathsf{T}}$$

where the column vector $\mathbf{x} = [25 \quad 20 \quad 55]^{\mathsf{T}}$ is the given 2004 state vector. Note that the sum of the entries of $\mathbf{y}$ is 100 [%]. Similarly, you may verify that for 2014 and 2019 we get the state vectors

$$\mathbf{z} = \mathbf{Ay} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}^2\mathbf{x} = [17.05 \quad 43.80 \quad 39.15]^{\mathsf{T}}$$
$$\mathbf{u} = \mathbf{Az} = \mathbf{A}^2\mathbf{y} = \mathbf{A}^3\mathbf{x} = [16.315 \quad 50.660 \quad 33.025]^{\mathsf{T}}.$$

[1]ANDREI ANDREJEVITCH MARKOV (1856–1922), Russian mathematician, known for his work in probability theory.

Answer. In 2009 the commercial area will be 19.5% (11.7 mi^2), the industrial 34% (20.4 mi^2) and the residential 46.5% (27.9 mi^2). For 2014 the corresponding figures are 17.05%, 43.80%, 39.15%. For 2019 they are 16.315%, 50.660%, 33.025%. (In Sec. 8.2 we shall see what happens in the limit, assuming that those probabilities remain the same. In the meantime, can you experiment or guess?) ■

PROBLEM SET 7.2

1–14 MULTIPLICATION, ADDITION, AND TRANSPOSITION OF MATRICES AND VECTORS

Let

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & -2 \\ 10 & -3 & 1 \\ -10 & 5 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 9 & 4 & -4 \\ 4 & 7 & 0 \\ -4 & 0 & 11 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 3 & 1 \\ 0 & -2 \\ 4 & 0 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{b} = [3 \quad 0 \quad 8].$$

Calculate the following products and sums or give reasons why they are not defined. (Show all intermediate results.)

1. $\mathbf{Aa}$, $\mathbf{Ab}$, $\mathbf{Ab}^\mathsf{T}$, $\mathbf{AB}$
2. $\mathbf{Ab}^\mathsf{T} + \mathbf{Bb}^\mathsf{T}$, $(\mathbf{A} + \mathbf{B})\mathbf{b}^\mathsf{T}$, $\mathbf{bA}$, $\mathbf{B} - \mathbf{B}^\mathsf{T}$
3. $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{AA}^\mathsf{T}$, $\mathbf{A}^\mathsf{T}\mathbf{A}$
4. $\mathbf{A}^2$, $\mathbf{B}^2$, $(\mathbf{A}^\mathsf{T})^2$, $(\mathbf{A}^2)^\mathsf{T}$
5. $\mathbf{a}^\mathsf{T}\mathbf{A}$, $\mathbf{bA}$, $5\mathbf{B}(3\mathbf{a} + 2\mathbf{b}^\mathsf{T})$, $15\mathbf{Ba} + 10\mathbf{Bb}^\mathsf{T}$
6. $\mathbf{A}^\mathsf{T}\mathbf{b}$, $\mathbf{b}^\mathsf{T}\mathbf{B}$, $(3\mathbf{A} - 2\mathbf{B})^\mathsf{T}\mathbf{a}$, $\mathbf{a}^\mathsf{T}(3\mathbf{A} - 2\mathbf{B})$
7. $\mathbf{ab}$, $\mathbf{ba}$, $(\mathbf{ab})\mathbf{A}$, $\mathbf{a}(\mathbf{bA})$
8. $\mathbf{ab} - \mathbf{ba}$, $-(4\mathbf{b})(7\mathbf{a})$, $-28\mathbf{ba}$, $5\mathbf{abB}$
9. $(\mathbf{A} + \mathbf{B})^2$, $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$, $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$
10. $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$, $\mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2$, $\mathbf{A}^2 - \mathbf{B}^2$
11. $\mathbf{A}^2\mathbf{B}$, $\mathbf{A}^3$, $(\mathbf{AB})^2$, $\mathbf{A}^2\mathbf{B}^2$
12. $\mathbf{B}^3$, $\mathbf{BC}$, $(\mathbf{BC})^2$, $(\mathbf{BC})(\mathbf{BC})^\mathsf{T}$
13. $\mathbf{a}^\mathsf{T}\mathbf{Aa}$, $\mathbf{a}^\mathsf{T}(\mathbf{A} + \mathbf{A}^\mathsf{T})\mathbf{a}$, $\mathbf{bBb}^\mathsf{T}$, $\mathbf{b}(\mathbf{B} - \mathbf{B}^\mathsf{T})\mathbf{b}^\mathsf{T}$
14. $\mathbf{a}^\mathsf{T}\mathbf{CC}^\mathsf{T}\mathbf{a}$, $\mathbf{a}^\mathsf{T}\mathbf{C}^2\mathbf{a}$, $\mathbf{bC}^\mathsf{T}\mathbf{Cb}^\mathsf{T}$, $\mathbf{bCC}^\mathsf{T}\mathbf{b}^\mathsf{T}$

15. **(General rules)** Prove (2) for 2×2 matrices $\mathbf{A} = [a_{jk}]$, $\mathbf{B} = [b_{jk}]$, $\mathbf{C} = [c_{jk}]$ and a general scalar.

16. **(Commutativity)** Find all 2×2 matrices $\mathbf{A} = [a_{jk}]$ that commute with $\mathbf{B} = [b_{jk}]$, where $b_{jk} = j + k$.

17. **(Product)** Write $\mathbf{AB}$ in Probs. 1–14 in terms of row and column vectors.

18. **(Product)** Calculate $\mathbf{AB}$ in Prob. 1 columnwise. (See Example 6.)

19. **TEAM PROJECT. Symmetric and Skew-Symmetric Matrices.** These matrices occur quite frequently in applications, so it is worthwhile to study some of their most important properties.

(a) Verify the claims in (11) that $a_{kj} = a_{jk}$ for a symmetric matrix, and $a_{kj} = -a_{jk}$ for a skew-symmetric matrix. Give examples.

(b) Show that for every square matrix $\mathbf{C}$ the matrix $\mathbf{C} + \mathbf{C}^\mathsf{T}$ is symmetric and $\mathbf{C} - \mathbf{C}^\mathsf{T}$ is skew-symmetric. Write $\mathbf{C}$ in the form $\mathbf{C} = \mathbf{S} + \mathbf{T}$, where $\mathbf{S}$ is symmetric and $\mathbf{T}$ is skew-symmetric and find $\mathbf{S}$ and $\mathbf{T}$ in terms of $\mathbf{C}$. Represent $\mathbf{A}$ and $\mathbf{B}$ in Probs. 1–14 in this form.

(c) A **linear combination** of matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{M}$ of the same size is an expression of the form

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + \cdots + m\mathbf{M}, \tag{14}$$

where $a, \cdots, m$ are any scalars. Show that if these matrices are square and symmetric, so is (14); similarly, if they are skew-symmetric, so is (14).

(d) Show that $\mathbf{AB}$ with symmetric $\mathbf{A}$ and $\mathbf{B}$ is symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ **commute,** that is, $\mathbf{AB} = \mathbf{BA}$.

(e) Under what condition is the product of skew-symmetric matrices skew-symmetric?

20. **(Idempotent and nilpotent matrices)** By definition, $\mathbf{A}$ is **idempotent** if $\mathbf{A}^2 = \mathbf{A}$, and $\mathbf{B}$ is **nilpotent** if $\mathbf{B}^m = \mathbf{0}$ for some positive integer m. Give examples (different from $\mathbf{0}$ or $\mathbf{I}$). Also give examples such that $\mathbf{A}^2 = \mathbf{I}$ (the unit matrix).

21. **(Triangular matrices)** Let $\mathbf{U}_1$, $\mathbf{U}_2$ be upper triangular and $\mathbf{L}_1$, $\mathbf{L}_2$ lower triangular. Which of the following are triangular? Give examples. How can you save half of your work by transposition?

$$\mathbf{U}_1 + \mathbf{U}_2,\ \mathbf{U}_1\mathbf{U}_2,\ \mathbf{U}_1{}^2,\ \mathbf{U}_1 + \mathbf{L}_1,\ \mathbf{U}_1\mathbf{L}_1,\ \mathbf{L}_1 + \mathbf{L}_2,$$
$$\mathbf{L}_1\mathbf{L}_2,\ \mathbf{L}_1{}^2$$

22. **(Transposition of products)** Prove (10a)–(10c). Illustrate the basic formula (10d) by examples of your own. Then prove it.

APPLICATIONS

23. **(Markov process)** If the transition matrix $\mathbf{A}$ has the entries $a_{11} = 0.5$, $a_{12} = 0.3$, $a_{21} = 0.5$, $a_{22} = 0.7$ and the initial state is $[1 \quad 1]^\mathsf{T}$, what will the next three states be?

24. **(Concert subscription)** In a community of 300 000 adults, subscribers to a concert series tend to renew their subscription with probability 90% and persons presently not subscribing will subscribe for the next season with probability 0.1%. If the present number of subscribers is 2000, can one predict an increase, decrease, or no change over each of the next three seasons?

25. CAS Experiment. Markov Process. Write a program for a Markov process. Use it to calculate further steps in Example 13 of the text. Experiment with other stochastic 3×3 matrices, also using different starting values.

26. (Production) In a production process, let N mean "no trouble" and T "trouble." Let the transition probabilities from one day to the next be 0.9 for $N \to N$, hence 0.1 for $N \to T$, and 0.5 for $T \to N$, hence 0.5 for $T \to T$. If today there is no trouble, what is the probability of N two days after today? Three days after today?

27. (Profit vector) Two factory outlets F_1 and F_2 in New York and Los Angeles sell sofas (S), chairs (C), and tables (T) with a profit of \$110, \$45, and \$80, respectively. Let the sales in a certain week be given by the matrix

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} S & C & T \end{array} \\ \begin{bmatrix} 600 & 400 & 100 \\ 300 & 820 & 205 \end{bmatrix} \end{array} \begin{array}{c} F_1 \\ F_2 \end{array}$$

Introduce a "profit vector" $\mathbf{p}$ such that the components of $\mathbf{v} = \mathbf{Ap}$ give the total profits of F_1 and F_2.

28. TEAM PROJECT. Special Linear Transformations. Rotations have various applications. We show in this project how they can be handled by matrices.

(a) Rotation in the plane. Show that the linear transformation $\mathbf{y} = \mathbf{Ax}$ with matrix

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is a counterclockwise rotation of the Cartesian x_1x_2-coordinate system in the plane about the origin, where θ is the angle of rotation.

(b) Rotation through $n\theta$. Show that in (a)

$$\mathbf{A}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Is this plausible? Explain this in words.

(c) Addition formulas for cosine and sine. By geometry we should have

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}.$$

Derive from this the addition formulas (6) in App. A3.1.

(d) Computer graphics. To visualize a three-dimensional object with plane faces (e.g., a cube), we may store the position vectors of the vertices with respect to a suitable $x_1x_2x_3$-coordinate system (and a list of the connecting edges) and then obtain a two-dimensional image on a video screen by projecting the object onto a coordinate plane, for instance, onto the x_1x_2-plane by setting $x_3 = 0$. To change the appearance of the image, we can impose a linear transformation on the position vectors stored. Show that a diagonal matrix $\mathbf{D}$ with main diagonal entries $3, 1, \frac{1}{2}$ gives from an $\mathbf{x} = [x_j]$ the new position vector $\mathbf{y} = \mathbf{Dx}$, where $y_1 = 3x_1$ (stretch in the x_1-direction by a factor 3), $y_2 = x_2$ (unchanged), $y_3 = \frac{1}{2}x_3$ (contraction in the x_3-direction). What effect would a scalar matrix have?

(e) Rotations in space. Explain $\mathbf{y} = \mathbf{Ax}$ geometrically when $\mathbf{A}$ is one of the three matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

$$\begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}, \quad \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What effect would these transformations have in situations such as that described in (d)?

7.3 Linear Systems of Equations. Gauss Elimination

The most important use of matrices occurs in the solution of *systems of linear equations,* briefly called **linear systems.** Such systems model various problems, for instance, in frameworks, electrical networks, traffic flow, economics, statistics, and many others. In this section we show an important solution method, the *Gauss elimination.* General properties of solutions will be discussed in the next sections.

Linear System, Coefficient Matrix, Augmented Matrix

A **linear system of m equations in n unknowns** $x_1, \cdots, x_n$ is a set of equations of the form

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots&\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned} \tag{1}
$$

The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line. $a_{11}, \cdots, a_{mn}$ are given numbers, called the **coefficients** of the system. $b_1, \cdots, b_m$ on the right are also given numbers. If all the b_j are zero, then (1) is called a **homogeneous system.** If at least one b_j is not zero, then (1) is called a **nonhomogeneous system.**

A **solution** of (1) is a set of numbers $x_1, \cdots, x_n$ that satisfies all the m equations. A **solution vector** of (1) is a vector $\mathbf{x}$ whose components form a solution of (1). If the system (1) is homogeneous, it has at least the **trivial solution** $x_1 = 0, \cdots, x_n = 0$.

Matrix Form of the Linear System (1). From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation

$$\mathbf{Ax} = \mathbf{b} \tag{2}$$

where the **coefficient matrix** $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}
$$

are column vectors. We assume that the coefficients a_{jk} are not all zero, so that $\mathbf{A}$ is not a zero matrix. Note that $\mathbf{x}$ has n components, whereas $\mathbf{b}$ has m components. The matrix

$$
\widetilde{\mathbf{A}} = \left[\begin{array}{cccc|c} a_{11} & \cdots & a_{1n} & & b_1 \\ \cdot & \cdots & \cdot & & \cdot \\ \cdot & \cdots & \cdot & & \cdot \\ a_{m1} & \cdots & a_{mn} & & b_m \end{array}\right]
$$

is called the **augmented matrix** of the system (1). The dashed vertical line could be omitted (as we shall do later); it is merely a reminder that the last column of $\widetilde{\mathbf{A}}$ does not belong to $\mathbf{A}$.

The augmented matrix $\widetilde{\mathbf{A}}$ determines the system* (1) *completely because it contains all the given numbers appearing in (1).

EXAMPLE 1 **Geometric Interpretation. Existence and Uniqueness of Solutions**

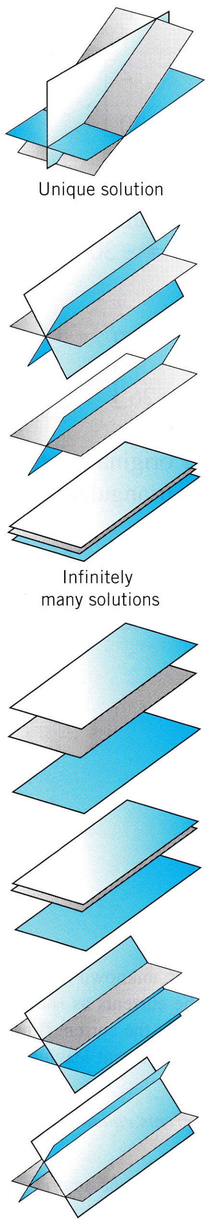


Fig. 156. Three equations in three unknowns interpreted as planes in space

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

If we interpret x_1, x_2 as coordinates in the x_1x_2-plane, then each of the two equations represents a straight line, and (x_1, x_2) is a solution if and only if the point P with coordinates x_1, x_2 lies on both lines. Hence there are three possible cases:

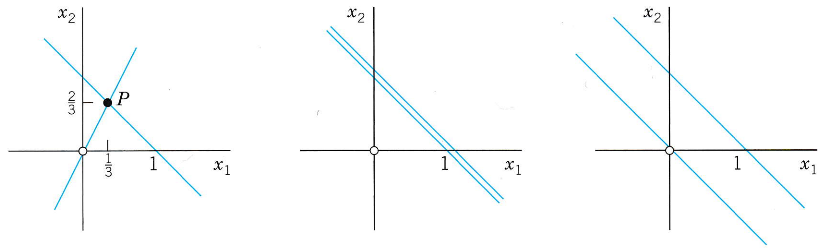
(*a*) Precisely one solution if the lines intersect.

(*b*) Infinitely many solutions if the lines coincide.

(*c*) No solution if the lines are parallel

For instance,

Case (*a*)	Case (*b*)	Case (*c*)
$x_1 + x_2 = 1$	$x_1 + x_2 = 1$	$x_1 + x_2 = 1$
$2x_1 - x_2 = 0$	$2x_1 + 2x_2 = 2$	$x_1 + x_2 = 0$



If the system is homogenous, Case (c) cannot happen, because then those two straight lines pass through the origin, whose coordinates 0, 0 constitute the trivial solution. If you wish, consider three equations in three unknowns as representations of three planes in space and discuss the various possible cases in a similar fashion. See Fig. 156. ■

Our simple example illustrates that a system (1) may perhaps have no solution. This poses the following problem. Does a given system (1) have a solution? Under what conditions does it have precisely one solution? If it has more than one solution, how can we characterize the set of all solutions? How can we actually obtain the solutions? Perhaps the last question is the most immediate one from a practical viewpoint. We shall answer it first and discuss the other questions in Sec. 7.5.

Gauss Elimination and Back Substitution

This is a standard elimination method for solving linear systems that proceeds systematically irrespective of particular features of the coefficients. It is a method of great practical importance and is reasonable with respect to computing time and storage demand (two aspects we shall consider in Sec. 20.1 in the chapter on numeric linear algebra). We begin by motivating the method. If a system is in **"triangular form,"** say,

$$2x_1 + 5x_2 = 2$$
$$13x_2 = -26$$

we can solve it by **"back substitution,"** that is, solve the last equation for the variable, $x_2 = -26/13 = -2$, and then work backward, substituting $x_2 = -2$ into the first equation and solve it for x_1, obtaining $x_1 = \frac{1}{2}(2 - 5x_2) = \frac{1}{2}(2 - 5\cdot(-2)) = 6$. This gives us the idea of first reducing a general system to triangular form. For instance, let the given system be

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30. \end{aligned} \qquad \text{Its augmented matrix is} \qquad \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}.$$

We leave the first equation as it is. We eliminate x_1 from the second equation, to get a triangular system. For this we add twice the first equation to the second, and we do the same operation on the ***rows*** of the augmented matrix. This gives $-4x_1 + 4x_1 + 3x_2 + 10x_2 = -30 + 2\cdot 2$, that is,

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \end{aligned} \qquad \begin{matrix} \\ \text{Row 2 + 2 Row 1} \end{matrix} \qquad \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

where Row 2 + 2 Row 1 means "Add twice Row 1 to Row 2" in the original matrix. This is the **Gauss elimination** (for 2 equations in 2 unknowns) giving the triangular form, from which back substitution now yields $x_2 = -2$ and $x_1 = 6$, as before.

Since a linear system is completely determined by its augmented matrix, ***Gauss elimination can be done by merely considering the matrices,*** as we have just indicated. We do this again in the next example, emphasizing the matrices by writing them first and the equations behind them, just as a help in order not to lose track.

EXAMPLE 2 Gauss Elimination. Electrical Network

Solve the linear system

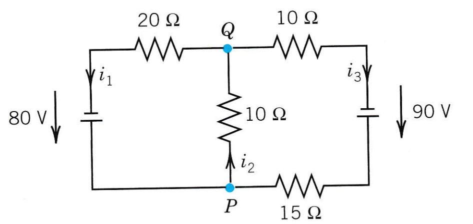
$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \qquad &= 80. \end{aligned}$$

Derivation from the circuit in Fig. 157 (Optional). This is the system for the unknown currents $x_1 = i_1$, $x_2 = i_2$, $x_3 = i_3$ in the electrical network in Fig. 157. To obtain it, we label the currents as shown, choosing directions arbitrarily; if a current will come out negative, this will simply mean that the current flows against the direction of our arrow. The current entering each battery will be the same as the current leaving it. The equations for the currents result from Kirchhoff's laws:

Kirchhoff's current law (KCL). *At any point of a circuit, the sum of the inflowing currents equals the sum of the outflowing currents.*

Kirchhoff's voltage law (KVL). *In any closed loop, the sum of all voltage drops equals the impressed electromotive force.*

Node P gives the first equation, node Q the second, the right loop the third, and the left loop the fourth, as indicated in the figure.



Node P: $\quad i_1 - i_2 + i_3 = 0$

Node Q: $\quad -i_1 + i_2 - i_3 = 0$

Right loop: $\quad 10i_2 + 25i_3 = 90$

Left loop: $\quad 20i_1 + 10i_2 = 80$

Fig. 157. Network in Example 2 and equations relating the currents

Solution by Gauss Elimination. This system could be solved rather quickly by noticing its particular form. But this is not the point. The point is that the Gauss elimination is systematic and will work in general, also for large systems. We apply it to our system and then do back substitution. As indicated let us write the augmented matrix of the system first and then the system itself:

Augmented Matrix $\tilde{\mathbf{A}}$

$$\begin{array}{l} \text{Pivot 1} \longrightarrow \\ \\ \text{Eliminate} \longrightarrow \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ \boxed{-1} & 1 & -1 & 0 \\ \boxed{0} & 10 & 25 & 90 \\ \boxed{20} & 10 & 0 & 80 \end{array}\right]$$

Equations

$$\begin{array}{l} \text{Pivot 1} \longrightarrow \\ \\ \text{Eliminate} \longrightarrow \\ \\ \end{array} \begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 0 \\ \boxed{-x_1} & + & x_2 & - & x_3 & = & 0 \\ & & 10x_2 & + & 25x_3 & = & 90 \\ \boxed{20x_1} & + & 10x_2 & & & = & 80. \end{array}$$

Step 1. Elimination of x_1

Call the first row of $\mathbf{A}$ the **pivot row** and the first equation the **pivot equation.** Call the coefficient 1 of its x_1-term the **pivot** in this step. Use this equation to eliminate x_1 (get rid of x_1) in the other equations. For this, do:

Add 1 times the pivot equation to the second equation.

Add -20 times the pivot equation to the fourth equation.

This corresponds to **row operations** on the augmented matrix as indicated in BLUE behind the ***new matrix*** in (3). So the operations are performed on the ***preceding matrix.*** The result is

$$\text{(3)} \qquad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \begin{array}{l} \\ \text{Row 2 + Row 1} \\ \\ \text{Row 4} - \text{20 Row 1} \end{array} \qquad \begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 0 \\ & & 10x_2 & + & 25x_3 & = & 90 \\ & & 30x_2 & - & 20x_3 & = & 80. \end{array}$$

Step 2. Elimination of x_2

The first equation remains as it is. We want the new second equation to serve as the next pivot equation. But since it has no x_2-term (in fact, it is $0 = 0$), we must first change the order of the equations and the corresponding rows of the new matrix. We put $0 = 0$ at the end and move the third equation and the fourth equation one place up. This is called **partial pivoting** (as opposed to the rarely used *total pivoting*, in which also the order of the unknowns is changed). It gives

$$\begin{array}{l} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate 30} \longrightarrow \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & \boxed{30} & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate } 30x_2 \longrightarrow \\ \\ \end{array} \begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 0 \\ & & 10x_2 & + & 25x_3 & = & 90 \\ & & \boxed{30x_2} & - & 20x_3 & = & 80 \\ & & & & 0 & = & 0 \end{array}$$

To eliminate x_2, do:

Add -3 times the pivot equation to the third equation.

The result is

$$\text{(4)} \qquad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - \text{3 Row 2} \\ \\ \end{array} \qquad \begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 0 \\ & & 10x_2 & + & 25x_3 & = & 90 \\ & & & - & 95x_3 & = & -190 \\ & & & & 0 & = & 0 \end{array}$$

Back Substitution. ***Determination of x_3, x_2, x_1 (in this order)***

Working backward from the last to the first equation of this "triangular" system (4), we can now readily find x_3, then x_2, and then x_1:

$$\begin{array}{rcrcrcr} & & & & -95x_3 & = & -190 \\ & & 10x_2 & + & 25x_3 & = & 90 \\ x_1 & - & x_2 & + & x_3 & = & 0 \end{array} \qquad \begin{array}{l} x_3 = i_3 = 2 \text{ [A]} \\ x_2 = \frac{1}{10}(90 - 25x_3) = i_2 = 4 \text{ [A]} \\ x_1 = x_2 - x_3 = i_1 = 2 \text{ [A]} \end{array}$$

where A stands for "amperes." This is the answer to our problem. The solution is unique. ■

Elementary Row Operations. Row-Equivalent Systems

Example 2 illustrates the operations of the Gauss elimination. These are the first two of three operations, which are called

Elementary Row Operations for Matrices:

Interchange of two rows

Addition of a constant multiple of one row to another row

Multiplication of a row by a ***nonzero*** *constant c.*

CAUTION! These operations are for rows, ***not for columns!*** They correspond to the following

Elementary Operations for Equations:

Interchange of two equations

Addition of a constant multiple of one equation to another equation

Multiplication of an equation by a ***nonzero*** *constant c.*

Clearly, the interchange of two equations does not alter the solution set. Neither does that addition because we can undo it by a corresponding subtraction. Similarly for that multiplication, which we can undo by multiplying the new equation by $1/c$ (since $c \neq 0$), producing the original equation.

We now call a linear system S_1 **row-equivalent** to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations. Thus we have proved the following result, which also justifies the Gauss elimination.

THEOREM 1

Row-Equivalent Systems

Row-equivalent linear systems have the same set of solutions.

Because of this theorem, systems having the same solution sets are often called *equivalent systems*. But note well that we are dealing with ***row operations.*** No column operations on the augmented matrix are permitted in this context because they would generally alter the solution set.

A linear system (1) is called **overdetermined** if it has more equations than unknowns, as in Example 2, **determined** if $m = n$, as in Example 1, and **underdetermined** if it has fewer equations than unknowns.

Furthermore, a system (1) is called **consistent** if it has at least one solution (thus, one solution or infinitely many solutions), but **inconsistent** if it has no solutions at all, as $x_1 + x_2 = 1$, $x_1 + x_2 = 0$ in Example 1.

Gauss Elimination: The Three Possible Cases of Systems

The Gauss elimination can take care of linear systems with a unique solution (see Example 2), with infinitely many solutions (Example 3, below), and without solutions (inconsistent systems; see Example 4).

EXAMPLE 3 **Gauss Elimination if Infinitely Many Solutions Exist**

Solve the following linear systems of three equations in four unknowns whose augmented matrix is

$$(5)\qquad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]. \qquad \text{Thus,} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ \boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1. \end{aligned}$$

Solution. As in the previous example, we circle pivots and box terms of equations and corresponding entries to be eliminated. We indicate the operations in terms of equations and operate on both equations and matrices.

Step 1. Elimination of x_1 from the second and third equations by adding

$$-0.6/3.0 = -0.2 \text{ times the first equation to the second equation,}$$

$$-1.2/3.0 = -0.4 \text{ times the first equation to the third equation.}$$

This gives the following, in which the pivot of the next step is circled.

$$(6)\qquad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - 0.2 \text{ Row 1} \\ \text{Row 3} - 0.4 \text{ Row 1} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ \boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned}$$

Step 2. Elimination of x_2 from the third equation of (6) by adding

$$1.1/1.1 = 1 \text{ times the second equation to the third equation.}$$

This gives

$$(7)\qquad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} + \text{Row 2} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0\ . \end{aligned}$$

Back Substitution. From the second equation, $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions. If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

On Notation. If unknowns remain arbitrary, it is also customary to denote them by other letters $t_1, t_2, \cdots$. In this example we may thus write $x_1 = 2 - x_4 = 2 - t_2$, $x_2 = 1 - x_3 + 4x_4 = 1 - t_1 + 4t_2$, $x_3 = t_1$ (first arbitrary unknown), $x_4 = t_2$ (second arbitrary unknown). ■

EXAMPLE 4 **Gauss Elimination if no Solution Exists**

What will happen if we apply the Gauss elimination to a linear system that has no solution? The answer is that in this case the method will show this fact by producing a contradiction. For instance, consider

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ \boxed{2x_1} + x_2 + x_3 &= 0 \\ \boxed{6x_1} + 2x_2 + 4x_3 &= 6. \end{aligned}$$

Step 1. Elimination of x_1 from the second and third equations by adding

$$-\tfrac{2}{3} \text{ times the first equation to the second equation,}$$

$$-\tfrac{6}{3} = -2 \text{ times the first equation to the third equation.}$$

This gives

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - \frac{2}{3} \text{ Row 1} \\ \text{Row 3} - 2 \text{ Row 1} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ \boxed{-2x_2} + 2x_3 &= 0. \end{aligned}$$

Step 2. Elimination of x_2 from the third equation gives

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \quad \text{Row 3} - 6 \text{ Row 2} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12. \end{aligned}$$

The false statement $0 = 12$ shows that the system has no solution. ■

Row Echelon Form and Information From It

At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix, and the system itself are called the **row echelon form.** In it, rows of zeros, if present, are the last rows, and in each nonzero row the leftmost nonzero entry is farther to the right than in the previous row. For instance, in Example 4 the coefficient matrix and its augmented in row echelon form are

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

Note that we do not require that the leftmost nonzero entries be 1 since this would have no theoretic or numeric advantage. (The so-called *reduced echelon form*, in which those entries *are* 1, will be discussed in Sec. 7.8.)

At the end of the Gauss elimination (before the back substitution) the row echelon form of the augmented matrix will be

$$\text{(8)} \qquad \left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ & c_{22} & \cdots & \cdots & c_{2n} & \tilde{b}_2 \\ & & \ddots & & \vdots & \vdots \\ & & & k_{rr} \;\; \cdots & k_{rn} & \tilde{b}_r \\ & & & & & \tilde{b}_{r+1} \\ & & & & & \vdots \\ & & & & & \tilde{b}_m \end{array}\right].$$

Here, $r \leqq m$ and $a_{11} \neq 0$, $c_{22} \neq 0, \cdots, k_{rr} \neq 0$, and all the entries in the blue triangle as well as in the blue rectangle are zero. From this we see that with respect to solutions of the system with augmented matrix (8) (and thus with respect to the originally given system) there are three possible cases:

(a) ***Exactly one solution*** if $r = n$ and $\tilde{b}_{r+1}, \cdots, \tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n = \tilde{b}_n$) for x_n, then the $(n-1)$st equation for x_{n-1}, and so on up the line. See Example 2, where $r = n = 3$ and $m = 4$.

(b) ***Infinitely many solutions*** if $r < n$ and $\tilde{b}_{r+1}, \cdots, \tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of $x_{r+1}, \cdots, x_n$ arbitrarily. Then solve the rth equation for x_r, then the $(r-1)$st equation for x_{r-1}, and so on up the line. See Example 3.

(c) ***No solution*** if $r < m$ and one of the entries $\tilde{b}_{r+1}, \cdots, \tilde{b}_m$ is not zero. See Example 4, where $r = 2 < m = 3$ and $\tilde{b}_{r+1} = \tilde{b}_3 = 12$.

PROBLEM SET 7.3

1–16 GAUSS ELIMINATION AND BACK SUBSTITUTION

Solve the following systems or indicate the nonexistence of solutions. (Show the details of your work.)

1. $$\begin{aligned} 5x - 2y &= 20.9 \\ -x + 4y &= -19.3 \end{aligned}$$

2. $$\begin{aligned} 3.0x + 6.2y &= 0.2 \\ 2.1x + 8.5y &= 4.3 \end{aligned}$$

3. $$\begin{aligned} 0.5x + 3.5y &= 5.7 \\ -x + 5.0y &= 7.8 \end{aligned}$$

4. $$\begin{aligned} 4y - 2z &= 2 \\ 6x - 2y + z &= 29 \\ 4x + 8y - 4z &= 24 \end{aligned}$$

5. $$\begin{aligned} 0.8x + 1.2y - 0.6z &= -7.8 \\ 2.6x + 1.7z &= 15.3 \\ 4.0x - 7.3y - 1.5z &= 1.1 \end{aligned}$$

6. $$\begin{aligned} 14x - 2y - 4z &= 0 \\ 18x - 2y - 6z &= 0 \\ 4x + 8y - 14z &= 0 \end{aligned}$$

7. $$\begin{aligned} y + z &= -2 \\ 4y + 6z &= -12 \\ x + y + z &= 2 \end{aligned}$$

8. $$\begin{aligned} 2x + y - 3z &= 8 \\ 5x + 2z &= 3 \\ 8x - y + 7z &= 0 \end{aligned}$$

9. $$\begin{aligned} 4y + 4z &= 24 \\ 3x - 11y - 2z &= -6 \\ 6x - 17y + z &= 18 \end{aligned}$$

10. $$\begin{aligned} 0.6x + 0.3y - 0.4z &= -1.9 \\ -4.6x + 0.5y + 1.2z &= -1.3 \end{aligned}$$

11. $$\begin{aligned} 2x - y + 3z &= -1 \\ -4x + 2y - 6z &= 2 \end{aligned}$$

12. $$\begin{aligned} -2y - 2z &= -8 \\ 3x + 4y - 5z &= 13 \end{aligned}$$

13. $$\begin{aligned} x + y - 2z &= 0 \\ -4w - x - y + 2z &= -4 \\ -2w + 3x + 3y - 6z &= -2 \end{aligned}$$

14. $$\begin{aligned} w - 2x + 5y - 3z &= 0 \\ -3w + 6x + y + z &= 0 \\ 2w - 4x + 3y - z &= 3 \end{aligned}$$

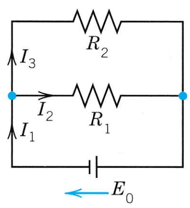
15. $$\begin{aligned} 3x + 7y - 4z &= -46 \\ 5w + 4x + 8y + z &= 7 \\ 8w + 4y - 2z &= 0 \\ -w + 6x + 2z &= 13 \end{aligned}$$

16. $$\begin{aligned} -2w - 17x + 4y + 3z &= 0 \\ 7w + 3y - 2z &= 0 \\ 2x + 8y - 6z &= -20 \\ 5w - 13x - y + 5z &= 16 \end{aligned}$$

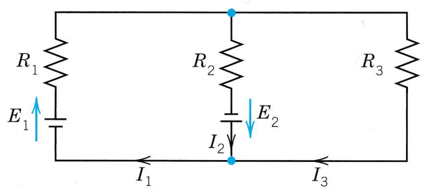
17–19 MODELS OF ELECTRICAL NETWORKS

Using Kirchhoff's laws (see Example 2), find the currents. (Show the details of your work.)

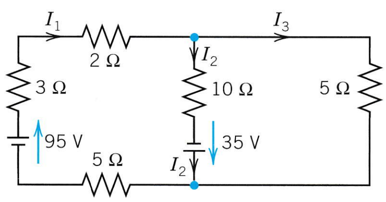
17.

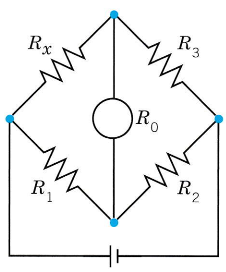


18.



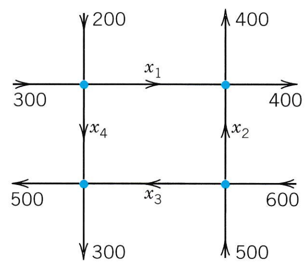
19.





Wheatstone bridge

(Prob. 20, next page)



Net of one-way streets

(Prob. 21, next page)

20. (Wheatstone bridge) Show that if $R_x/R_3 = R_1/R_2$ in the figure, then $I = 0$. (R_0 is the resistance of the instrument by which I is measured.) This bridge is a method for determining R_x. R_1, R_2, R_3 are known. R_3 is variable. To get R_x, make $I = 0$ by varing R_3. Then calculate $R_x = R_3R_1/R_2$.

21. (Traffic flow) Methods of electrical circuit analysis have applications to other fields. For instance, applying the analog of Kirchhoff's current law, find the traffic flow (cars per hour) in the net of one-way streets (in the directions indicated by the arrows) shown in the figure. Is the solution unique?

22. (Models of markets) Determine the equilibrium solution ($D_1 = S_1$, $D_2 = S_2$) of the two-commodity market with linear model (D, S, P = demand, supply, price; index 1 = first commodity, index 2 = second commodity)

$$D_1 = 60 - 2P_1 - P_2, \qquad S_1 = 4P_1 - 2P_2 + 14$$
$$D_2 = 4P_1 - P_2 + 10, \qquad S_2 = 5P_2 - 2.$$

23. (Equivalence relation) By definition, an *equivalence relation* on a set is a relation satisfying three conditions (named as indicated):

(i) Each element A of the set is equivalent to itself (*"Reflexivity"*).

(ii) If A is equivalent to B, then B is equivalent to A (*"Symmetry"*).

(iii) If A is equivalent to B and B is equivalent to C, then A is equivalent to C (*"Transitivity"*).

Show that row equivalence of matrices satisfies these three conditions. *Hint.* Show that for each of the three elementary row operations these conditions hold.

24. PROJECT. Elementary Matrices. The idea is that elementary operations can be accomplished by matrix multiplication. If $\mathbf{A}$ is an $m \times n$ matrix on which we want to do an elementary operation, then there is a matrix $\mathbf{E}$ such that $\mathbf{EA}$ is the new matrix after the operation. Such an $\mathbf{E}$ is called an **elementary matrix.** This idea can be helpful, for instance, in the design of algorithms. (*Computationally,* it is generally preferable to do row operations *directly,* rather than by multiplication by $\mathbf{E}$.)

(a) Show that the following are elementary matrices, for interchanging Rows 2 and 3, for adding -5 times the first row to the third, and for multiplying the fourth row by 8.

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

Apply $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ to a vector and to a 4×3 matrix of your choice. Find $\mathbf{B} = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}$, where $\mathbf{A} = [a_{jk}]$ is the general 4×2 matrix. Is $\mathbf{B}$ equal to $\mathbf{C} = \mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\mathbf{A}$?

(b) Conclude that $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are obtained by doing the corresponding elementary operations on the 4×4 unit matrix. Prove that *if* $\mathbf{M}$ *is obtained from* $\mathbf{A}$ *by an elementary row operation, then*

$$\mathbf{M} = \mathbf{EA},$$

where $\mathbf{E}$ *is obtained from the* $n \times n$ *unit matrix* $\mathbf{I}_n$ *by the same row operation.*

25. CAS PROJECT. Gauss Elimination and Back Substitution. Write a program for Gauss elimination and back substitution (a) that does not include pivoting, (b) that does include pivoting. Apply the programs to Probs. 13–16 and to some larger systems of your choice.

7.4 Linear Independence. Rank of a Matrix. Vector Space

In the last section we explained the Gauss elimination with back substitution, the most important numeric solution method for linear systems of equations. It appeared that such a system may have a unique solution or infinitely many solutions, or it may be inconsistent, that is, have no solution at all. Hence we are confronted with the questions of **existence and uniqueness of solutions.** We shall answer these questions in the next section. As the key concept for this (and other questions) we introduce the ***rank of a matrix.*** To define rank, we first need the following concepts, which are of general importance.

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ (with the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

where $c_1, c_2, \cdots, c_m$ are any scalars. Now consider the equation

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}. \tag{1}$$

Clearly, this vector equation (1) holds if we choose all c_j's zero, because then it becomes $\mathbf{0} = \mathbf{0}.$ If this is the only m-tuple of scalars for which (1) holds, then our vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ are said to form a *linearly independent set* or, more briefly, we call them **linearly independent.** Otherwise, if (1) also holds with scalars not all zero, we call these vectors **linearly dependent,** because then we can express (at least) one of them as a linear combination of the others. For instance, if (1) holds with, say, $c_1 \neq 0$, we can solve (1) for $\mathbf{a}_{(1)}$:

$$\mathbf{a}_{(1)} = k_2\mathbf{a}_{(2)} + \cdots + k_m\mathbf{a}_{(m)} \qquad \text{where } k_j = -c_j/c_1.$$

(Some k_j's may be zero. Or even all of them, namely, if $\mathbf{a}_{(1)} = \mathbf{0}.$)

Why is this important? Well, in the case of linear dependence we can get rid of some of the vectors until we arrive at a linearly independent set that is optimal to work with because it is smallest possible in the sense that it consists only of the "really essential" vectors, which can no longer be expressed linearly in terms of each other. This motivates the idea of a "basis" used in various contexts, notably later in our present section.

EXAMPLE 1 Linear Independence and Dependence

The three vectors

$$\begin{aligned} \mathbf{a}_{(1)} &= [\begin{matrix} 3 & 0 & 2 & 2 \end{matrix}] \\ \mathbf{a}_{(2)} &= [\begin{matrix} -6 & 42 & 24 & 54 \end{matrix}] \\ \mathbf{a}_{(3)} &= [\begin{matrix} 21 & -21 & 0 & -15 \end{matrix}] \end{aligned}$$

are linearly dependent because

$$6\mathbf{a}_{(1)} - \tfrac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0}.$$

Although this is easily checked (do it!), it is not so easy to discover. However, a systematic method for finding out about linear independence and dependence follows below.

The first two of the three vectors are linearly independent because $c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} = \mathbf{0}$ implies $c_2 = 0$ (from the second components) and then $c_1 = 0$ (from any other component of $\mathbf{a}_{(1)}$). ■

Rank of a Matrix

DEFINITION

The **rank** of a matrix **A** is the maximum number of linearly independent row vectors of **A.** It is denoted by rank **A.**

Our further discussion will show that the rank of a matrix is an important key concept for understanding general properties of matrices and linear systems of equations.

EXAMPLE 2 Rank

The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \tag{2}$$

has rank 2, because Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Note further that rank $\mathbf{A} = 0$ if and only if $\mathbf{A} = \mathbf{0}$. This follows directly from the definition. ■

We call a matrix $\mathbf{A}_1$ **row-equivalent** to a matrix $\mathbf{A}_2$ if $\mathbf{A}_1$ can be obtained from $\mathbf{A}_2$ by (finitely many!) elementary row operations.

Now the maximum number of linearly independent row vectors of a matrix does not change if we change the order of rows or multiply a row by an nonzero c or take a linear combination by adding a multiple of a row to another row. This proves that rank is **invariant** under elementary row operations:

THEOREM 1 Row-Equivalent Matrices

Row-equivalent matrices have the same rank.

Hence we can determine the rank of a matrix by reduction to row-echelon form (Sec. 7.3) and then see the rank directly.

EXAMPLE 3 Determination of Rank

For the matrix in Example 2 we obtain successively

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} \\ \text{Row 2} + 2 \text{ Row 1} \\ \text{Row 3} - 7 \text{ Row 1} \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ \\ \text{Row 3} + \tfrac{1}{2} \text{ Row 2} \end{matrix}$$

■

Since rank is defined in terms of two vectors, we immediately have the useful

THEOREM 2 Linear Independence and Dependence of Vectors

p vectors with n components each are linearly independent if the matrix with these vectors as row vectors has rank p, but they are linearly dependent if that rank is less than p.

Further important properties will result from the basic

THEOREM 3

Rank in Terms of Column Vectors

*The rank r of a matrix $\mathbf{A}$ equals the maximum number of linearly independent **column** vectors of $\mathbf{A}$.*

Hence $\mathbf{A}$ and its transpose $\mathbf{A}^{\mathsf{T}}$ have the same rank.

PROOF In this proof we write simply "rows" and "columns" for row and column vectors. Let $\mathbf{A}$ be an $m \times n$ matrix of rank $\mathbf{A} = r$. Then by definition of rank, $\mathbf{A}$ has r linearly independent rows which we denote by $\mathbf{v}_{(1)}, \cdots, \mathbf{v}_{(r)}$ (regardless of their position in $\mathbf{A}$), and all the rows $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ of $\mathbf{A}$ are linear combinations of those, say,

$$\begin{aligned} \mathbf{a}_{(1)} &= c_{11}\mathbf{v}_{(1)} + c_{12}\mathbf{v}_{(2)} + \cdots + c_{1r}\mathbf{v}_{(r)} \\ \mathbf{a}_{(2)} &= c_{21}\mathbf{v}_{(1)} + c_{22}\mathbf{v}_{(2)} + \cdots + c_{2r}\mathbf{v}_{(r)} \\ &\vdots \\ \mathbf{a}_{(m)} &= c_{m1}\mathbf{v}_{(1)} + c_{m2}\mathbf{v}_{(2)} + \cdots + c_{mr}\mathbf{v}_{(r)}. \end{aligned} \tag{3}$$

These are vector equations for rows. To switch to columns, we write (3) in terms of components as n such systems, with $k = 1, \cdots, n$,

$$\begin{aligned} a_{1k} &= c_{11}v_{1k} + c_{12}v_{2k} + \cdots + c_{1r}v_{rk} \\ a_{2k} &= c_{21}v_{1k} + c_{22}v_{2k} + \cdots + c_{2r}v_{rk} \\ &\vdots \\ a_{mk} &= c_{m1}v_{1k} + c_{m2}v_{2k} + \cdots + c_{mr}v_{rk} \end{aligned} \tag{4}$$

and collect components in columns. Indeed, we can write (4) as

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{bmatrix} + \cdots + v_{rk} \begin{bmatrix} c_{1r} \\ c_{2r} \\ \vdots \\ c_{mr} \end{bmatrix} \tag{5}$$

where $k = 1, \cdots, n$. Now the vector on the left is the kth column vector of $\mathbf{A}$. We see that each of these n columns is a linear combination of the same r columns on the right. Hence $\mathbf{A}$ cannot have more linearly independent columns than rows, whose number is rank $\mathbf{A} = r$. Now rows of $\mathbf{A}$ are columns of the transpose $\mathbf{A}^{\mathsf{T}}$. For $\mathbf{A}^{\mathsf{T}}$ our conclusion is that $\mathbf{A}^{\mathsf{T}}$ cannot have more linearly independent columns than rows, so that $\mathbf{A}$ cannot have more linearly independent rows than columns. Together, the number of linearly independent columns of $\mathbf{A}$ must be r, the rank of $\mathbf{A}$. This completes the proof. ■

EXAMPLE 4 **Illustration of Theorem 3**

The matrix in (2) has rank 2. From Example 3 we see that the first two row vectors are linearly independent and by "working backward" we can verify that Row 3 = 6 Row 1 $-\frac{1}{2}$ Row 2. Similarly, the first two columns are linearly independent, and by reducing the last matrix in Example 3 by columns we find that

Column 3 = $\frac{2}{3}$ Column 1 + $\frac{2}{3}$ Column 2 and Column 4 = $\frac{2}{3}$ Column 1 + $\frac{29}{21}$ Column 2. ■

Combining Theorems 2 and 3 we obtain

THEOREM 4

Linear Dependence of Vectors

p vectors with $n < p$ components are always linearly dependent.

PROOF The matrix $\mathbf{A}$ with those p vectors as row vectors has p rows and $n < p$ columns; hence by Theorem 3 it has rank $\mathbf{A} \leqq n < p$, which implies linear dependence by Theorem 2. ■

Vector Space

The following related concepts are of general interest in linear algebra. In the present context they provide a clarification of essential properties of matrices and their role in connection with linear systems.

A **vector space** is a (nonempty) set V of vectors such that with any two vectors $\mathbf{a}$ and $\mathbf{b}$ in V all their linear combinations $\alpha\mathbf{a} + \beta\mathbf{b}$ (α, β any real numbers) are elements of V, and these vectors satisfy the laws (3) and (4) in Sec. 7.1 (written in lowercase letters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\cdots$, which is our notation for vectors). (This definition is presently sufficient. General vector spaces will be discussed in Sec. 7.9.)

The maximum number of linearly independent vectors in V is called the **dimension** of V and is denoted by dim V. Here we assume the dimension to be finite; infinite dimension will be defined in Sec. 7.9.

A linearly independent set in V consisting of a maximum possible number of vectors in V is called a **basis** for V. Thus the number of vectors of a basis for V equals dim V.

The set of all linear combinations of given vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(p)}$ with the same number of components is called the **span** of these vectors. Obviously, a span is a vector space.

By a **subspace** of a vector space V we mean a nonempty subset of V (including V itself) that forms itself a vector space with respect to the two algebraic operations (addition and scalar multiplication) defined for the vectors of V.

EXAMPLE 5 **Vector Space, Dimension, Basis**

The span of the three vectors in Example 1 is a vector space of dimension 2, and a basis is $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$, for instance, or $\mathbf{a}_{(1)}$, $\mathbf{a}_{(3)}$, etc. ■

We further note the simple

THEOREM 5

Vector Space R^n

The vector space R^n consisting of all vectors with n components (n real numbers) has dimension n.

PROOF A basis of n vectors is $\mathbf{a}_{(1)} = [1 \quad 0 \quad \cdots \quad 0]$, $\mathbf{a}_{(2)} = [0 \quad 1 \quad 0 \quad \cdots \quad 0], \cdots$, $\mathbf{a}_{(n)} = [0 \quad \cdots \quad 0 \quad 1]$. ■

In the case of a matrix $\mathbf{A}$ we call the span of the row vectors the **row space** of $\mathbf{A}$ and the span of the column vectors the **column space** of $\mathbf{A}$.

Now, Theorem 3 shows that a matrix **A** has as many linearly independent rows as columns. By the definition of dimension, their number is the dimension of the row space or the column space of **A**. This proves

THEOREM 6 **Row Space and Column Space**

The row space and the column space of a matrix **A** *have the same dimension, equal to rank* **A**.

Finally, for a given matrix **A** the solution set of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ is a vector space, called the **null space** of **A**, and its dimension is called the **nullity** of **A**. In the next section we motivate and prove the basic relation

$$\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = \text{Number of columns of } \mathbf{A}. \tag{6}$$

PROBLEM SET 7.4

1–12 RANK, ROW SPACE, COLUMN SPACE

Find the rank and a basis for the row space and for the column space. *Hint.* Row-reduce the matrix and its transpose. (You may omit obvious factors from the vectors of these bases.)

1. $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -3 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 8 & 2 & 5 \\ 16 & 6 & 29 \\ 4 & 0 & -7 \end{bmatrix}$

3. $\begin{bmatrix} 0 & -2 & 1 & 3 \\ 1 & 4 & 0 & 7 \\ 5 & 5 & 5 & 5 \end{bmatrix}$

4. $\begin{bmatrix} a & b & c \\ b & a & c \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -2 & 3 & -4 \\ 2 & -3 & 4 & -1 \\ 3 & -4 & 1 & -2 \\ 4 & -1 & 2 & -3 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 5 & 8 & -37 \\ 3 & 8 & 7 & 0 \\ 0 & -37 & 0 & 37 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \\ 2 & 16 & 8 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 0 & -7 & 1 \\ 0 & 0 & 5 & 0 \\ -7 & 5 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

13–20 LINEAR INDEPENDENCE

Are the following sets of vectors linearly independent? (Show the details.)

13. $[3 \quad -2 \quad 0 \quad 4]$, $[5 \quad 0 \quad 0 \quad 1]$, $[-6 \quad 1 \quad 0 \quad 1]$, $[2 \quad 0 \quad 0 \quad 3]$

14. $[1 \quad 1 \quad 0]$, $[1 \quad 0 \quad 0]$, $[1 \quad 1 \quad 1]$

15. $[6 \quad 0 \quad 3 \quad 1 \quad 4 \quad 2]$, $[0 \quad -1 \quad 2 \quad 7 \quad 0 \quad 5]$, $[12 \quad 3 \quad 0 \quad -19 \quad 8 \quad -11]$

16. $[3 \quad 4 \quad 7]$, $[2 \quad 0 \quad 3]$, $[8 \quad 2 \quad 3]$, $[5 \quad 5 \quad 6]$

17. $[0.2 \quad 1.2 \quad 5.3 \quad 2.8 \quad 1.6]$, $[4.3 \quad 3.4 \quad 0.9 \quad 2.0 \quad -4.3]$

18. $[3 \quad 2 \quad 1]$, $[0 \quad 0 \quad 0]$, $[4 \quad 3 \quad 6]$

19. $[1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4}]$, $[\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5}]$, $[\frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6}]$, $[\frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7}]$

20. $[1 \quad 2 \quad 3 \quad 4]$, $[2 \quad 3 \quad 4 \quad 5]$, $[3 \quad 4 \quad 5 \quad 6]$, $[4 \quad 5 \quad 6 \quad 7]$

21. CAS Experiment. Rank. **(a)** Show experimentally that the $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ with $a_{jk} = j + k - 1$ has rank 2 for any n. (Problem 20 shows $n = 4$.) Try to prove it.

(b) Do the same when $a_{jk} = j + k + c$, where c is any positive integer.

(c) What is rank $\mathbf{A}$ if $a_{jk} = 2^{j+k-2}$? Try to find other large matrices of low rank independent of n.

22–26 PROPERTIES OF RANK AND CONSEQUENCES

Show the following.

22. rank $\mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$ = rank $\mathbf{AB}$. (Note the order!)

23. rank $\mathbf{A}$ = rank $\mathbf{B}$ does *not* imply rank $\mathbf{A}^2$ = rank $\mathbf{B}^2$. (Give a counterexample.)

24. If $\mathbf{A}$ is not square, either the row vectors or the column vectors of $\mathbf{A}$ are linearly dependent.

25. If the row vectors of a square matrix are linearly independent, so are the column vectors, and conversely.

26. Give examples showing that the rank of a product of matrices cannot exceed the rank of either factor.

27–36 VECTOR SPACES

Is the given set of vectors a vector space? (Give reason.) If your answer is yes, determine the dimension and find a basis. ($v_1, v_2, \cdots$ denote components.)

27. All vectors in R^3 such that $v_1 + v_2 = 0$

28. All vectors in R^4 such that $2v_2 - 3v_4 = k$

29. All vectors in R^3 with $v_1 \geqq 0$, $v_2 = -4v_3$

30. All vectors in R^2 with $v_1 \leqq v_2$

31. All vectors in R^3 with $4v_1 + v_3 = 0$, $3v_2 = v_3$

32. All vectors in R^4 with $v_1 - v_2 = 0$, $v_3 = 5v_1$, $v_4 = 0$

33. All vectors in R^n with $|v_j| \leqq 1$ for $j = 1, \cdots, n$

34. All ordered quadruples of positive real numbers

35. All vectors in R^5 with $v_1 = 2v_2 = 3v_3 = 4v_4 = 5v_5$

36. All vectors in R^4 with $3v_1 - v_3 = 0$, $2v_1 + 3v_2 - 4v_4 = 0$

7.5 Solutions of Linear Systems: Existence, Uniqueness

Rank as just defined gives complete information about existence, uniqueness, and general structure of the solution set of linear systems as follows.

A linear system of equations in n unknowns has a unique solution if the coefficient matrix and the augmented matrix have the same rank n, and infinitely many solution if that common rank is less than n. The system has no solution if those two matrices have different rank.

To state this precisely and prove it, we shall use the (generally important) concept of a **submatrix** of $\mathbf{A}$. By this we mean any matrix obtained from $\mathbf{A}$ by omitting some rows or columns (or both). By definition this includes $\mathbf{A}$ itself (as the matrix obtained by omitting no rows or columns); this is practical.

THEOREM 1

Fundamental Theorem for Linear Systems

(a) Existence. *A linear system of m equations in n unknowns $x_1, \cdots, x_n$*

$$(1) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is **consistent,** *that is, has solutions, if and only if the coefficient matrix* **A** *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank. Here,*

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \textit{and} \quad \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

(b) Uniqueness. *The system* (1) *has precisely one solution if and only if this common rank r of* **A** *and* $\tilde{\mathbf{A}}$ *equals n.*

(c) Infinitely many solutions. *If this common rank r is less than n, the system* (1) *has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns* (*whose submatrix of coefficients must have rank r*) *in terms of the remaining n − r unknowns, to which arbitrary values can be assigned.* (See Example 3 in Sec. 7.3.)

(d) Gauss elimination (Sec. 7.3). *If solutions exist, they can all be obtained by the Gauss elimination.* (This method will automatically reveal whether or not solutions exist; see Sec. 7.3.)

PROOF **(a)** We can write the system (1) in vector form $\mathbf{Ax} = \mathbf{b}$ or in terms of column vectors $\mathbf{c}_{(1)}, \cdots, \mathbf{c}_{(n)}$ of **A:**

$$\mathbf{c}_{(1)}x_1 + \mathbf{c}_{(2)}x_2 + \cdots + \mathbf{c}_{(n)}x_n = \mathbf{b}. \tag{2}$$

$\tilde{\mathbf{A}}$ is obtained by augmenting **A** by a single column **b.** Hence, by Theorem 3 in Sec. 7.4, rank $\tilde{\mathbf{A}}$ equals rank **A** or rank $\mathbf{A} + 1$. Now if (1) has a solution **x,** then (2) shows that **b** must be a linear combination of those column vectors, so that $\tilde{\mathbf{A}}$ and **A** have the same maximum number of linearly independent column vectors and thus the same rank.

Conversely, if rank $\tilde{\mathbf{A}}$ = rank **A,** then **b** must be a linear combination of the column vectors of **A,** say,

$$\mathbf{b} = \alpha_1\mathbf{c}_{(1)} + \cdots + \alpha_n\mathbf{c}_{(n)} \tag{2*}$$

since otherwise rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A} + 1$. But (2*) means that (1) has a solution, namely, $x_1 = \alpha_1, \cdots, x_n = \alpha_n$, as can be seen by comparing (2*) and (2).

(b) If rank $\mathbf{A} = n$, the n column vectors in (2) are linearly independent by Theorem 3 in Sec. 7.4. We claim that then the representation (2) of **b** is unique because otherwise

$$\mathbf{c}_{(1)}x_1 + \cdots + \mathbf{c}_{(n)}x_n = \mathbf{c}_{(1)}\tilde{x}_1 + \cdots + \mathbf{c}_{(n)}\tilde{x}_n.$$

This would imply (take all terms to the left, with a minus sign)

$$(x_1 - \tilde{x}_1)\mathbf{c}_{(1)} + \cdots + (x_n - \tilde{x}_n)\mathbf{c}_{(n)} = \mathbf{0}$$

and $x_1 - \tilde{x}_1 = 0, \cdots, x_n - \tilde{x}_n = 0$ by linear independence. But this means that the scalars $x_1, \cdots, x_n$ in (2) are uniquely determined, that is, the solution of (1) is unique.

(c) If rank $\mathbf{A}$ = rank $\tilde{\mathbf{A}} = r < n$, then by Theorem 3 in Sec. 7.4 there is a linearly independent set K of r column vectors of $\mathbf{A}$ such that the other $n - r$ column vectors of $\mathbf{A}$ are linear combinations of those vectors. We renumber the columns and unknowns, denoting the renumbered quantities by $\hat{}$, so that $\{\hat{\mathbf{c}}_{(1)}, \cdots, \hat{\mathbf{c}}_{(r)}\}$ is that linearly independent set K. Then (2) becomes

$$\hat{\mathbf{c}}_{(1)}\hat{x}_1 + \cdots + \hat{\mathbf{c}}_{(r)}\hat{x}_r + \hat{\mathbf{c}}_{(r+1)}\hat{x}_{r+1} + \cdots + \hat{\mathbf{c}}_{(n)}\hat{x}_n = \mathbf{b},$$

$\hat{\mathbf{c}}_{(r+1)}, \cdots, \hat{\mathbf{c}}_{(n)}$ are linear combinations of the vectors of K, and so are the vectors $\hat{x}_{r+1}\hat{\mathbf{c}}_{(r+1)}, \cdots, \hat{x}_n\hat{\mathbf{c}}_{(n)}$. Expressing these vectors in terms of the vectors of K and collecting terms, we can thus write the system in the form

$$\hat{\mathbf{c}}_{(1)}y_1 + \cdots + \hat{\mathbf{c}}_{(r)}y_r = \mathbf{b} \tag{3}$$

with $y_j = \hat{x}_j + \beta_j$, where β_j results from the $n - r$ terms $\hat{\mathbf{c}}_{(r+1)}\hat{x}_{r+1}, \cdots, \hat{\mathbf{c}}_{(n)}\hat{x}_n$; here, $j = 1, \cdots, r$. Since the system has a solution, there are $y_1, \cdots, y_r$ satisfying (3). These scalars are unique since K is linearly independent. Choosing $\hat{x}_{r+1}, \cdots, \hat{x}_n$ fixes the β_j and corresponding $\hat{x}_j = y_j - \beta_j$, where $j = 1, \cdots, r$.

(d) This was discussed in Sec. 7.3 and is restated here as a reminder. ■

The theorem is illustrated in Sec. 7.3. In Example 2 there is a unique solution since rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A} = n = 3$ (as can be seen from the last matrix in the example). In Example 3 we have rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A} = 2 < n = 4$ and can choose x_3 and x_4 arbitrarily. In Example 4 there is no solution because rank $\mathbf{A} = 2 <$ rank $\tilde{\mathbf{A}} = 3$.

Homogeneous Linear System

Recall from Sec. 7.3 that a linear system (1) is called **homogeneous** if all the b_j's are zero, and **nonhomogeneous** if one or several b_j's are not zero. For the homogeneous system we obtain from the Fundamental Theorem the following results.

THEOREM 2

Homogeneous Linear System

A homogeneous linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{4}$$

always has the **trivial solution** $x_1 = 0, \cdots, x_n = 0$. *Nontrivial solutions exist if and only if rank* $\mathbf{A} < n$. *If* rank $\mathbf{A} = r < n$, *these solutions, together with* $\mathbf{x} = \mathbf{0}$, *form a vector space* (see Sec. 7.4) *of dimension* $n - r$, called the **solution space** of (4).

In particular, if $\mathbf{x}_{(1)}$ *and* $\mathbf{x}_{(2)}$ *are solution vectors of* (4), *then* $\mathbf{x} = \mathbf{c}_1\mathbf{x}_{(1)} + \mathbf{c}_2\mathbf{x}_{(2)}$ *with any scalars* c_1 *and* c_2 *is a solution vector of* (4). (This ***does not hold*** for nonhomogeneous systems. Also, the term *solution space* is used for *homogeneous* systems only.)

PROOF The first proposition can be seen directly from the system. It agrees with the fact that $\mathbf{b} = \mathbf{0}$ implies that rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A}$, so that a homogeneous system is always ***consistent.*** If rank $\mathbf{A} = n$, the trivial solution is the unique solution according to (b) in Theorem 1. If rank $\mathbf{A} < n$, there are nontrivial solutions according to (c) in Theorem 1. The solutions form a vector space because if $\mathbf{x}_{(1)}$ and $\mathbf{x}_{(2)}$ are any of them, then $\mathbf{A}\mathbf{x}_{(1)} = \mathbf{0}$, $\mathbf{A}\mathbf{x}_{(2)} = \mathbf{0}$, and this implies $\mathbf{A}(\mathbf{x}_{(1)} + \mathbf{x}_{(2)}) = \mathbf{A}\mathbf{x}_{(1)} + \mathbf{A}\mathbf{x}_{(2)} = \mathbf{0}$ as well as $\mathbf{A}(c\mathbf{x}_{(1)}) = c\mathbf{A}\mathbf{x}_{(1)} = \mathbf{0}$, where c is arbitrary. If rank $\mathbf{A} = r < n$, Theorem 1 (c) implies that we can choose $n - r$ suitable unknowns, call them $x_{r+1}, \cdots, x_n$, in an arbitrary fashion, and every solution is obtained in this way. Hence a basis for the solution space, briefly called a **basis of solutions** of (4), is $\mathbf{y}_{(1)}, \cdots, \mathbf{y}_{(n-r)}$, where the basis vector $\mathbf{y}_{(j)}$ is obtained by choosing $x_{r+j} = 1$ and the other $x_{r+1}, \cdots, x_n$ zero; the corresponding first r components of this solution vector are then determined. Thus the solution space of (4) has dimension $n - r$. This proves Theorem 2. ■

The solution space of (4) is also called the **null space** of $\mathbf{A}$ because $\mathbf{A}\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ in the solution space of (4). Its dimension is called the **nullity** of $\mathbf{A}$. Hence Theorem 2 states that

$$\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n \tag{5}$$

where n is the number of unknowns (number of columns of $\mathbf{A}$).

Furthermore, by the definition of rank we have rank $\mathbf{A} \leqq m$ in (4). Hence if $m < n$, then rank $\mathbf{A} < n$. By Theorem 2 this gives the practically important

THEOREM 3

Homogeneous Linear System with Fewer Equations Than Unknowns

A homogeneous linear system with fewer equations than unknowns has always nontrivial solutions.

Nonhomogeneous Linear Systems

The characterization of all solutions of the linear system (1) is now quite simple, as follows.

THEOREM 4

Nonhomogeneous Linear System

If a nonhomogeneous linear system (1) *is consistent, then all of its solutions are obtained as*

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h \tag{6}$$

where $\mathbf{x}_0$ *is any* (fixed) *solution of* (1) *and* $\mathbf{x}_h$ *runs through all the solutions of the corresponding homogeneous system* (4).

PROOF The difference $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_0$ of any two solutions of (1) is a solution of (4) because $\mathbf{A}\mathbf{x}_h = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Since $\mathbf{x}$ is any solution of (1), we get all the solutions of (1) if in (6) we take any solution $\mathbf{x}_0$ of (1) and let $\mathbf{x}_h$ vary throughout the solution space of (4). ■

7.6 For Reference: Second- and Third-Order Determinants

We explain these determinants separately from the general theory in Sec. 7.7 because they will be sufficient for many of our examples and problems. Since this section is for reference, ***go on to the next section, consulting this material only when needed.***

A **determinant of second order** is denoted and defined by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{1}$$

So here we have ***bars*** (whereas a matrix has ***brackets***).

Cramer's rule for solving linear systems of two equations in two unknowns

$$\begin{aligned} &\text{(a)} \quad a_{11}x_1 + a_{12}x_2 = b_1 \\ &\text{(b)} \quad a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned} \tag{2}$$

is

$$\begin{aligned} x_1 &= \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D}, \\ x_2 &= \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - b_1a_{21}}{D}. \end{aligned} \tag{3}$$

with D as in (1), provided

$$D \neq 0.$$

The value $D = 0$ appears for inconsistent nonhomogeneous systems and for homogeneous systems with nontrivial solutions.

PROOF We prove (3). To eliminate x_2, multiply (2a) by a_{22} and (2b) by $-a_{12}$ and add,

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2.$$

Similarly, to eliminate x_1, multiply (2a) by $-a_{21}$ and (2b) by a_{11} and add,

$$(a_{11}a_{22} - a_{12}a_{21})x_2 = a_{11}b_2 - b_1a_{21}.$$

Assuming that $D = a_{11}a_{22} - a_{12}a_{21} \neq 0$, dividing, and writing the right sides of these two equations as determinants, we obtain (3). ∎

EXAMPLE 1 Cramer's Rule for Two Equations

If $\begin{aligned} 4x_1 + 3x_2 &= 12 \\ 2x_1 + 5x_2 &= -8 \end{aligned}$ then $x_1 = \dfrac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{84}{14} = 6, \quad x_2 = \dfrac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{-56}{14} = -4.$ ■

Third-Order Determinants

A **determinant of third order** can be defined by

$$\textbf{(4)} \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Note the following. The signs on the right are $+ \; - \; +$. Each of the three terms on the right is an entry in the first column of D times its **minor,** that is, the second-order determinant obtained from D by deleting the row and column of that entry; thus, for a_{11} delete the first row and first column, and so on.

If we write out the minors in (4), we obtain

$$(4^*) \qquad D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}.$$

Cramer's Rule for Linear Systems of Three Equations

$$(5) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

is

$$\textbf{(6)} \qquad x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}, \qquad x_3 = \frac{D_3}{D} \qquad (D \neq 0)$$

with the *determinant D of the system* given by (4) and

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3, respectively, by the column of the right sides of (5).

Cramer's rule (6) can be derived by eliminations similar to those for (3), but it also follows from the general case (Theorem 4) in the next section.

7.7 Determinants. Cramer's Rule

Determinants were originally introduced for solving linear systems. Although ***impractical in computations,*** they have important engineering applications in eigenvalue problems (Sec. 8.1), differential equations, vector algebra (Sec. 9.3), and so on. They can be introduced in several equivalent ways. Our definition is particularly practical in connection with linear systems.

A **determinant of order** n is a scalar associated with an $n \times n$ (hence ***square!***) matrix $\mathbf{A} = [a_{jk}]$, which is written

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{1}$$

and is defined for $n = 1$ by

$$D = a_{11} \tag{2}$$

and for $n \geqq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad (j = 1, 2, \cdots, \text{ or } n) \tag{3a}$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \qquad (k = 1, 2, \cdots, \text{ or } n) \tag{3b}$$

Here,

$$C_{jk} = (-1)^{j+k}M_{jk}$$

and M_{jk} is a determinant of order $n - 1$, namely, the determinant of the submatrix of $\mathbf{A}$ obtained from $\mathbf{A}$ by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

In this way, D is defined in terms of n determinants of order $n - 1$, each of which is, in turn, defined in terms of $n - 1$ determinants of order $n - 2$, and so on; we finally arrive at second-order determinants, in which those submatrices consist of single entries whose determinant is defined to be the entry itself.

From the definition it follows that *we may **expand** D by any row or column,* that is, choose in (3) the entries in any row or column, similarly when expanding the C_{jk}'s in (3), and so on.

This definition is unambiguous, that is, yields the same value for D no matter which columns or rows we choose in expanding. A proof is given in App. 4.

Terms used in connection with determinants are taken from matrices. In D we have n^2 **entries** a_{jk}, also n **rows** and n **columns,** and a **main diagonal** on which $a_{11}, a_{22}, \cdots, a_{nn}$ stand. Two terms are new:

M_{jk} is called the **minor** *of* a_{jk} *in* D, and C_{jk} the **cofactor** *of* a_{jk} *in* D.

For later use we note that (3) may also be written in terms of minors

(4a)
$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{ or } n)$$

(4b)
$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (k = 1, 2, \cdots, \text{ or } n).$$

EXAMPLE 1 **Minors and Cofactors of a Third-Order Determinant**

In (4) of the previous section the minors and cofactors of the entries in the first column can be seen directly. For the entries in the second row the minors are

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the cofactors are $C_{21} = -M_{21}$, $C_{22} = +M_{22}$, and $C_{23} = -M_{23}$. Similarly for the third row—write these down yourself. And verify that the signs in C_{jk} form a **checkerboard pattern**

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

■

EXAMPLE 2 **Expansions of a Third-Order Determinant**

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12,$$

Verify that the other four expansions also give the value -12. ■

EXAMPLE 3 **Determinant of a Triangular Matrix**

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60.$$

Inspired by this, can you formulate a little theorem on determinants of triangular matrices? Of diagonal matrices? ■

General Properties of Determinants

To obtain the value of a determinant (1), we can first simplify it systematically by elementary row operations, similar to those for matrices in Sec. 7.3, as follows.

THEOREM 1

Behavior of an nth-Order Determinant under Elementary Row Operations

(a) *Interchange of two rows multiplies the value of the determinant by* -1.

(b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*

(c) *Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.* (This holds also when $c = 0$, but gives no longer an elementary row operation.)

PROOF **(a)** By induction. The statement holds for $n = 2$ because

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \text{but} \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad.$$

We now make the induction hypothesis that (a) holds for determinants of order $n - 1 \geqq 2$ and show that it then holds for determinants of order n. Let D be of order n. Let E be obtained from D by the interchange of two rows. Expand D and E by a row that is ***not*** one of those interchanged, call it the jth row. Then by (4a),

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}, \qquad E = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} N_{jk} \tag{5}$$

where N_{jk} is obtained from the minor M_{jk} of a_{jk} in D by the interchange of those two rows which have been interchanged in D (and which N_{jk} must both contain because we expand by another row!). Now these minors are of order $n - 1$. Hence the induction hypothesis applies and gives $N_{jk} = -M_{jk}$. Thus $E = -D$ by (5).

(b) Add c times Row i to Row j. Let $\tilde{D}$ be the new determinant. Its entries in Row j are $a_{jk} + ca_{ik}$. If we expand $\tilde{D}$ by this Row j, we see that we can write it as $\tilde{D} = D_1 + cD_2$, where $D_1 = D$ has in Row j the a_{jk}, whereas D_2 has in that Row j the a_{ik} from the addition. Hence D_2 has a_{ik} in both Row i and Row j. Interchanging these two rows gives D_2 back, but on the other hand it gives $-D_2$ by (a). Together $D_2 = -D_2 = 0$, so that $\tilde{D} = D_1 = D$.

(c) Expand the determinant by the row that has been multiplied.

CAUTION! $\det(c\mathbf{A}) = c^n \det \mathbf{A}$ (not $c \det \mathbf{A}$). Explain why. ■

EXAMPLE 4 **Evaluation of Determinants by Reduction to Triangular Form**

Because of Theorem 1 we may evaluate determinants by reduction to triangular form, as in the Gauss elimination for a matrix. For instance (with the blue explanations always referring to the ***preceding determinant***)

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{matrix} \\ \text{Row 2} - 2 \text{ Row 1} \\ \\ \text{Row 4} + 1.5 \text{ Row 1} \end{matrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{matrix} \\ \\ \text{Row 3} - 0.4 \text{ Row 2} \\ \text{Row 4} - 1.6 \text{ Row 2} \end{matrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -0 & 47.25 \end{vmatrix} \begin{matrix} \\ \\ \\ \text{Row 4} + 4.75 \text{ Row 3} \end{matrix}$$

$$= 2 \cdot 5 \cdot 2.4 \cdot 47.25 = 1134.$$ ■

THEOREM 2

Further Properties of nth-Order Determinants

(a)–(c) *in Theorem* 1 *hold also for columns.*

(d) ***Transposition*** *leaves the value of a determinant unaltered.*

(e) ***A zero row or column*** *renders the value of a determinant zero.*

(f) ***Proportional rows or columns*** *render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

PROOF **(a)–(e)** follow directly from the fact that a determinant can be expanded by any row column. In **(d),** transposition is defined as for matrices, that is, the jth row becomes the jth column of the transpose.

(f) If Row $j = c$ times Row i, then $D = cD_1$, where D_1 has Row j = Row i. Hence an interchange of these rows reproduces D_1, but it also gives $-D_1$ by Theorem 1(a). Hence $D_1 = 0$ and $D = cD_1 = 0$. Similarly for columns. ■

It is quite remarkable that the important concept of the rank of a matrix $\mathbf{A}$, which is the maximum number of linearly independent row or column vectors of $\mathbf{A}$ (see Sec. 7.4), can be related to determinants. Here we may assume that rank $\mathbf{A} > 0$ because the only matrices with rank 0 are the zero matrices (see Sec. 7.4).

THEOREM 3

Rank in Terms of Determinants

An $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ has rank $r \geqq 1$ if and only if $\mathbf{A}$ has an $r \times r$ submatrix with nonzero determinant, whereas every square submatrix with more than r rows that $\mathbf{A}$ has (or does not have!) has determinant equal to zero.

In particular, if $\mathbf{A}$ is square, $n \times n$, it has rank n if and only if

$$\det \mathbf{A} \neq 0.$$

PROOF The key idea is that elementary row operations (Sec. 7.3) alter neither rank (by Theorem 1 in Sec. 7.4) nor the property of a determinant being nonzero (by Theorem 1 in this section). The echelon form $\hat{\mathbf{A}}$ of $\mathbf{A}$ (see Sec. 7.3) has r nonzero row vectors (which are the first r row vectors) if and only if rank $\mathbf{A} = r$. Let $\hat{\mathbf{R}}$ be the $r \times r$ submatrix in the left upper corner of $\hat{\mathbf{A}}$ (so that the entries of $\hat{\mathbf{R}}$ are in both the first r rows and r columns of $\hat{\mathbf{A}}$). Now $\hat{\mathbf{R}}$ is triangular, with all diagonal entries r_{jj} nonzero. Thus, det $\hat{\mathbf{R}} = r_{11} \cdots r_{rr} \neq 0$. Also det $\mathbf{R} \neq 0$ for the corresponding $r \times r$ submatrix $\mathbf{R}$ of $\mathbf{A}$ because $\hat{\mathbf{R}}$ results from $\mathbf{R}$ by elementary row operations. Similarly, det $\mathbf{S} = 0$ for any square submatrix $\mathbf{S}$ of $r + 1$ or more rows perhaps contained in $\mathbf{A}$ because the corresponding submatrix $\hat{\mathbf{S}}$ of $\hat{\mathbf{A}}$ must contain a row of zeros (otherwise we would have rank $\mathbf{A} \geqq r + 1$), so that det $\hat{\mathbf{S}} = 0$ by Theorem 2. This proves the theorem for an $m \times n$ matrix.

In particular, if $\mathbf{A}$ is square, $n \times n$, then rank $\mathbf{A} = n$ if and only if $\mathbf{A}$ contains an $n \times n$ submatrix with nonzero determinant. But the only such submatrix can be $\mathbf{A}$ itself, hence det $\mathbf{A} \neq 0$. ■

Cramer's Rule

Theorem 3 opens the way to the classical solution formula for linear systems known as **Cramer's rule**[2], which gives solutions as quotients of determinants. ***Cramer's rule is not practical in computations*** (for which the methods in Secs. 7.3 and 20.1–20.3 are suitable), but is of ***theoretical interest*** in differential equations (Secs. 2.10, 3.3) and other theories that have engineering applications.

THEOREM 4

Cramer's Theorem (Solution of Linear Systems by Determinants)

(a) *If a linear system of n equations in the same number of unknowns $x_1, \cdots, x_n$*

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\tag{6}
$$

has a nonzero coefficient determinant $D =$ det $\mathbf{A}$, the system has precisely one solution. This solution is given by the formulas

$$
x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \cdots, \quad x_n = \frac{D_n}{D} \qquad \textbf{(Cramer's rule)}
\tag{7}
$$

where D_k is the determinant obtained from D by replacing in D the kth column by the column with the entries $b_1, \cdots, b_n$.

(b) *Hence if the system (6) is **homogeneous** and $D \neq 0$, it has only the trivial solution $x_1 = 0, x_2 = 0, \cdots, x_n = 0$. If $D = 0$, the homogeneous system also has nontrivial solutions.*

---

[2]GABRIEL CRAMER (1704–1752), Swiss mathematician.

PROOF The augmented matrix $\tilde{\mathbf{A}}$ of the system (6) is of size $n \times (n+1)$. Hence its rank can be at most n. Now if

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0, \tag{8}$$

then rank $\mathbf{A} = n$ by Theorem 3. Thus rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A}$. Hence, by the Fundamental Theorem in Sec. 7.5, the system (6) has a unique solution.

Let us now prove (7). Expanding D by its kth column, we obtain

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}, \tag{9}$$

where C_{ik} is the cofactor of entry a_{ik} in D. If we replace the entries in the kth column of D by any other numbers, we obtain a new determinant, say, $\hat{D}$. Clearly, its expansion by the kth column will be of the form (9), with $a_{1k}, \cdots, a_{nk}$ replaced by those new numbers and the cofactors C_{ik} as before. In particular, if we choose as new numbers the entries $a_{1l}, \cdots, a_{nl}$ of the lth column of D (where $l \neq k$), we have a new determinant $\hat{D}$ which has twice the column $[a_{1l} \quad \cdots \quad a_{nl}]^{\mathsf{T}}$, once as its lth column, and once as its kth because of the replacement. Hence $\hat{D} = 0$ by Theorem 2(f). If we now expand $\hat{D}$ by the column that has been replaced (the kth column), we thus obtain

$$a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \qquad (l \neq k). \tag{10}$$

We now multiply the first equation in (6) by C_{1k} on both sides, the second by $C_{2k}, \cdots$, the last by C_{nk}, and add the resulting equations. This gives

$$\begin{aligned} C_{1k}(a_{11}x_1 + \cdots + a_{1n}x_n) + \cdots + C_{nk}(a_{n1}x_1 + \cdots + a_{nn}x_n) \\ = b_1C_{1k} + \cdots + b_nC_{nk}. \end{aligned} \tag{11}$$

Collecting terms with the same x_j, we can write the left side as

$$x_1(a_{11}C_{1k} + a_{21}C_{2k} + \cdots + a_{n1}C_{nk}) + \cdots + x_n(a_{1n}C_{1k} + a_{2n}C_{2k} + \cdots + a_{nn}C_{nk}).$$

From this we see that x_k is multiplied by

$$a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}.$$

Equation (9) shows that this equals D. Similarly, x_l is multiplied by

$$a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk}.$$

Equation (10) shows that this is zero when $l \neq k$. Accordingly, the left side of (11) equals simply x_kD, so that (11) becomes

$$x_kD = b_1C_{1k} + b_2C_{2k} + \cdots + b_nC_{nk}.$$

Now the right side of this is D_k as defined in the theorem, expanded by its kth column, so that division by D gives (7). This proves Cramer's rule.

If (6) is homogeneous and $D \neq 0$, then each D_k has a column of zeros, so that $D_k = 0$ by Theorem 2(e), and (7) gives the trivial solution.

Finally, if (6) is homogeneous and $D = 0$, then rank $\mathbf{A} < n$ by Theorem 3, so that nontrivial solutions exist by Theorem 2 in Sec. 7.5. ■

Illustrations of Theorem 4 for $n = 2$ and 3 are given in Sec. 7.6, and an important application of the present formulas will follow in the next section.

PROBLEM SET 7.7

1. **(Second-order determinant)** Expand a general second-order determinant in four possible ways and show that the results agree.
2. **(Minors, cofactors)** Complete the list of minors and cofactors in Example 1.
3. **(Third-order determinant)** Do the task indicated in Example 2. Also evaluate D by reduction to triangular form.
4. **(Scalar multiplication)** Show that $\det(k\mathbf{A}) = k^n \det \mathbf{A}$ (not $k \det \mathbf{A}$), where $\mathbf{A}$ is any $n \times n$ matrix. Give an example.

5–16 EVALUATION OF DETERMINANTS

Evaluate, showing the details of your work.

5. $\begin{vmatrix} 13 & 8 \\ -2 & 7 \end{vmatrix}$

6. $\begin{vmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{vmatrix}$

7. $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$

8. $\begin{vmatrix} 14 & 2 & 5 \\ 2 & 0 & 8 \\ 5 & 8 & -2 \end{vmatrix}$

9. $\begin{vmatrix} 70.4 & 0.3 & 0.8 \\ 0 & 0.5 & 2.6 \\ 0 & 0 & -1.9 \end{vmatrix}$

10. $\begin{vmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{vmatrix}$

11. $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & -4 \\ 1 & 4 & 0 \end{vmatrix}$

12. $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

13. $\begin{vmatrix} u & v & w \\ w & u & v \\ v & w & u \end{vmatrix}$

14. $\begin{vmatrix} 1 & -2 & 0 & 0 \\ 4 & 3 & 5 & 0 \\ 0 & 2 & 7 & 5 \\ 0 & 0 & 2 & 4 \end{vmatrix}$

15. $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{vmatrix}$

16. $\begin{vmatrix} 0 & -2 & 1 & 0 \\ 2 & 0 & -2 & 4 \\ -1 & 2 & 0 & 1 \\ 0 & -4 & -1 & 0 \end{vmatrix}$

17. **(Expansion numerically impractical)** Show that the computation of an nth-order determinant by expansion involves $n!$ multiplications, which if a multiplication takes 10^{-9} sec would take these times:

n	10	15	20	25
Time	0.004 sec	22 min	77 years	$0.5 \cdot 10^9$ years

18–20 CRAMER'S RULE

Solve by Cramer's rule and check by Gauss elimination and back substitution. (Show details.)

18. $$\begin{aligned} 2x - 5y &= 23 \\ 4x + 6y &= -2 \end{aligned}$$

19. $$\begin{aligned} 3y + 4z &= 14.8 \\ 4x + 2y - z &= -6.3 \\ x - y + 5z &= 13.5 \end{aligned}$$

20. $$\begin{aligned} w + 2x \qquad - 3z &= 30 \\ 4x - 5y + 2z &= 13 \\ 2w + 8x - 4y + z &= 42 \\ 3w \qquad + y - 5z &= 35 \end{aligned}$$

21–23 RANK BY DETERMINANTS

Find the rank by Theorem 3 (which is not a very practical way) and check by row reduction. (Show details.)

21. $\begin{bmatrix} 8 & 4 \\ -2 & -1 \\ 6 & 3 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 1 & 0 \\ 13 & -13 & 12 \\ -3 & 5 & -4 \end{bmatrix}$

23. $\begin{bmatrix} 0.4 & 0 & -2.4 & 3.0 \\ 1.2 & 0.6 & 0 & 0.3 \\ 0 & 1.2 & 1.2 & 0 \end{bmatrix}$

24. TEAM PROJECT. Geometrical Applications: Curves and Surfaces Through Given Points. The idea is to get an equation from the vanishing of the determinant of a homogeneous linear system as the condition for a nontrivial solution in Cramer's theorem. We explain the trick for obtaining such a system for the case of a line L through two given points P_1: (x_1, y_1) and P_2: (x_2, y_2). The unknown line is $ax + by = -c$, say. We write it as $ax + by + c \cdot 1 = 0$. To get a nontrivial solution a, b, c, the determinant of the "coefficients" $x, y, 1$ must be zero. The system is

$$
\begin{aligned}
&ax + by + c \cdot 1 = 0 \quad &&(\text{Line } L)\\
(12)\qquad &ax_1 + by_1 + c \cdot 1 = 0 \quad &&(P_1 \text{ on } L)\\
&ax_2 + by_2 + c \cdot 1 = 0 \quad &&(P_2 \text{ on } L).
\end{aligned}
$$

(a) Line through two points. Derive from $D = 0$ in (12) the familiar formula

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

(b) Plane. Find the analog of (12) for a plane through three given points. Apply it when the points are (1, 1, 1), (3, 2, 6), (5, 0, 5).

(c) Circle. Find a similar formula for a circle in the plane through three given points. Find and sketch the circle through (2, 6), (6, 4), (7, 1).

(d) Sphere. Find the analog of the formula in (c) for a sphere through four given points. Find the sphere through (0, 0, 5), (4, 0, 1), (0, 4, 1), (0, 0, −3) by this formula or by inspection.

(e) General conic section. Find a formula for a general conic section (the vanishing of a determinant of 6th order). Try it out for a quadratic parabola and for a more general conic section of your own choice.

25. WRITING PROJECT. General Properties of Determinants. Illustrate each statement in Theorems 1 and 2 with an example of your choice.

26. CAS EXPERIMENT. Determinant of Zeros and Ones. Find the value of the determinant of the $n \times n$ matrix $\mathbf{A}_n$ with main diagonal entries all 0 and all others 1. Try to find a formula for this. Try to prove it by induction. Interpret $\mathbf{A}_3$ and $\mathbf{A}_4$ as *"incidence matrices"* (as in Problem Set 7.1 but without the minuses) of a triangle and a tetrahedron, respectively; similarly for an *"n-simplex"*, having n vertices and $n(n-1)/2$ edges (and spanning R^{n-1}, $n = 5, 6, \cdots$).

7.8 Inverse of a Matrix. Gauss–Jordan Elimination

In this section we consider* square *matrices exclusively.

The **inverse** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is denoted by $\mathbf{A}^{-1}$ and is an $n \times n$ matrix such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \qquad \textbf{(1)}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix (see Sec. 7.2).

If $\mathbf{A}$ has an inverse, then $\mathbf{A}$ is called a **nonsingular matrix.** If $\mathbf{A}$ has no inverse, then $\mathbf{A}$ is called a **singular matrix.**

If $\mathbf{A}$ *has an inverse, the inverse is unique.*

Indeed, if both $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$, then $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, so that we obtain the uniqueness from

$$\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}.$$

We prove next that $\mathbf{A}$ has an inverse (is nonsingular) if and only if it has maximum possible rank n. The proof will also show that $\mathbf{Ax} = \mathbf{b}$ implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ provided $\mathbf{A}^{-1}$ exists, and will thus give a motivation for the inverse as well as a relation to linear systems. (But this will ***not*** give a good method of solving $\mathbf{Ax} = \mathbf{b}$ ***numerically*** because the Gauss elimination in Sec. 7.3 requires fewer computations.)

THEOREM 1

Existence of the Inverse

The inverse $\mathbf{A}^{-1}$ *of an* $n \times n$ *matrix* $\mathbf{A}$ *exists if and only if* rank $\mathbf{A} = n$, *thus* (by Theorem 3, Sec. 7.7) *if and only if* det $\mathbf{A} \neq 0$. *Hence* $\mathbf{A}$ *is nonsingular if* rank $\mathbf{A} = n$, *and is singular if* rank $\mathbf{A} < n$.

PROOF Let $\mathbf{A}$ be a given $n \times n$ matrix and consider the linear system

$$\mathbf{Ax} = \mathbf{b}. \tag{2}$$

If the inverse $\mathbf{A}^{-1}$ exists, then multiplication from the left on both sides and use of (1) gives

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

This shows that (2) has a unique solution $\mathbf{x}$. Hence $\mathbf{A}$ must have rank n by the Fundamental Theorem in Sec. 7.5.

Conversely, let rank $\mathbf{A} = n$. Then by the same theorem, the system (2) has a unique solution $\mathbf{x}$ for any $\mathbf{b}$. Now the back substitution following the Gauss elimination (Sec. 7.3) shows that the components x_j of $\mathbf{x}$ are linear combinations of those of $\mathbf{b}$. Hence we can write

$$\mathbf{x} = \mathbf{Bb} \tag{3}$$

with $\mathbf{B}$ to be determined. Substitution into (2) gives

$$\mathbf{Ax} = \mathbf{A}(\mathbf{Bb}) = (\mathbf{AB})\mathbf{b} = \mathbf{Cb} = \mathbf{b} \qquad (\mathbf{C} = \mathbf{AB})$$

for any $\mathbf{b}$. Hence $\mathbf{C} = \mathbf{AB} = \mathbf{I}$, the unit matrix. Similarly, if we substitute (2) into (3) we get

$$\mathbf{x} = \mathbf{Bb} = \mathbf{B}(\mathbf{Ax}) = (\mathbf{BA})\mathbf{x}$$

for any $\mathbf{x}$ (and $\mathbf{b} = \mathbf{Ax}$). Hence $\mathbf{BA} = \mathbf{I}$. Together, $\mathbf{B} = \mathbf{A}^{-1}$ exists. ■

---

[3]WILHELM JORDAN (1842–1899), German mathematician and geodesist. [See *American Mathematical Monthly* **94** (1987), 130–142.]

We do ***not recommend*** it as a method for solving systems of linear equations, since the number of operations in addition to those of the Gauss elimination is larger than that for back substitution, which the Gauss–Jordan elimination avoids. See also Sec. 20.1.

Determination of the Inverse by the Gauss–Jordan Method

For the practical determination of the inverse $\mathbf{A}^{-1}$ of a nonsingular $n \times n$ matrix $\mathbf{A}$ we can use the Gauss elimination (Sec. 7.3), actually a variant of it, called the **Gauss-Jordan elimination**[3] (footnote of p. 316). The idea of the method is as follows.

Using $\mathbf{A}$, we form n linear systems

$$\mathbf{A}\mathbf{x}_{(1)} = \mathbf{e}_{(1)}, \qquad \cdots, \qquad \mathbf{A}\mathbf{x}_{(n)} = \mathbf{e}_{(n)}$$

where $\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(n)}$ are the columns of the $n \times n$ unit matrix $\mathbf{I}$; thus, $\mathbf{e}_{(1)} = [1 \quad 0 \quad \cdots \quad 0]^{\mathsf{T}}$, $\mathbf{e}_{(2)} = [0 \quad 1 \quad 0 \quad \cdots \quad 0]^{\mathsf{T}}$, etc. These are n vector equations in the unknown vectors $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(n)}$. We combine them into a single matrix equation $\mathbf{AX} = \mathbf{I}$, with the unknown matrix $\mathbf{X}$ having the columns $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(n)}$. Correspondingly, we combine the n augmented matrices $[\mathbf{A} \quad \mathbf{e}_{(1)}], \cdots, [\mathbf{A} \quad \mathbf{e}_{(n)}]$ into one $n \times 2n$ "augmented matrix" $\widetilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{I}]$. Now multiplication of $\mathbf{AX} = \mathbf{I}$ by $\mathbf{A}^{-1}$ from the left gives $\mathbf{X} = \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$. Hence, to solve $\mathbf{AX} = \mathbf{I}$ for $\mathbf{X}$, we can apply the Gauss elimination to $\widetilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{I}]$. This gives a matrix of the form $[\mathbf{U} \quad \mathbf{H}]$ with upper triangular $\mathbf{U}$ because the Gauss elimination triangularizes systems. The Gauss–Jordan method reduces $\mathbf{U}$ by further elementary row operations to diagonal form, in fact to the unit matrix $\mathbf{I}$. This is done by eliminating the entries of $\mathbf{U}$ above the main diagonal and making the diagonal entries all 1 by multiplication (see the example below). Of course, the method operates on the entire matrix $[\mathbf{U} \quad \mathbf{H}]$, transforming $\mathbf{H}$ into some matrix $\mathbf{K}$, hence the entire $[\mathbf{U} \quad \mathbf{H}]$ to $[\mathbf{I} \quad \mathbf{K}]$. This is the "augmented matrix" of $\mathbf{IX} = \mathbf{K}$. Now $\mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}$, as shown before. By comparison, $\mathbf{K} = \mathbf{A}^{-1}$, so that we can read $\mathbf{A}^{-1}$ directly from $[\mathbf{I} \quad \mathbf{K}]$.

The following example illustrates the practical details of the method.

EXAMPLE 1 Inverse of a Matrix. Gauss–Jordan Elimination

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Solution. We apply the Gauss elimination (Sec. 7.3) to the following $n \times 2n = 3 \times 6$ matrix, where BLUE always refers to the previous matrix.

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} + 3\text{ Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - \text{Row 2} \end{array}$$

This is [**U** **H**] as produced by the Gauss elimination. Now follow the additional Gauss–Jordan steps, reducing **U** to **I**, that is, to diagonal form with entries 1 on the main diagonal.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} -\text{Row } 1 \\ 0.5 \text{ Row } 2 \\ -0.2 \text{ Row } 3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row } 1 + 2 \text{ Row } 3 \\ \text{Row } 2 - 3.5 \text{ Row } 3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row } 1 + \text{Row } 2 \\ \\ \end{array}$$

The last three columns constitute $\mathbf{A}^{-1}$. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Similarly, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. ■

Useful Formulas for Inverses

The explicit formula (4) in the following theorem is often useful in theoretical studies (as opposed to *computing* inverses). In fact, the special case $n = 2$ occurs quite frequently in geometrical and other applications.

THEOREM 2

Inverse of a Matrix

The inverse of a nonsingular $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is given by

$$\text{(4)} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} [C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

where C_{jk} is the cofactor of a_{jk} in det **A** (see Sec. 7.7). (CAUTION! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in **A**.)

In particular, the inverse of

$$\text{(4*)} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \textit{is} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

PROOF We denote the right side of (4) by $\mathbf{B}$ and show that $\mathbf{BA} = \mathbf{I}$. We first write

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \tag{5}$$

and then show that $\mathbf{G} = \mathbf{I}$. Now by the definition of matrix multiplication and because of the form of $\mathbf{B}$ in (4), we obtain (CAUTION! C_{sk}, not C_{ks})

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl} = \frac{1}{\det \mathbf{A}} (a_{1l}C_{1k} + \cdots + a_{nl}C_{nk}). \tag{6}$$

Now (9) and (10) in Sec. 7.7 show that the sum $(\cdots)$ on the right is $D = \det \mathbf{A}$ when $l = k$, and is zero when $l \neq k$. Hence

$$g_{kk} = \frac{1}{\det \mathbf{A}} \det \mathbf{A} = 1,$$

$$g_{kl} = 0 \quad (l \neq k),$$

In particular, for $n = 2$ we have in (4) in the first row $C_{11} = a_{22}$, $C_{21} = -a_{12}$ and in the second row $C_{12} = -a_{21}$, $C_{22} = a_{11}$. This gives (4*). ■

EXAMPLE 2 **Inverse of a 2 × 2 Matrix**

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$ ■

EXAMPLE 3 **Further Illustration of Theorem 2**

Using (4), find the inverse of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Solution. We obtain $\det \mathbf{A} = -1(-7) - 1 \cdot 13 + 2 \cdot 8 = 10$, and in (4),

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \qquad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \qquad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \qquad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

so that by (4), in agreement with Example 1,

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$ ■

Diagonal matrices $\mathbf{A} = [a_{jk}]$, $a_{jk} = 0$ when $j \neq k$, have an inverse if and only if all $a_{jj} \neq 0$. Then $\mathbf{A}^{-1}$ is diagonal, too, with entries $1/a_{11}, \cdots, 1/a_{nn}$.

PROOF For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D} = \frac{a_{22} \cdots a_{nn}}{a_{11}a_{22} \cdots a_{nn}} = \frac{1}{a_{11}}, \qquad \text{etc.}$$ ■

EXAMPLE 4 Inverse of a Diagonal Matrix

Let

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Then the inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ■

Products can be inverted by taking the inverse of each factor and multiplying these inverses ***in reverse order,***

(7) $$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

Hence for more than two factors,

(8) $$(\mathbf{AC} \cdots \mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} \cdots \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

PROOF The idea is to start from (1) for $\mathbf{AC}$ instead of $\mathbf{A}$, that is, $\mathbf{AC}(\mathbf{AC})^{-1} = \mathbf{I}$, and multiply it on both sides from the left, first by $\mathbf{A}^{-1}$, which because of $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ gives

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{AC}(\mathbf{AC})^{-1} &= \mathbf{C}(\mathbf{AC})^{-1} \\ &= \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}, \end{aligned}$$

and then multiplying this on both sides from the left, this time by $\mathbf{C}^{-1}$ and by using $\mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$,

$$\mathbf{C}^{-1}\mathbf{C}(\mathbf{AC})^{-1} = (\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

This proves (7), and from it, (8) follows by induction. ■

We also note that *the inverse of the inverse is the given matrix,* as you may prove,

(9) $$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

Unusual Properties of Matrix Multiplication. Cancellation Laws

Section 7.2 contains warnings that some properties of matrix multiplication deviate from those for numbers, and we are now able to explain the restricted validity of the so-called **cancellation laws** [2.] and [3.] below, using rank and inverse, concepts that were not yet available in Sec. 7.2. The deviations from the usual are of great practical importance and must be carefully observed. They are as follows.

[1.] Matrix multiplication is not commutative, that is, in general we have

$$\mathbf{AB} \neq \mathbf{BA}.$$

[2.] $\mathbf{AB} = \mathbf{0}$ does not generally imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ (or $\mathbf{BA} = \mathbf{0}$); for example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

[3.] $\mathbf{AC} = \mathbf{AD}$ does not generally imply $\mathbf{C} = \mathbf{D}$ (even when $\mathbf{A} \neq \mathbf{0}$).

Complete answers to [2.] and [3.] are contained in the following theorem.

THEOREM 3

Cancellation Laws

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be* $n \times n$ *matrices. Then:*

(a) *If* rank $\mathbf{A} = n$ *and* $\mathbf{AB} = \mathbf{AC}$, *then* $\mathbf{B} = \mathbf{C}$.

(b) *If* rank $\mathbf{A} = n$, *then* $\mathbf{AB} = \mathbf{0}$ *implies* $\mathbf{B} = \mathbf{0}$. *Hence if* $\mathbf{AB} = \mathbf{0}$, *but* $\mathbf{A} \neq \mathbf{0}$ *as well as* $\mathbf{B} \neq \mathbf{0}$, *then* rank $\mathbf{A} < n$ *and* rank $\mathbf{B} < n$.

(c) *If* $\mathbf{A}$ *is singular, so are* $\mathbf{BA}$ *and* $\mathbf{AB}$.

PROOF **(a)** The inverse of $\mathbf{A}$ exists by Theorem 1. Multiplication by $\mathbf{A}^{-1}$ from the left gives $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC}$, hence $\mathbf{B} = \mathbf{C}$.

(b) Let rank $\mathbf{A} = n$. Then $\mathbf{A}^{-1}$ exists, and $\mathbf{AB} = \mathbf{0}$ implies $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{B} = \mathbf{0}$. Similarly when rank $\mathbf{B} = n$. This implies the second statement in (b).

(c_1) Rank $\mathbf{A} < n$ by Theorem 1. Hence $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions by Theorem 2 in Sec. 7.5. Multiplication by $\mathbf{B}$ shows that these solutions are also solutions of $\mathbf{BAx} = \mathbf{0}$, so that rank $(\mathbf{BA}) < n$ by Theorem 2 in Sec. 7.5 and $\mathbf{BA}$ is singular by Theorem 1.

(c_2) $\mathbf{A}^{\mathsf{T}}$ is singular by Theorem 2(d) in Sec. 7.7. Hence $\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ is singular by part (c_1), and is equal to $(\mathbf{AB})^{\mathsf{T}}$ by (10d) in Sec. 7.2. Hence $\mathbf{AB}$ is singular by Theorem 2(d) in Sec. 7.7. ■

Determinants of Matrix Products

The determinant of a matrix product $\mathbf{AB}$ or $\mathbf{BA}$ can be written as the product of the determinants of the factors, and it is interesting that det $\mathbf{AB}$ = det $\mathbf{BA}$, although $\mathbf{AB} \neq \mathbf{BA}$ in general. The corresponding formula (10) is needed occasionally and can be obtained by Gauss-Jordan elimination (see Example 1) and from the theorem just proved.

THEOREM 4

Determinant of a Product of Matrices

For any $n \times n$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}. \tag{10}$$

PROOF If $\mathbf{A}$ or $\mathbf{B}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$ by Theorem 3(c), and (10) reduces to $0 = 0$ by Theorem 3 in Sec. 7.7.

Now let $\mathbf{A}$ and $\mathbf{B}$ be nonsingular. Then we can reduce $\mathbf{A}$ to a diagonal matrix $\hat{\mathbf{A}} = [a_{jk}]$ by Gauss–Jordan steps. Under these operations, det $\mathbf{A}$ retains its value, by Theorem 1 in Sec. 7.7, (a) and (b) [not (c)] except perhaps for a sign reversal in row interchanging when pivoting. But the same operations reduce $\mathbf{AB}$ to $\hat{\mathbf{A}}\mathbf{B}$ with the same effect on $\det(\mathbf{AB})$. Hence it remains to prove (10) for $\hat{\mathbf{A}}\mathbf{B}$; written out,

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{bmatrix}.$$

We now take the determinant $\det(\hat{\mathbf{A}}\mathbf{B})$. On the right we can take out a factor $\hat{a}_{11}$ from the first row, $\hat{a}_{22}$ from the second, $\cdots$, $\hat{a}_{nn}$ from the nth. But this product $\hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn}$ equals $\det \hat{\mathbf{A}}$ because $\hat{\mathbf{A}}$ is diagonal. The remaining determinant is det $\mathbf{B}$. This proves (10) for $\det(\mathbf{AB})$, and the proof for $\det(\mathbf{BA})$ follows by the same idea. ■

This completes our discussion of linear systems (Secs. 7.3–7.8). Section 7.9 on vector spaces and linear transformations is optional. ***Numeric methods*** are discussed in Secs. 20.1–20.4, which are independent of other sections on numerics.

PROBLEM SET 7.8

1–12 INVERSE

Find the inverse by Gauss–Jordan [or by (4*) if $n = 2$] or state that it does not exist. Check by using (1).

1. $\begin{bmatrix} 1.20 & 4.64 \\ 0.50 & 3.60 \end{bmatrix}$

2. $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$

3. $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

4. $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$

5. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 29 & -11 & 10 \\ -160 & 61 & -55 \\ 55 & -21 & 19 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 8 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 10 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & -9 \\ -2 & -4 & 19 \\ 0 & -1 & 2 \end{bmatrix}$

13. (Triangular matrix) Is the inverse of a triangular matrix always triangular (as in Prob. 7)? Give reason.

14. (Rotation) Give an application of the matrix in Prob. 3 that makes the form of its inverse obvious.

15. (Inverse of the square) Verify $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$ for $\mathbf{A}$ in Prob. 5.

16. Prove the formula in Prob. 15.

17. (Inverse of the transpose) Verify $(\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}}$ for $\mathbf{A}$ in Prob. 5.

18. Prove the formula in Prob. 17.

19. (Inverse of the inverse) Prove that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

20. (Row interchange) Same question as in Prob. 14 for the matrix in Prob. 9.

21–23 EXPLICIT FORMULA (4) FOR THE INVERSE

Formula (4) is generally not very practical. To understand its use, apply it:

21. To Prob. 9. **22.** To Prob. 4. **23.** To Prob. 7.

7.9 Vector Spaces, Inner Product Spaces, Linear Transformations *Optional*

In Sec. 7.4 we have seen that ***special vector spaces*** arise quite naturally in connection with matrices and linear systems, that their elements, called *vectors,* satisfy rules quite similar to those for numbers [(3) and (4) in Sec. 7.1], and that they are often obtained as *spans* (sets of linear combinations) of finitely many given vectors. Each such vector has n real numbers as its *components.* Look this up before going on.

Now if we take *all* vectors with n real numbers as components (*"real vectors"*), we obtain the very important **real n-dimensional vector space** R^n. This is a standard name and notation. Thus, each vector in R^n is an ordered n-tuple of real numbers.

Particular cases are R^2, the space of all ordered pairs (**"vectors in the plane"**) and R^3, the space of all ordered triples (**"vectors in 3-space"**). These vectors have wide applications in mechanics, geometry, and calculus that are basic to the engineer and physicist.

Similarly, if we take all ordered n-tuples of *complex numbers* as vectors and complex numbers as scalars, we obtain the **complex vector space** C^n, which we shall consider in Sec. 8.5.

This is not all. There are other sets of practical interest (sets of matrices, functions, transformations, etc.) for which addition and scalar multiplication can be defined in a natural way so that they form a "vector space". This suggests to create from the ***"concrete model"*** R^n the ***"abstract concept"*** of a ***"real vector space"*** V by taking the basic properties (3) and (4) in Sec. 7.1 as axioms. These axioms guarantee that one obtains a useful and applicable theory of those more general situations. Note that each axiom expresses a simple property of R^n or, as a matter of fact, of R^3. Selecting good axioms needs experience and is a process of trial and error that often extends over a long period of time.

DEFINITION

Real Vector Space

A nonempty set V of elements $\mathbf{a}, \mathbf{b}, \cdots$ is called a **real vector space** (or *real linear space*), and these elements are called **vectors** (regardless of their nature, which will come out from the context or will be left arbitrary) if in V there are defined two algebraic operations (called *vector addition* and *scalar multiplication*) as follows.

I. Vector addition associates with every pair of vectors $\mathbf{a}$ and $\mathbf{b}$ of V a unique vector of V, called the *sum* of $\mathbf{a}$ and $\mathbf{b}$ and denoted by $\mathbf{a} + \mathbf{b}$, such that the following axioms are satisfied.

I.1 *Commutativity.* For any two vectors $\mathbf{a}$ and $\mathbf{b}$ of V,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

I.2 *Associativity.* For any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of V,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \qquad (\text{written } \mathbf{u} + \mathbf{v} + \mathbf{w}).$$

I.3 There is a unique vector in V, called the *zero vector* and denoted by $\mathbf{0}$, such that for every $\mathbf{a}$ in V,

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

I.4 For every $\mathbf{a}$ in V there is a unique vector in V that is denoted by $-\mathbf{a}$ and is such that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

II. Scalar multiplication. The real numbers are called **scalars.** Scalar multiplication associates with every $\mathbf{a}$ in V and every scalar c a unique vector of V, called the *product* of c and $\mathbf{a}$ and denoted by $c\mathbf{a}$ (or $\mathbf{a}c$) such that the following axioms are satisfied.

II.1 *Distributivity.* For every scalar c and vectors $\mathbf{a}$ and $\mathbf{b}$ in V,

$$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}.$$

II.2 *Distributivity.* For all scalars c and k and every $\mathbf{a}$ in V,

$$(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}.$$

II.3 *Associativity.* For all scalars c and k and every $\mathbf{a}$ in V,

$$c(k\mathbf{a}) = (ck)\mathbf{a} \qquad (\text{written } ck\mathbf{a}).$$

II.4 For every $\mathbf{a}$ in V,

$$1\mathbf{a} = \mathbf{a}.$$

A **complex vector space** is obtained if, instead of real numbers, we take complex numbers as scalars.

Basic concepts related to the concept of a vector space are defined as in Sec. 7.4.

A **linear combination** of vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ in a vector space V is an expression

$$c_1\mathbf{a}_{(1)} + \cdots + c_m\mathbf{a}_{(m)} \qquad (c_1, \cdots, c_m \text{ any scalars}).$$

These vectors form a **linearly independent set** (briefly, they are called **linearly independent**) if

$$c_1\mathbf{a}_{(1)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \tag{1}$$

implies that $c_1 = 0, \cdots, c_m = 0$. Otherwise, if (1) also holds with scalars not all zero, the vectors are called **linearly dependent.**

Note that (1) with $m = 1$ is $c\mathbf{a} = \mathbf{0}$ and shows that a single vector $\mathbf{a}$ is linearly independent if and only if $\mathbf{a} \neq \mathbf{0}$.

V has **dimension n,** or is **n-dimensional,** if it contains a linearly independent set of n vectors, whereas any set of more than n vectors in V is linearly dependent. That set of n linearly independent vectors is called a **basis** for V. Then every vector in V can be written as a linear combination of the basis vectors; for a given basis, this representation is unique (see Prob. 14).

EXAMPLE 1 Vector Space of Matrices

The real 2×2 matrices form a four-dimensional real vector space. A basis is

$$\mathbf{B}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

because any 2×2 matrix $\mathbf{A} = [a_{jk}]$ has a unique representation $\mathbf{A} = a_{11}\mathbf{B}_{11} + a_{12}\mathbf{B}_{12} + a_{21}\mathbf{B}_{21} + a_{22}\mathbf{B}_{22}$. Similarly, the real $m \times n$ matrices with fixed m and n form an mn-dimensional vector space. What is the dimension of the vector space of all 3×3 skew-symmetric matrices? Can you find a basis? ■

EXAMPLE 2 Vector Space of Polynomials

The set of all constant, linear, and quadratic polynomials in x together is a vector space of dimension 3 with basis $\{1, x, x^2\}$ under the usual addition and multiplication by real numbers because these two operations give polynomials not exceeding degree 2. What is the dimension of the vector space of all polynomials of degree not exceeding a given fixed n? Can you find a basis? ■

If a vector space V contains a linearly independent set of n vectors for every n, no matter how large, then V is called **infinite dimensional,** as opposed to a *finite dimensional* (n-dimensional) vector space just defined. An example of an infinite dimensional vector space is the space of all continuous functions on some interval $[a, b]$ of the x-axis, as we mention without proof.

Inner Product Spaces

If $\mathbf{a}$ and $\mathbf{b}$ are vectors in R^n, regarded as column vectors, we can form the product $\mathbf{a}^\mathsf{T}\mathbf{b}$. This is a 1×1 matrix, which we can identify with its single entry, that is, with a number. This product is called the **inner product** or **dot product** of $\mathbf{a}$ and $\mathbf{b}$. Other notations for it are $(\mathbf{a}, \mathbf{b})$ and $\mathbf{a} \cdot \mathbf{b}$. Thus

$$\mathbf{a}^\mathsf{T}\mathbf{b} = (\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = [a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{l=1}^{n} a_l b_l = a_1 b_1 + \cdots + a_n b_n.$$

We now extend this concept to general real vector spaces by taking basic properties of $(\mathbf{a}, \mathbf{b})$ as axioms for an "abstract inner product" $(\mathbf{a}, \mathbf{b})$ as follows.

DEFINITION

Real Inner Product Space

A real vector space V is called a **real inner product space** (or *real pre-Hilbert*[4] *space*) if it has the following property. With every pair of vectors **a** and **b** in V there is associated a real number, which is denoted by $(\mathbf{a}, \mathbf{b})$ and is called the **inner product** of **a** and **b**, such that the following axioms are satisfied.

I. For all scalars q_1 and q_2 and all vectors **a, b, c** in V,

$$(q_1\mathbf{a} + q_2\mathbf{b}, \mathbf{c}) = q_1(\mathbf{a}, \mathbf{c}) + q_2(\mathbf{b}, \mathbf{c}) \qquad \textit{(Linearity)}.$$

II. For all vectors **a** and **b** in V,

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a}) \qquad \textit{(Symmetry)}.$$

III. For every **a** in V,

$$\left.\begin{aligned} &(\mathbf{a}, \mathbf{a}) \geqq 0, \\ &(\mathbf{a}, \mathbf{a}) = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{aligned}\right\} \qquad \textit{(Positive-definiteness)}.$$

Vectors whose inner product is zero are called **orthogonal.**

The *length* or **norm** of a vector in V is defined by

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} \qquad (\geqq 0). \tag{2}$$

A vector of norm 1 is called a **unit vector.**

From these axioms and from (2) one can derive the basic inequality

$$|(\mathbf{a}, \mathbf{b})| \leqq \|\mathbf{a}\|\,\|\mathbf{b}\| \qquad \textit{(Cauchy–Schwarz}^{5}\textit{ inequality)}. \tag{3}$$

From this follows

$$\|\mathbf{a} + \mathbf{b}\| \leqq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad \textit{(Triangle inequality)}. \tag{4}$$

A simple direct calculation gives

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2) \qquad \textit{(Parallelogram equality)}. \tag{5}$$

[4]DAVID HILBERT (1862–1943), great German mathematician, taught at Königsberg and Göttingen and was the creator of the famous Göttingen mathematical school. He is known for his basic work in algebra, the calculus of variations, integral equations, functional analysis, and mathematical logic. His "Foundations of Geometry" helped the axiomatic method to gain general recognition. His famous 23 problems (presented in 1900 at the International Congress of Mathematicians in Paris) considerably influenced the development of modern mathematics.

If V is finite dimensional, it is actually a so-called *Hilbert space;* see Ref. [GR7], p. 73, listed in App. I.

[5]HERMANN AMANDUS SCHWARZ (1843–1921). German mathematician, known by his work in complex analysis (conformal mapping) and differential geometry. For Cauchy see Sec. 2.5.

EXAMPLE 3 ***n*-Dimensional Euclidean Space**

R^n with the inner product

(6) $$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^\mathsf{T}\mathbf{b} = a_1b_1 + \cdots + a_nb_n$$

(where both **a** and **b** are *column* vectors) is called the ***n*-dimensional Euclidean space** and is denoted by E^n or again simply by R^n. Axioms I–III hold, as direct calculation shows. Equation (2) gives the **"Euclidean norm"**

(7) $$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} = \sqrt{\mathbf{a}^\mathsf{T}\mathbf{a}} = \sqrt{a_1^{\,2} + \cdots + a_n^{\,2}}.$$ ■

EXAMPLE 4 **An Inner Product for Functions. Function Space**

The set of all real-valued continuous functions $f(x), g(x), \cdots$ on a given interval $\alpha \leqq x \leqq \beta$ is a real vector space under the usual addition of functions and multiplication by scalars (real numbers). On this **"function space"** we can define an inner product by the integral

(8) $$(f, g) = \int_\alpha^\beta f(x)\, g(x)\, dx.$$

Axioms I–III can be verified by direct calculation. Equation (2) gives the norm

(9) $$\|f\| = \sqrt{(f, f)} = \sqrt{\int_\alpha^\beta f(x)^2\, dx}.$$ ■

Our examples give a first impression of the great generality of the abstract concepts of vector spaces and inner product spaces. Further details belong to more advanced courses (on functional analysis, meaning abstract modern analysis; see Ref. [GR7] listed in App. 1) and cannot be discussed here. Instead we now take up a related topic where matrices play a central role.

Linear Transformations

Let X and Y be any vector spaces. To each vector **x** in X we assign a unique vector **y** in Y. Then we say that a **mapping** (or **transformation** or **operator**) of X into Y is given. Such a mapping is denoted by a capital letter, say F. The vector **y** in Y assigned to a vector **x** in X is called the **image** of **x** under F and is denoted by $F(\mathbf{x})$ [or $F\mathbf{x}$, without parentheses].

F is called a **linear mapping** or **linear transformation** if for all vectors **v** and **x** in X and scalars c,

(10) $$\begin{aligned} F(\mathbf{v} + \mathbf{x}) &= F(\mathbf{v}) + F(\mathbf{x}) \\ F(c\mathbf{x}) &= cF(\mathbf{x}). \end{aligned}$$

Linear Transformation of Space R^n into Space R^m

From now on we let $X = R^n$ and $Y = R^m$. Then any real $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ gives a transformation of R^n into R^m,

(11) $$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

Since $\mathbf{A}(\mathbf{u} + \mathbf{x}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{x}$ and $\mathbf{A}(c\mathbf{x}) = c\mathbf{A}\mathbf{x}$, this transformation is linear.

We show that, conversely, every linear transformation F of R^n into R^m can be given in terms of an $m \times n$ matrix **A**, after a basis for R^n and a basis for R^m have been chosen. This can be proved as follows.

Let $\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(n)}$ be any basis for R^n. Then every $\mathbf{x}$ in R^n has a unique representation

$$\mathbf{x} = x_1\mathbf{e}_{(1)} + \cdots + x_n\mathbf{e}_{(n)}.$$

Since F is linear, this representation implies for the image $F(\mathbf{x})$:

$$F(\mathbf{x}) = F(x_1\mathbf{e}_{(1)} + \cdots + x_n\mathbf{e}_{(n)}) = x_1F(\mathbf{e}_{(1)}) + \cdots + x_nF(\mathbf{e}_{(n)}).$$

Hence F is uniquely determined by the images of the vectors of a basis for R^n. We now choose for R^n the **"standard basis"**

$$\mathbf{e}_{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \mathbf{e}_{(n)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{12}$$

where $\mathbf{e}_{(j)}$ has its jth component equal to 1 and all others 0. We show that we can now determine an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ such that for every $\mathbf{x}$ in R^n and image $\mathbf{y} = F(\mathbf{x})$ in R^m,

$$\mathbf{y} = F(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

Indeed, from the image $\mathbf{y}^{(1)} = F(\mathbf{e}_{(1)})$ of $\mathbf{e}_{(1)}$ we get the condition

$$\mathbf{y}^{(1)} = \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ \vdots \\ y_m^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

from which we can determine the first column of $\mathbf{A}$, namely $a_{11} = y_1^{(1)}$, $a_{21} = y_2^{(1)}, \cdots,$ $a_{m1} = y_m^{(1)}$. Similarly, from the image of $\mathbf{e}_{(2)}$ we get the second column of $\mathbf{A}$, and so on. This completes the proof. ■

We say that **A represents** F, or *is a representation of* F, with respect to the bases for R^n and R^m. Quite generally, the purpose of a **"representation"** is the replacement of one object of study by another object whose properties are more readily apparent.

In three-dimensional Euclidean space E^3 the standard basis is usually written $\mathbf{e}_{(1)} = \mathbf{i}$, $\mathbf{e}_{(2)} = \mathbf{j}$, $\mathbf{e}_{(3)} = \mathbf{k}$. Thus,

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{13}$$

These are the three unit vectors in the positive directions of the axes of the **Cartesian coordinate system in space,** that is, the usual coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes.

EXAMPLE 5 Linear Transformations

Interpreted as transformations of Cartesian coordinates in the plane, the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

represent a reflection in the line $x_2 = x_1$, a reflection in the x_1-axis, a reflection in the origin, and a stretch (when $a > 1$, or a contraction when $0 < a < 1$) in the x_1-direction, respectively. ■

EXAMPLE 6 Linear Transformations

Our discussion preceding Example 5 is simpler than it may look at first sight. To see this, find $\mathbf{A}$ representing the linear transformation that maps (x_1, x_2) onto $(2x_1 - 5x_2, 3x_1 + 4x_2)$.

Solution. Obviously, the transformation is

$$y_1 = 2x_1 - 5x_2$$
$$y_2 = 3x_1 + 4x_2.$$

From this we can directly see that the matrix is

$$\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix}. \qquad \text{Check:} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 5x_2 \\ 3x_1 + 4x_2 \end{bmatrix}.$$ ■

If $\mathbf{A}$ in (11) is square, $n \times n$, then (11) maps R^n into R^n. If this $\mathbf{A}$ is nonsingular, so that $\mathbf{A}^{-1}$ exists (see Sec. 7.8), then multiplication of (11) by $\mathbf{A}^{-1}$ from the left and use of $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ gives the **inverse transformation**

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}. \tag{14}$$

It maps every $\mathbf{y} = \mathbf{y}_0$ onto that $\mathbf{x}$, which by (11) is mapped onto $\mathbf{y}_0$. *The inverse of a linear transformation is itself linear,* because it is given by a matrix, as (14) shows.

PROBLEM SET 7.9

1–12 VECTOR SPACES

(Additional problems in Problem Set 7.4.)

Is the given set (taken with the usual addition and scalar multiplication) a vector space? (Give a reason.) If your answer is yes, find the dimension and a basis.

1. All vectors in R^3 satisfying $5v_1 - 3v_2 + 2v_3 = 0$
2. All vectors in R^3 satisfying $2v_1 + 3v_2 - v_3 = 0$, $v_1 - 4v_2 + v_3 = 0$
3. All 2×3 matrices with all entries nonnegative
4. All symmetric 3×3 matrices
5. All vectors in R^5 with the first three components 0
6. All vectors in R^4 with $v_1 + v_2 = 0$, $v_3 - v_4 = 1$
7. All skew-symmetric 2×2 matrices
8. All $n \times n$ matrices $\mathbf{A}$ with fixed n and $\det \mathbf{A} = 0$
9. All polynomials with positive coefficients and degree 3 or less
10. All functions $f(x) = a \cos x + b \sin x$ with any constants a and b
11. All functions $f(x) = (ax + b)e^{-x}$ with any constants a and b
12. All 2×3 matrices with the second row any multiple of $[4 \quad 0 \quad -9]$

13. (Different bases) Find three bases for R^2.

14. (Uniqueness) Show that the representation $\mathbf{v} = c_1\mathbf{a}_{(1)} + \cdots + c_n\mathbf{a}_{(n)}$ of any given vector in an n-dimensional vector space V in terms of a given basis $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(n)}$ for V is unique.

15–20 LINEAR TRANSFORMATIONS

Find the inverse transformation. (Show the details of your work.)

15. $y_1 = x_1 - 2x_2$, $\quad y_2 = 4x_1 - 3x_2$

16. $y_1 = 5x_1 - x_2$, $\quad y_2 = 3x_1 - x_2$

17. $y_1 = 3x_1 - x_2$, $\quad y_2 = -5x_1 + 2x_2$

18. $y_1 = 0.25x_1 - 0.1x_3$, $\quad y_2 = x_2 - 0.8x_3$, $\quad y_3 = 0.2x_3$

19. $y_1 = 2x_1 - 3x_2$, $\quad y_2 = -10x_1 + 16x_2 + x_3$, $\quad y_3 = -7x_1 + 11x_2 + x_3$

20. $y_1 = x_1 + x_2 - 2x_3$, $\quad y_2 = x_1 + x_2 + 2x_3$, $\quad y_3 = -2x_1 + 2x_2 + 4x_3$

21–26 INNER PRODUCT. ORTHOGONALITY

Find the Euclidean norm of the vectors

21. $[4 \quad 2 \quad -6]^\mathsf{T}$

22. $[0 \quad -3 \quad 3 \quad 0 \quad 5 \quad 1]^\mathsf{T}$

23. $[16 \quad -32 \quad 0]^\mathsf{T}$

24. $[\frac{1}{4} \quad \frac{3}{8} \quad \frac{1}{2} \quad 2]^\mathsf{T}$

25. $[0 \quad 1 \quad 0 \quad 0 \quad -1 \quad 1 \quad -1]^\mathsf{T}$

26. $[\frac{2}{3} \quad -\frac{2}{3} \quad \frac{1}{3}]^\mathsf{T}$

27. (Orthogonality) Show that the vectors in Probs. 21 and 23 are orthogonal.

28. Find all vectors $\mathbf{v}$ in R^3 orthogonal to $[2 \quad 0 \quad 1]^\mathsf{T}$.

29. (Unit vectors) Find all unit vectors orthogonal to $[4 \quad -3]^\mathsf{T}$. Make a sketch.

30. (Triangle inequality) Verify (4) for the vectors in Probs. 21 and 23.

CHAPTER 7 REVIEW QUESTIONS AND PROBLEMS

1. What properties of matrix multiplication differ from those of the multiplication of numbers? What about division of matrices?

2. Let **A** be a 50 × 50 matrix and **B** a 50 × 20 matrix. Are the following expressions defined or not? $\mathbf{A} + \mathbf{B}$, $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{AA}^\mathsf{T}$, $\mathbf{B}^\mathsf{T}\mathbf{A}$, $\mathbf{B}^\mathsf{T}\mathbf{B}$, $\mathbf{BB}^\mathsf{T}$, $\mathbf{B}^\mathsf{T}\mathbf{AB}$. (Give reasons.)

3. How is matrix multiplication motivated?

4. Are there any linear systems without solutions? With one solution? With more than one solution? Give simple examples.

5. How can you give the rank of a matrix in terms of row vectors? Of column vectors? Of determinants?

6. What is the role of rank in connection with solving linear systems?

7. What is the row space of a matrix? The column space? The null space?

8. What is the idea of Gauss elimination and back substitution?

9. What is the inverse of a matrix? When does it exist? How would you determine it?

10. What is Cramer's rule? When would you apply it?

11–19 LINEAR SYSTEMS

Find all solutions or indicate that no solution exists. (Show the details of your work.)

11. $9x - 3y = 15$, $\quad 5x + 4y = 48$

12. $-2x - 4y + 7z = -6$, $\quad x + 2y + 16z = 3$

13. $3x + 5y - 8z = 18$, $\quad x + 2y - 3z = 6$

14. $5x - 10y = 2$, $\quad 3x + y = 13$, $\quad -x + 6y = 6$

15. $-8x + 2z = 1$, $\quad 6y + 4z = 3$, $\quad 12x + 2y = 2$

16. $2y + z = -1$, $\quad 2x + 3y - z = -12$, $\quad 5x - 4y + 3z = 32$

17. $3x + 7y = 0$, $\quad 5x - 4y = 47$, $\quad 6x + 9y = 15$

18. $-x + 4y - 2z = 1$, $\quad 3x + 4y + 6z = 1$, $\quad x - 2y + 2z = -\frac{1}{2}$

19. $7x + 9y - 14z = 36$, $\quad -x - 3y + 2z = -12$, $\quad 2x + y - 4z = 4$

20–30 CALCULATIONS WITH MATRICES AND VECTORS

Calculate the following expressions (showing the details of your work) or indicate why they do not exist, when

$$\mathbf{A} = \begin{bmatrix} 9 & 2 & 8 \\ 2 & 18 & 10 \\ 8 & 10 & 15 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & 6 \\ -2 & 0 & -3 \\ -6 & 3 & 0 \end{bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

20. $\mathbf{AB}$, $\mathbf{BA}$

21. $\mathbf{A} - \mathbf{A}^\mathsf{T}$

22. $\mathbf{A}^2 + \mathbf{B}^2$

23. $\det \mathbf{A}$, $\det \mathbf{B}$, $\det \mathbf{AB}$

24. $\mathbf{AA}^\mathsf{T}$, $\mathbf{A}^\mathsf{T}\mathbf{A}$

25. $0.2\mathbf{BB}^\mathsf{T}$

26. $\mathbf{Aa}$, $\mathbf{a}^\mathsf{T}\mathbf{A}$, $\mathbf{a}^\mathsf{T}\mathbf{Aa}$

27. $\mathbf{a}^\mathsf{T}\mathbf{b}$, $\mathbf{b}^\mathsf{T}\mathbf{a}$, $\mathbf{ab}^\mathsf{T}$

28. $\mathbf{b}^\mathsf{T}\mathbf{Bb}$

29. $\mathbf{a}^\mathsf{T}\mathbf{B}$, $\mathbf{B}^\mathsf{T}\mathbf{a}$

30. $0.1(\mathbf{A} + \mathbf{A}^\mathsf{T})(\mathbf{B} - \mathbf{B}^\mathsf{T})$

31–36 RANK

Determine the ranks of the coefficient matrix and the augmented matrix and state how many solutions the linear system will have.

31. In Prob. 13 **32.** In Prob. 12 **33.** In Prob. 17

34. In Prob. 14 **35.** In Prob. 19 **36.** In Prob. 18

37–42 INVERSE

Find the inverse or state why it does not exist. (Show details.)

37. Of the coefficient matrix in Prob. 11

38. Of the coefficient matrix in Prob. 15

39. Of the coefficient matrix in Prob. 16

40. Of the coefficient matrix in Prob. 18

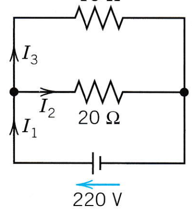
41. Of the augmented matrix in Prob. 14

42. Of the diagonal matrix with entries 3, -1, 5

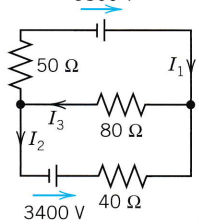
43–45 NETWORKS

Find the currents in the following networks.

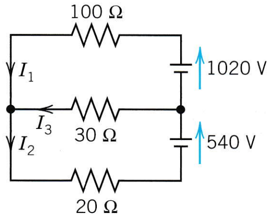
43.



44.



45.



SUMMARY OF CHAPTER 7

Linear Algebra: Matrices, Vectors, Determinants Linear Systems of Equations

An $m \times n$ **matrix** $\mathbf{A} = [a_{jk}]$ is a rectangular array of numbers or functions ("entries", "elements") arranged in m horizontal **rows** and n vertical **columns.** If $m = n$, the matrix is called **square.** A $1 \times n$ matrix is called a **row vector** and an $m \times 1$ matrix a **column vector** (Sec. 7.1).

The **sum** $\mathbf{A} + \mathbf{B}$ of matrices of the same **size** (i.e., both $m \times n$) is obtained by adding corresponding entries. The **product** of $\mathbf{A}$ by a scalar c is obtained by multiplying each a_{jk} by c (Sec. 7.1).

The **product** $\mathbf{C} = \mathbf{AB}$ of an $m \times n$ matrix $\mathbf{A}$ by an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined only when $r = n$, and is the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \text{(row } j \text{ of } \mathbf{A} \text{ times column } k \text{ of } \mathbf{B}). \tag{1}$$

This multiplication is motivated by the composition of **linear transformations** (Secs. 7.2, 7.9). It is associative, but is ***not commutative:*** if **AB** is defined, **BA** may not be defined, but even if **BA** is defined, $\mathbf{AB} \neq \mathbf{BA}$ in general. Also $\mathbf{AB} = \mathbf{0}$ may not imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$ (Secs. 7.2, 7.8). Illustrations:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [11], \qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}.$$

The **transpose** $\mathbf{A}^\mathsf{T}$ of a matrix $\mathbf{A} = [a_{jk}]$ is $\mathbf{A}^\mathsf{T} = [a_{kj}]$; rows become columns and conversely (Sec. 7.2). Here, **A** need not be square. If it is and $\mathbf{A} = \mathbf{A}^\mathsf{T}$, then **A** is called **symmetric;** if $\mathbf{A} = -\mathbf{A}^\mathsf{T}$, it is called **skew-symmetric.** For a product, $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$ (Sec. 7.2).

A main application of matrices concerns **linear systems of equations**

$$\mathbf{Ax} = \mathbf{b} \qquad \text{(2)} \qquad \text{(Sec. 7.3)}$$

(m equations in n unknowns $x_1, \cdots, x_n$; **A** and **b** given). The most important method of solution is the **Gauss elimination** (Sec. 7.3), which reduces the system to "triangular" form by *elementary row operations,* which leave the set of solutions unchanged. (Numeric aspects and variants, such as *Doolittle's* and *Cholesky's methods,* are discussed in Secs. 20.1 and 20.2)

Cramer's rule (Secs. 7.6, 7.7) represents the unknowns in a system (2) of n equations in n unknowns as quotients of determinants; for numeric work it is impractical. **Determinants** (Sec. 7.7) have decreased in importance, but will retain their place in eigenvalue problems, elementary geometry, etc.

The **inverse** $\mathbf{A}^{-1}$ of a square matrix satisfies $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. It exists if and only if $\det \mathbf{A} \neq 0$. It can be computed by the *Gauss–Jordan elimination* (Sec. 7.8).

The **rank** r of a matrix **A** is the maximum number of linearly independent rows or columns of **A** or, equivalently, the number of rows of the largest square submatrix of **A** with nonzero determinant (Secs. 7.4, 7.7).

The system (2) has solutions if and only if rank $\mathbf{A}$ = rank $[\mathbf{A} \;\; \mathbf{b}]$, where $[\mathbf{A} \;\; \mathbf{b}]$ is the **augmented matrix** (Fundamental Theorem, Sec. 7.5).

The **homogeneous system**

$$\mathbf{Ax} = \mathbf{0} \qquad \text{(3)}$$

has solutions $\mathbf{x} \neq \mathbf{0}$ ("nontrivial solutions") if and only if rank $\mathbf{A} < n$, in the case $m = n$ equivalently if and only if $\det \mathbf{A} = 0$ (Secs. 7.6, 7.7).

Vector spaces, inner product spaces, and linear transformations are discussed in Sec. 7.9. See also Sec. 7.4.

CHAPTER 8

Linear Algebra: Matrix Eigenvalue Problems

Matrix eigenvalue problems concern the solutions of vector equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

where $\mathbf{A}$ is a given ***square*** matrix and vector $\mathbf{x}$ and scalar λ are unknown. Clearly, $\mathbf{x} = \mathbf{0}$ is a solution of (1), giving $\mathbf{0} = \mathbf{0}$. But this of no interest, and we want to find solution vectors $\mathbf{x} \neq \mathbf{0}$ of (1), called **eigenvectors** of $\mathbf{A}$. We shall see that eigenvectors can be found only for certain values of the scalar λ; these values λ for which an eigenvector exists are called the **eigenvalues** of $\mathbf{A}$. Geometrically, solving (1) in this way means that we are looking for vectors $\mathbf{x}$ for which the multiplication of $\mathbf{x}$ by the matrix $\mathbf{A}$ has the same effect as the multiplication of $\mathbf{x}$ by a scalar λ, giving a vector $\lambda\mathbf{x}$ with components proportional to those of $\mathbf{x}$, and λ as the factor of proportionality.

Eigenvalue problems are of greatest practical interest to the engineer, physicist, and mathematician, and we shall see that their theory makes up a beautiful chapter in linear algebra that has found numerous applications.

We shall explain how to solve that vector equation (1) in Sec. 8.1, show a few typical applications in Sec. 8.2, and then discuss eigenvalue problems for symmetric, skew-symmetric, and orthogonal matrices in Sec. 8.3. In Sec. 8.4 we show how to obtain eigenvalues by diagonalization of a matrix. We also consider the complex counterparts of those matrices (Hermitian, skew-Hermitian, and unitary matrices, Sec. 8.5), which play a role in modern physics.

COMMENT. ***Numerics for eigenvalues* (Secs. 20.6–20.9) *can be studied immediately after this chapter.***

Prerequisite: Chap. 7.
Sections that may be omitted in a shorter course: 8.4, 8.5
References and Answers to Problems: App. 1 Part B, App. 2.

8.1 Eigenvalues, Eigenvectors

From the viewpoint of engineering applications, eigenvalue problems are among the most important problems in connection with matrices, and the student should follow the present discussion with particular attention. We begin by defining the basic concepts and show how to solve these problems, by examples as well as in general. Then we shall turn to applications.

Let $\mathbf{A} = [a_{jk}]$ be a given $n \times n$ matrix and consider the vector equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}. \tag{1}$$

Here $\mathbf{x}$ is an unknown vector and λ an unknown scalar. Our task is to determine $\mathbf{x}$'s and λ's that satisfy (1). Geometrically, we are looking for vectors $\mathbf{x}$ for which the multiplication by $\mathbf{A}$ has the same effect as the multiplication by a scalar λ; in other words, $\mathbf{A}\mathbf{x}$ should be proportional to $\mathbf{x}$.

Clearly, the zero vector $\mathbf{x} = \mathbf{0}$ is a solution of (1) for any value of λ, because $\mathbf{A}\mathbf{0} = \mathbf{0}$. This is of no interest. A value of λ for which (1) has a solution $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvalue** or *characteristic value* (or *latent root*) of the matrix $\mathbf{A}$. ("Eigen" is German and means "proper" or "characteristic.") The corresponding solutions $\mathbf{x} \neq \mathbf{0}$ of (1) are called the **eigenvectors** or *characteristic vectors* of $\mathbf{A}$ corresponding to that eigenvalue λ. The set of all the eigenvalues of $\mathbf{A}$ is called the **spectrum** of $\mathbf{A}$. We shall see that the spectrum consists of at least one eigenvalue and at most of n numerically different eigenvalues. The largest of the absolute values of the eigenvalues of $\mathbf{A}$ is called the *spectral radius* of $\mathbf{A}$, a name to be motivated later.

How to Find Eigenvalues and Eigenvectors

The problem of determining the eigenvalues and eigenvectors of a matrix is called an **eigenvalue problem.** (More precisely: an *algebraic* eigenvalue problem, as opposed to an eigenvalue problem involving an ODE, PDE (see Secs. 5.7 and 12.3) or integral equation.) Such problems occur in physical, technical, geometric, and other applications, as we shall see. We show how to solve them, first by an example and then in general. Some typical applications will follow afterwards.

EXAMPLE 1 Determination of Eigenvalues and Eigenvectors

We illustrate all the steps in terms of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

Solution. **(a)** ***Eigenvalues.*** These must be determined ***first.*** Equation (1) is

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \text{in components,} \qquad \begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2. \end{aligned}$$

Transferring the terms on the right to the left, we get

$$\begin{aligned} (-5 - \lambda)x_1 + \quad 2x_2 \quad &= 0 \\ 2x_1 \quad + (-2 - \lambda)x_2 &= 0. \end{aligned} \tag{2*}$$

This can be written in matrix notation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{3*}$$

because (1) is $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which gives (3*). We see that this is a ***homogeneous*** linear system. By Cramer's theorem in Sec. 7.7 it has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ (an eigenvector of $\mathbf{A}$ we are looking for) if and only if its coefficient determinant is zero, that is,

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0. \tag{4*}$$

We call $D(\lambda)$ the **characteristic determinant** or, if expanded, the **characteristic polynomial**, and $D(\lambda) = 0$ the **characteristic equation** of **A**. The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$. These are the eigenvalues of **A**.

($\mathbf{b_1}$) ***Eigenvector of* A *corresponding to* $\boldsymbol{\lambda_1}$.** This vector is obtained from (2*) with $\lambda = \lambda_1 = -1$, that is,

$$\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 2x_1 - x_2 &= 0. \end{aligned}$$

A solution is $x_2 = 2x_1$, as we see from either of the two equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad \text{Check:} \qquad \mathbf{Ax}_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\mathbf{x}_1 = \lambda_1\mathbf{x}_1.$$

($\mathbf{b_2}$) ***Eigenvector of* A *corresponding to* $\boldsymbol{\lambda_2}$.** For $\lambda = \lambda_2 = -6$, equation (2*) becomes

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0. \end{aligned}$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Thus an eigenvector of **A** corresponding to $\lambda_2 = -6$ is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \qquad \text{Check:} \qquad \mathbf{Ax}_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = (-6)\mathbf{x}_2 = \lambda_2\mathbf{x}_2. \quad ■$$

This example illustrates the general case as follows. Equation (1) written in components is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n. \end{aligned}$$

Transferring the terms on the right side to the left side, we have

$$\begin{array}{ccccccccc} (a_{11}-\lambda)x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & (a_{22}-\lambda)x_2 & + \cdots + & a_{2n}x_n & = 0 \\ \cdots & & \cdots & & \cdots & \\ a_{n1}x_1 & + & a_{n2}x_2 & + \cdots + & (a_{nn}-\lambda)x_n & = 0. \end{array} \tag{2}$$

In matrix notation,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{3}$$

By Cramer's theorem in Sec. 7.7, this homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

$$\textbf{(4)} \quad D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

$\mathbf{A} - \lambda\mathbf{I}$ is called the **characteristic matrix** and $D(\lambda)$ the **characteristic determinant** of **A.** Equation (4) is called the **characteristic equation** of **A.** By developing $D(\lambda)$ we obtain a polynomial of nth degree in λ. This is called the **characteristic polynomial** of **A.**

This proves the following important theorem.

THEOREM 1

Eigenvalues

The eigenvalues of a square matrix **A** *are the roots of the characteristic equation* (4) *of* **A.**

Hence an $n \times n$ matrix has at least one eigenvalue and at most n numerically different eigenvalues.

For larger n, the actual computation of eigenvalues will in general require the use of Newton's method (Sec. 19.2) or another numeric approximation method in Secs. 20.7–20.9.

The eigenvalues must be determined first. Once these are known, corresponding eigen*vectors* are obtained from the system (2), for instance, by the Gauss elimination, where λ is the eigenvalue for which an eigenvector is wanted. This is what we did in Example 1 and shall do again in the examples below. (To prevent misunderstandings: *numeric approximation methods* (Sec. 20.8) may determine eigen*vectors first.*)

Eigenvectors have the following properties.

THEOREM 2

Eigenvectors, Eigenspace

If **w** *and* **x** *are eigenvectors of a matrix* **A** *corresponding to* **the same** *eigenvalue λ, so are* $\mathbf{w} + \mathbf{x}$ (provided $\mathbf{x} \neq -\mathbf{w}$) *and* $k\mathbf{x}$ *for any* $k \neq 0$.

Hence the eigenvectors corresponding to one and the same eigenvalue λ of **A,** *together with* **0,** *form a vector space* (cf. Sec. 7.4), *called the* **eigenspace** *of* **A** *corresponding to that λ.*

PROOF $\mathbf{Aw} = \lambda\mathbf{w}$ and $\mathbf{Ax} = \lambda\mathbf{x}$ *imply* $\mathbf{A}(\mathbf{w} + \mathbf{x}) = \mathbf{Aw} + \mathbf{Ax} = \lambda\mathbf{w} + \lambda\mathbf{x} = \lambda(\mathbf{w} + \mathbf{x})$ and $\mathbf{A}(k\mathbf{w}) = k(\mathbf{Aw}) = k(\lambda\mathbf{w}) = \lambda(k\mathbf{w})$; hence $\mathbf{A}(k\mathbf{w} + \ell\mathbf{x}) = \lambda(k\mathbf{w} + \ell\mathbf{x})$. ■

In particular, *an eigenvector* **x** *is determined only up to a constant factor.* Hence we can **normalize x,** that is, multiply it by a scalar to get a unit vector (see Sec. 7.9). For instance, $\mathbf{x}_1 = [1 \quad 2]^\mathsf{T}$ in Example 1 has the length $\|\mathbf{x}_1\| = \sqrt{1^2 + 2^2} = \sqrt{5}$; hence $[1/\sqrt{5} \quad 2/\sqrt{5}]^\mathsf{T}$ is a normalized eigenvector (a unit eigenvector).

Examples 2 and 3 will illustrate that an $n \times n$ matrix may have n linearly independent eigenvectors, or it may have fewer than n. In Example 4 we shall see that a *real* matrix may have *complex* eigenvalues and eigenvectors.

EXAMPLE 2 Multiple Eigenvalues

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Solution. For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$$

The roots (eigenvalues of $\mathbf{A}$) are $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$. To find eigenvectors, we apply the Gauss elimination (Sec. 7.3) to the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, first with $\lambda = 5$ and then with $\lambda = -3$. For $\lambda = 5$ the characteristic matrix is

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}. \qquad \text{It row-reduces to} \qquad \begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it has rank 2. Choosing $x_3 = -1$ we have $x_2 = 2$ from $-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$ and then $x_1 = 1$ from $-7x_1 + 2x_2 - 3x_3 = 0$. Hence an eigenvector of $\mathbf{A}$ coresponding to $\lambda = 5$ is $\mathbf{x}_1 = [1 \quad 2 \quad -1]^{\mathsf{T}}$.

For $\lambda = -3$ the characteristic matrix

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \qquad \text{row-reduces to} \qquad \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it has rank 1. From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$. Choosing $x_2 = 1$, $x_3 = 0$ and $x_2 = 0$, $x_3 = 1$, we obtain two linearly independent eigenvectors of $\mathbf{A}$ corresponding to $\lambda = -3$ [as they must exist by (5), Sec. 7.5, with rank $= 1$ and $n = 3$],

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

The order M_λ of an eigenvalue λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ. The number m_λ of linearly independent eigenvectors corresponding to λ is called the **geometric multiplicity** of λ. Thus m_λ is the dimension of the eigenspace corresponding to this λ. Since the characteristic polynomial has degree n, the sum of all the algebraic multiplicities must equal n. In Example 2 for $\lambda = -3$ we have $m_\lambda = M_\lambda = 2$. In general, $m_\lambda \leqq M_\lambda$, as can be shown. The difference $\Delta_\lambda = M_\lambda - m_\lambda$ is called the **defect** of λ. Thus $\Delta_{-3} = 0$ in Example 2, but positive defects Δ_λ can easily occur:

EXAMPLE 3 Algebraic Multiplicity, Geometric Multiplicity. Positive Defect

The characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0.$$

Hence $\lambda = 0$ is an eigenvalue of algebraic multiplicity $M_0 = 2$. But its geometric multiplicity is only $m_0 = 1$, since eigenvectors result from $-0x_1 + x_2 = 0$, hence $x_2 = 0$, in the form $[x_1 \quad 0]^\mathsf{T}$. Hence for $\lambda = 0$ the defect is $\Delta_0 = 1$.

Similarly, the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0.$$

Hence $\lambda = 3$ is an eigenvalue of algebraic multiplicity $M_3 = 2$, but its geometric multiplicity is only $m_3 = 1$, since eigenvectors result from $0x_1 + 2x_2 = 0$ in the form $[x_1 \quad 0]^\mathsf{T}$. ■

EXAMPLE 4 Real Matrices with Complex Eigenvalues and Eigenvectors

Since real polynomials may have complex roots (which then occur in conjugate pairs), a real matrix may have complex eigenvalues and eigenvectors. For instance, the characteristic equation of the skew-symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

It gives the eigenvalues $\lambda_1 = i \ (=\sqrt{-1})$, $\lambda_2 = -i$. Eigenvectors are obtained from $-ix_1 + x_2 = 0$ and $ix_1 + x_2 = 0$, respectively, and we can choose $x_1 = 1$ to get

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$ ■

In the next section we shall need the following simple theorem.

THEOREM 3 Eigenvalues of the Transpose

The transpose $\mathbf{A}^\mathsf{T}$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

PROOF Transposition does not change the value of the characteristic determinant, as follows from Theorem 2d in Sec. 7.7. ■

Having gained a first impression of matrix eigenvalue problems, in the next section we illustrate their importance with some typical applications.

PROBLEM SET 8.1

1–25 EIGENVALUES AND EIGENVECTORS

Find the eigenvalues and eigenvectors of the following matrices. (Use the given λ or factors.)

1. $\begin{bmatrix} -2 & 0 \\ 0 & 0.4 \end{bmatrix}$

2. $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

3. $\begin{bmatrix} 4 & 0 \\ 2 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$

6. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

7. $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

10. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

12. $\begin{bmatrix} 85 & -28 & -28 \\ -10 & -11 & -11 \\ -46 & -2 & -2 \end{bmatrix}$

13. $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$, $\lambda = 3$

14. $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & -2 \\ -2 & -2 & 1 \end{bmatrix}$, $\lambda = 1$

15. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 6 & 4 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0 & 1.0 & 1.5 \\ 0 & 0 & 3.5 \end{bmatrix}$

17. $\begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}$, $\lambda = -3$

18. $\begin{bmatrix} 3 & 0 & 12 \\ -6 & 3 & 0 \\ 9 & 6 & 3 \end{bmatrix}$, $\lambda = 9$

19. $\begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 0 & -5 & 7 \\ 0 & 0 & 7 & -5 \\ 0 & 0 & 19 & -1 \\ 0 & 0 & -1 & 19 \end{bmatrix}$

21. $\begin{bmatrix} 0 & -2 & 2 & 0 \\ -4 & 2 & -2 & 4 \\ 0 & 2 & 2 & -4 \\ 0 & 2 & -6 & 4 \end{bmatrix}$, $\lambda = 4$

22. $\begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix}$, $(\lambda - 3)^2$

23. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 4 & 2 & -6 \end{bmatrix}$

24. $\begin{bmatrix} -3 & 0 & -2 & 8 \\ 0 & 1 & 4 & -2 \\ -4 & 10 & -1 & -2 \\ 6 & -4 & -2 & 3 \end{bmatrix}$, $\begin{matrix} \lambda = 3 \\ \lambda = -5 \end{matrix}$

25. $\begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$, $(\lambda + 1)^2$

26. (Multiple eigenvalues) Find further 2×2 and 3×3 matrices with multiple eigenvalues. (See Example 2.)

27. (Nonzero defect) Find further 2×2 and 3×3 matrices with positive defect. (See Example 3.)

28. (Transpose) Illustrate Theorem 3 with examples of your own.

29. (Complex eigenvalues) Show that the eigenvalues of a real matrix are real or complex conjugate in pairs.

30. (Inverse) Show that the inverse $\mathbf{A}^{-1}$ exists if and only if none of the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $\mathbf{A}$ is zero, and then $\mathbf{A}^{-1}$ has the eigenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.

8.2 Some Applications of Eigenvalue Problems

In this section we discuss a few typical examples from the range of applications of matrix eigenvalue problems, which is incredibly large. Chapter 4 shows matrix eigenvalue problems related to ODEs governing mechanical systems and electrical networks. To keep our present discussion independent of Chap. 4, we include a typical application of that kind as our last example.

EXAMPLE 1 Stretching of an Elastic Membrane

An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ (Fig. 158) is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by

$$(1) \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \text{in components,} \qquad \begin{aligned} y_1 &= 5x_1 + 3x_2 \\ y_2 &= 3x_1 + 5x_2. \end{aligned}$$

Find the **principal directions,** that is, the directions of the position vector $\mathbf{x}$ of P for which the direction of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Solution. We are looking for vectors $\mathbf{x}$ such that $\mathbf{y} = \lambda\mathbf{x}$. Since $\mathbf{y} = \mathbf{A}\mathbf{x}$, this gives $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, the equation of an eigenvalue problem. In components, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is

$$(2) \qquad \begin{aligned} 5x_1 + 3x_2 &= \lambda x_1 \\ 3x_1 + 5x_2 &= \lambda x_2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} (5-\lambda)x_1 + \quad 3x_2 \quad &= 0 \\ 3x_1 \quad + (5-\lambda)x_2 &= 0. \end{aligned}$$

The characteristic equation is

$$(3) \qquad \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0.$$

Its solutions are $\lambda_1 = 8$ and $\lambda_2 = 2$. These are the eigenvalues of our problem. For $\lambda = \lambda_1 = 8$, our system (2) becomes

$$\begin{aligned} -3x_1 + 3x_2 &= 0, \\ 3x_1 - 3x_2 &= 0. \end{aligned} \qquad \left| \quad \begin{aligned} &\text{Solution } x_2 = x_1, \quad x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = x_2 = 1. \end{aligned} \right.$$

For $\lambda_2 = 2$, our system (2) becomes

$$\begin{aligned} 3x_1 + 3x_2 &= 0, \\ 3x_1 + 3x_2 &= 0. \end{aligned} \qquad \left| \quad \begin{aligned} &\text{Solution } x_2 = -x_1, \quad x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = 1,\ x_2 = -1. \end{aligned} \right.$$

We thus obtain as eigenvectors of $\mathbf{A}$, for instance, $[1 \ \ 1]^\mathsf{T}$ corresponding to λ_1 and $[1 \ \ -1]^\mathsf{T}$ corresponding to λ_2 (or a nonzero scalar multiple of these). These vectors make 45° and 135° angles with the positive x_1-direction. They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively; see Fig. 158.

Accordingly, if we choose the principal directions as directions of a new Cartesian u_1u_2-coordinate system, say, with the positive u_1-semi-axis in the first quadrant and the positive u_2-semi-axis in the second quadrant of the x_1x_2-system, and if we set $u_1 = r\cos\phi$, $u_2 = r\sin\phi$, then a boundary point of the unstretched circular membrane has coordinates $\cos\phi$, $\sin\phi$. Hence, after the stretch we have

$$z_1 = 8\cos\phi, \qquad z_2 = 2\sin\phi.$$

Since $\cos^2\phi + \sin^2\phi = 1$, this shows that the deformed boundary is an ellipse (Fig. 158)

$$(4) \qquad \frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = 1. \qquad \blacksquare$$

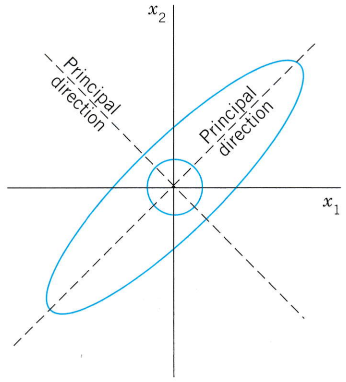


Fig. 158. Undeformed and deformed membrane in Example 1

EXAMPLE 2 **Eigenvalue Problems Arising from Markov Processes**

Markov processes as considered in Example 13 of Sec. 7.2 lead to eigenvalue problems if we ask for the limit state of the process in which the state vector **x** is reproduced under the multiplication by the stochastic matrix **A** governing the process, that is, $\mathbf{Ax} = \mathbf{x}$. Hence **A** should have the eigenvalue 1, and **x** should be a corresponding eigenvector. This is of practical interest because it shows the long-term tendency of the development modeled by the process.

In that example,

$$\mathbf{A} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}. \qquad \text{For the transpose,} \qquad \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $\mathbf{A}^\mathsf{T}$ has the eigenvalue 1, and the same is true for **A** by Theorem 3 in Sec. 8.1. An eigenvector **x** of **A** for $\lambda = 1$ is obtained from

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \qquad \text{row-reduced to} \qquad \begin{bmatrix} -3/10 & 1/10 & 0 \\ 0 & -1/30 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $x_3 = 1$, we get $x_2 = 6$ from $-x_2/30 + x_3/5 = 0$ and then $x_1 = 2$ from $-3x_1/10 + x_2/10 = 0$. This gives $\mathbf{x} = [2 \quad 6 \quad 1]^\mathsf{T}$. It means that in the long run, the ratio Commercial : Industrial : Residential will approach 2 : 6 : 1, provided that the probabilities given by **A** remain (about) the same. (We switched to ordinary fractions to avoid rounding errors.) ■

EXAMPLE 3 **Eigenvalue Problems Arising from Population Models. Leslie Model**

The Leslie model describes age-specified population growth, as follows. Let the oldest age attained by the females in some animal population be 9 years. Divide the population into three age classes of 3 years each. Let the *"Leslie matrix"* be

$$\mathbf{L} = [l_{jk}] = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \tag{5}$$

where l_{1k} is the average number of daughters born to a single female during the time she is in age class k, and $l_{j,j-1}$ $(j = 2, 3)$ is the fraction of females in age class $j - 1$ that will survive and pass into class j. (a) What is the number of females in each class after 3, 6, 9 years if each class initially consists of 400 females? (b) For what initial distribution will the number of females in each class change by the same proportion? What is this rate of change?

Solution. **(a)** Initially, $\mathbf{x}_{(0)}^{\mathsf{T}} = [400 \quad 400 \quad 400]$. After 3 years,

$$\mathbf{x}_{(3)} = \mathbf{L}\mathbf{x}_{(0)} = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix}.$$

Similarly, after 6 years the number of females in each class is given by $\mathbf{x}_{(6)}^{\mathsf{T}} = (\mathbf{L}\mathbf{x}_{(3)})^{\mathsf{T}} = [600 \quad 648 \quad 72]$, and after 9 years we have $\mathbf{x}_{(9)}^{\mathsf{T}} = (\mathbf{L}\mathbf{x}_{(6)})^{\mathsf{T}} = [1519.2 \quad 360 \quad 194.4]$.

(b) Proportional change means that we are looking for a distribution vector $\mathbf{x}$ such that $\mathbf{L}\mathbf{x} = \lambda\mathbf{x}$, where λ is the rate of change (growth if $\lambda > 1$, decrease if $\lambda < 1$). The characteristic equation is (develop the characteristic determinant by the first column)

$$\det(\mathbf{L} - \lambda\mathbf{I}) = -\lambda^3 - 0.6(-2.3\lambda - 0.3 \cdot 0.4) = -\lambda^3 + 1.38\lambda + 0.072 = 0.$$

A positive root is found to be (for instance, by Newton's method, Sec. 19.2) $\lambda = 1.2$. A corresponding eigenvector $\mathbf{x}$ can be determined from the characteristic matrix

$$\mathbf{A} - 1.2\mathbf{I} = \begin{bmatrix} -1.2 & 2.3 & 0.4 \\ 0.6 & -1.2 & 0 \\ 0 & 0.3 & -1.2 \end{bmatrix}, \qquad \text{say,} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 0.5 \\ 0.125 \end{bmatrix}$$

where $x_3 = 0.125$ is chosen, $x_2 = 0.5$ then follows from $0.3x_2 - 1.2x_3 = 0$, and $x_1 = 1$ from $-1.2x_1 + 2.3x_2 + 0.4x_3 = 0$. To get an initial population of 1200 as before, we multiply $\mathbf{x}$ by $1200/(1 + 0.5 + 0.125) = 738$. *Answer:* Proportional growth of the numbers of females in the three classes will occur if the initial values are 738, 369, 92 in classes 1, 2, 3, respectively. The growth rate will be 1.2 per 3 years. ■

EXAMPLE 4 Vibrating System of Two Masses on Two Springs (Fig. 159)

Mass–spring systems involving several masses and springs can be treated as eigenvalue problems. For instance, the mechanical system in Fig. 159 is governed by the system of ODEs

$$\begin{aligned} y_1'' &= -5y_1 + 2y_2 \\ y_2'' &= 2y_1 - 2y_2 \end{aligned} \tag{6}$$

where y_1 and y_2 are the displacements of the masses from rest, as shown in the figure, and primes denote derivatives with respect to time t. In vector form, this becomes

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \mathbf{A}\mathbf{y} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{7}$$

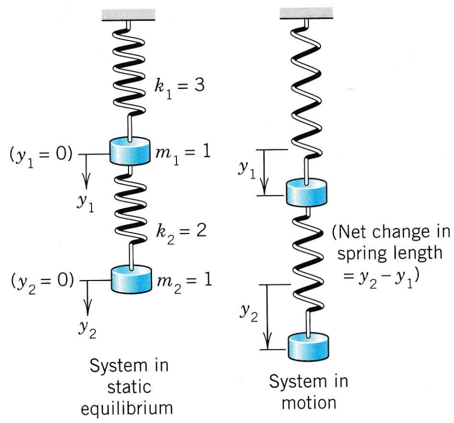


Fig. 159. Masses on springs in Example 4

We try a vector solution of the form

$$\mathbf{y} = \mathbf{x}e^{\omega t}. \tag{8}$$

This is suggested by a mechanical system of a single mass on a spring (Sec. 2.4), whose motion is given by exponential functions (and sines and cosines). Substitution into (7) gives

$$\omega^2 \mathbf{x} e^{\omega t} = \mathbf{A}\mathbf{x}e^{\omega t}.$$

Dividing by $e^{\omega t}$ and writing $\omega^2 = \lambda$, we see that our mechanical system leads to the eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \qquad \text{where } \lambda = \omega^2. \tag{9}$$

From Example 1 in Sec. 8.1 we see that $\mathbf{A}$ has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Consequently, $\omega = \sqrt{-1} = \pm i$ and $\sqrt{-6} = \pm i\sqrt{6}$, respectively. Corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \tag{10}$$

From (8) we thus obtain the four complex solutions [see (10), Sec. 2.2]

$$\mathbf{x}_1 e^{\pm it} = \mathbf{x}_1(\cos t \pm i \sin t),$$
$$\mathbf{x}_2 e^{\pm i\sqrt{6}t} = \mathbf{x}_2(\cos \sqrt{6}\, t \pm i \sin \sqrt{6}\, t).$$

By addition and subtraction (see Sec. 2.2) we get the four real solutions

$$\mathbf{x}_1 \cos t, \quad \mathbf{x}_1 \sin t, \quad \mathbf{x}_2 \cos \sqrt{6}\, t, \quad \mathbf{x}_2 \sin \sqrt{6}\, t.$$

A general solution is obtained by taking a linear combination of these,

$$\mathbf{y} = \mathbf{x}_1(a_1 \cos t + b_1 \sin t) + \mathbf{x}_2(a_2 \cos \sqrt{6}\, t + b_2 \sin \sqrt{6}\, t)$$

with arbitrary constants a_1, b_1, a_2, b_2 (to which values can be assigned by prescribing initial displacement and initial velocity of each of the two masses). By (10), the components of $\mathbf{y}$ are

$$y_1 = a_1 \cos t + b_1 \sin t + 2a_2 \cos \sqrt{6}\, t + 2b_2 \sin \sqrt{6}\, t$$
$$y_2 = 2a_1 \cos t + 2b_1 \sin t - a_2 \cos \sqrt{6}\, t - b_2 \sin \sqrt{6}\, t.$$

These functions describe harmonic oscillations of the two masses. Physically, this had to be expected because we have neglected damping. ■

PROBLEM SET 8.2

1–6 LINEAR TRANSFORMATIONS

Find the matrix $\mathbf{A}$ in the indicated linear transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$. Explain the geometric significance of the eigenvalues and eigenvectors of $\mathbf{A}$. Show the details.

1. Reflection about the y-axis in R^2
2. Reflection about the xy-plane in R^3
3. Orthogonal projection (perpendicular projection) of R^2 onto the x-axis
4. Orthogonal projection of R^3 onto the plane $y = x$
5. Dilatation (uniform stretching) in R^2 by a factor 5
6. Counterclockwise rotation through the angle $\pi/2$ about the origin in R^2

7–14 ELASTIC DEFORMATIONS

Given $\mathbf{A}$ in a deformation $\mathbf{y} = \mathbf{A}\mathbf{x}$, find the principal directions and corresponding factors of extension or contraction. Show the details.

7. $\begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 0.4 & 0.8 \\ 0.8 & 0.4 \end{bmatrix}$

9. $\begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 6.5 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 4 \\ 4 & 11 \end{bmatrix}$

11. $\begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$

12. $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$

13. $\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$

14. $\begin{bmatrix} 10.5 & 1/\sqrt{2} \\ 1/\sqrt{2} & 10.0 \end{bmatrix}$

15. (Leontief[1] input–output model) Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3×3 **consumption matrix**

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.6 & 0 & 0.3 \\ 0.2 & 0.5 & 0.7 \end{bmatrix}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j. Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to $\mathbf{Ap} = \mathbf{p}$, where $\mathbf{p} = [p_1 \quad p_2 \quad p_3]^\mathsf{T}$, and find a solution $\mathbf{p}$ with nonnegative p_1, p_2, p_3.

16. Show that a consumption matrix as considered in Prob. 15 must have column sums 1 and always has the eigenvalue 1.

17. (Open Leontief input–output model) If not the whole output but only a portion of it is consumed by the industries themselves, then instead of $\mathbf{Ax} = \mathbf{x}$ (as in Prob. 15), we have $\mathbf{x} - \mathbf{Ax} = \mathbf{y}$, where $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^\mathsf{T}$ is produced, $\mathbf{Ax}$ is consumed by the industries, and, thus, $\mathbf{y}$ is the net production available for other consumers. Find for what production $\mathbf{x}$ a given **demand vector** $\mathbf{y} = [0.136 \quad 0.272 \quad 0.136]^\mathsf{T}$ can be achieved if the consumption matrix is

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.3 & 0 & 0.1 \\ 0.2 & 0.4 & 0.5 \end{bmatrix}.$$

18–20 MARKOV PROCESSES

Find limit states of the Markov processes modeled by the following matrices. (Show the details.)

18. $\begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix}$

19. $\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$

20. $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

21–23 POPULATION MODEL WITH AGE SPECIFICATION

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. (Show details.)

21. $\begin{bmatrix} 0 & 3.45 & 0.60 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 12.0 & 0 \\ 0.75 & 0 & 0 \\ 0 & 0.30 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 7.280 & 2.975 \\ 0.560 & 0 & 0 \\ 0 & 0.420 & 0 \end{bmatrix}$

24. TEAM PROJECT. General Properties of Eigenvalues and Eigenvectors. Prove the following statements and illustrate them with examples of your own choice. Here, $\lambda_1, \cdots, \lambda_n$ are the (not necessarily distinct) eigenvalues of a given $n \times n$ matrix $\mathbf{A} = [a_{jk}]$.

(a) Trace. The sum of the main diagonal entries is called the **trace** of $\mathbf{A}$. It equals the sum of the eigenvalues.

(b) "Spectral shift." $\mathbf{A} - k\mathbf{I}$ has the eigenvalues $\lambda_1 - k, \cdots, \lambda_n - k$ and the same eigenvectors as $\mathbf{A}$.

(c) Scalar multiples, powers. $k\mathbf{A}$ has the eigenvalues $k\lambda_1, \cdots, k\lambda_n$. $\mathbf{A}^m$ $(m = 1, 2, \cdots)$ has the eigenvalues $\lambda_1{}^m, \cdots, \lambda_n{}^m$. The eigenvectors are those of $\mathbf{A}$.

(d) Spectral mapping theorem. The **"polynomial matrix"**

$$p(\mathbf{A}) = k_m\mathbf{A}^m + k_{m-1}\mathbf{A}^{m-1} + \cdots + k_1\mathbf{A} + k_0\mathbf{I}$$

has the eigenvalues

$$p(\lambda_j) = k_m\lambda_j{}^m + k_{m-1}\lambda_j{}^{m-1} + \cdots + k_1\lambda_j + k_0$$

where $j = 1, \cdots, n$, and the same eigenvectors as $\mathbf{A}$.

(e) Perron's theorem. Show that a Leslie matrix $\mathbf{L}$ with positive $l_{12}, l_{13}, l_{21}, l_{32}$ has a positive eigenvalue. (This is a special case of the famous Perron–Frobenius theorem in Sec. 20.7, which is difficult to prove in its general form.)

[1] WASSILY LEONTIEF (1906–1999). American economist at New York University. For his input–output analysis he was awarded the Nobel Prize in 1973.

8.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

We consider three classes of real square matrices that occur quite frequently in applications because they have several remarkable properties which we shall now discuss. The first two of these classes have already been mentioned in Sec. 7.2.

DEFINITIONS

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A ***real*** square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged,

(1) $$\mathbf{A}^{\mathsf{T}} = \mathbf{A}, \qquad \text{thus} \qquad a_{kj} = a_{jk},$$

skew-symmetric if transposition gives the negative of **A**,

(2) $$\mathbf{A}^{\mathsf{T}} = -\mathbf{A}, \qquad \text{thus} \qquad a_{kj} = -a_{jk},$$

orthogonal if transposition gives the inverse of **A**,

(3) $$\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}.$$

EXAMPLE 1 **Symmetric, Skew-Symmetric, and Orthogonal Matrices**

The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

are symmetric, skew-symmetric, and orthogonal, respectively, as you should verify. Every skew-symmetric matrix has all main diagonal entries zero. (Can you prove this?) ■

Any real square matrix **A** may be written as the sum of a symmetric matrix **R** and a skew-symmetric matrix **S**, where

(4) $$\mathbf{R} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}}) \qquad \text{and} \qquad \mathbf{S} = \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathsf{T}}).$$

EXAMPLE 2 **Illustration of Formula (4)**

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} = \mathbf{R} + \mathbf{S} = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix} + \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$ ■

THEOREM 1

Eigenvalues of Symmetric and Skew-Symmetric Matrices

(a) *The eigenvalues of a symmetric matrix are real.*

(b) *The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.*

This basic theorem (and an extension of it) will be proved in Sec. 8.5.

EXAMPLE 3 **Eigenvalues of Symmetric and Skew-Symmetric Matrices**

The matrices in (1) and (7) of Sec. 8.2 are symmetric and have real eigenvalues. The skew-symmetric matrix in Example 1 has the eigenvalues 0, $-25i$, and $25i$. (Verify this.) The following matrix has the real eigenvalues 1 and 5 but is not symmetric. Does this contradict Theorem 1?

$$\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$$ ■

Orthogonal Transformations and Orthogonal Matrices

Orthogonal transformations are transformations

(5) $$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{A} \text{ is an orthogonal matrix.}$$

With each vector $\mathbf{x}$ in R^n such a transformation assigns a vector $\mathbf{y}$ in R^n. For instance, the plane rotation through an angle θ

(6) $$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an orthogonal transformation. It can be shown that any orthogonal transformation in the plane or in three-dimensional space is a **rotation** (possibly combined with a reflection in a straight line or a plane, respectively).

The main reason for the importance of orthogonal matrices is as follows.

THEOREM 2

Invariance of Inner Product

An orthogonal transformation preserves the value of the **inner product** *of vectors* **a** *and* **b** *in* R^n, *defined by*

(7) $$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b} = [a_1 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

That is, for any **a** *and* **b** *in* R^n, *orthogonal* $n \times n$ *matrix* **A**, *and* $\mathbf{u} = \mathbf{A}\mathbf{a}$, $\mathbf{v} = \mathbf{A}\mathbf{b}$ *we have* $\mathbf{u}\cdot\mathbf{v} = \mathbf{a}\cdot\mathbf{b}$.

Hence the transformation also preserves the **length** *or* **norm** *of any vector* **a** *in* R^n *given by*

(8) $$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\mathbf{a}^{\mathsf{T}}\mathbf{a}}.$$

PROOF Let $\mathbf{A}$ be orthogonal. Let $\mathbf{u} = \mathbf{Aa}$ and $\mathbf{v} = \mathbf{Ab}$. We must show that $\mathbf{u} \bullet \mathbf{v} = \mathbf{a} \bullet \mathbf{b}$. Now $(\mathbf{Aa})^{\mathsf{T}} = \mathbf{a}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ by (10d) in Sec. 7.2 and $\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ by (3). Hence

$$\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^{\mathsf{T}}\mathbf{v} = (\mathbf{Aa})^{\mathsf{T}}\mathbf{Ab} = \mathbf{a}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ab} = \mathbf{a}^{\mathsf{T}}\mathbf{I}\,\mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b} = \mathbf{a} \bullet \mathbf{b}. \tag{9}$$

From this the invariance of $\|\mathbf{a}\|$ follows if we set $\mathbf{b} = \mathbf{a}$. ■

Orthogonal matrices have further interesting properties as follows.

THEOREM 3

Orthonormality of Column and Row Vectors

A real square matrix is orthogonal if and only if its column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ *(and also its row vectors) form an* **orthonormal system,** *that is,*

$$\mathbf{a}_j \bullet \mathbf{a}_k = \mathbf{a}_j^{\mathsf{T}}\mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases} \tag{10}$$

PROOF **(a)** Let $\mathbf{A}$ be orthogonal. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$, in terms of column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$,

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathsf{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathsf{T}} \\ \vdots \\ \mathbf{a}_n^{\mathsf{T}} \end{bmatrix} [\mathbf{a}_1 \cdots \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_1^{\mathsf{T}}\mathbf{a}_1 & \mathbf{a}_1^{\mathsf{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1^{\mathsf{T}}\mathbf{a}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_n^{\mathsf{T}}\mathbf{a}_1 & \mathbf{a}_n^{\mathsf{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_n^{\mathsf{T}}\mathbf{a}_n \end{bmatrix}. \tag{11}$$

The last equality implies (10), by the definition of the $n \times n$ unit matrix $\mathbf{I}$. From (3) it follows that the inverse of an orthogonal matrix is orthogonal (see CAS Experiment 20). Now the column vectors of $\mathbf{A}^{-1}$ ($= \mathbf{A}^{\mathsf{T}}$) are the row vectors of $\mathbf{A}$. Hence the row vectors of $\mathbf{A}$ also form an orthonormal system.

(b) Conversely, if the column vectors of $\mathbf{A}$ satisfy (10), the off-diagonal entries in (11) must be 0 and the diagonal entries 1. Hence $\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$, as (11) shows. Similarly, $\mathbf{AA}^{\mathsf{T}} = \mathbf{I}$. This implies $\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}$ because also $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$ and the inverse is unique. Hence $\mathbf{A}$ is orthogonal. Similarly when the row vectors of $\mathbf{A}$ form an orthonormal system, by what has been said at the end of part **(a)**. ■

THEOREM 4

Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value $+1$ *or* -1.

PROOF From det $\mathbf{AB}$ = det $\mathbf{A}$ det $\mathbf{B}$ (Sec. 7.8, Theorem 4) and det $\mathbf{A}^{\mathsf{T}}$ = det $\mathbf{A}$ (Sec. 7.7, Theorem 2d), we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det(\mathbf{AA}^{-1}) = \det(\mathbf{AA}^{\mathsf{T}}) = \det \mathbf{A} \det \mathbf{A}^{\mathsf{T}} = (\det \mathbf{A})^2. \quad ■$$

EXAMPLE 4 **Illustration of Theorems 3 and 4**

The last matrix in Example 1 and the matrix in (6) illustrate Theorems 3 and 4 because their determinants are -1 and $+1$, as you should verify. ■

THEOREM 5

Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix **A** *are real or complex conjugates in pairs and have absolute value* 1.

PROOF The first part of the statement holds for any real matrix **A** because its characteristic polynomial has real coefficients, so that its zeros (the eigenvalues of **A**) must be as indicated. The claim that $|\lambda| = 1$ will be proved in Sec. 8.5. ■

EXAMPLE 5 **Eigenvalues of an Orthogonal Matrix**

The orthogonal matrix in Example 1 has the characteristic equation

$$-\lambda^3 + \tfrac{2}{3}\lambda^2 + \tfrac{2}{3}\lambda - 1 = 0.$$

Now one of the eigenvalues must be real (why?), hence $+1$ or -1. Trying, we find -1. Division by $\lambda + 1$ gives $-(\lambda^2 - 5\lambda/3 + 1) = 0$ and the two eigenvalues $(5 + i\sqrt{11})/6$ and $(5 - i\sqrt{11})/6$, which have absolute value 1. Verify all of this. ■

Looking back at this section, you will find that the numerous basic results it contains have relatively short, straightforward proofs. This is typical of large portions of matrix eigenvalue theory.

PROBLEM SET 8.3

1. **(Verification)** Verify the statements in Example 1.
2. Verify the statements in Examples 3 and 4.
3. Are the eigenvalues of $\mathbf{A} + \mathbf{B}$ of the form $\lambda_j + \mu_j$, where λ_j and μ_j are the eigenvalues of **A** and **B**, respectively?
4. **(Orthogonality)** Prove that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Give an example.
5. **(Skew-symmetric matrix)** Show that the inverse of a skew-symmetric matrix is skew-symmetric.
6. Do there exist nonsingular skew-symmetric $n \times n$ matrices with odd n?
7. **(Orthogonal matrix)** Do there exist skew-symmetric orthogonal 3×3 matrices?
8. **(Symmetric matrix)** Do there exist nondiagonal symmetric 3×3 matrices that are orthogonal?

9–17 EIGENVALUES OF SYMMETRIC, SKEW-SYMMETRIC, AND ORTHOGONAL MATRICES

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find their spectrum (thereby illustrating Theorems 1 and 5). (Show the details of your work.)

9. $\begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$

10. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

11. $\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 14 & 4 & -2 \\ 4 & 14 & 2 \\ -2 & 2 & 17 \end{bmatrix}$

14. $\begin{bmatrix} 0 & -6 & -12 \\ 6 & 0 & -12 \\ 12 & 12 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} \frac{4}{9} & \frac{8}{9} & \frac{1}{9} \\ -\frac{7}{9} & \frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{1}{9} & \frac{8}{9} \end{bmatrix}$

17. $\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$

18. **(Rotation in space)** Give a geometric interpretation of the transformation $\mathbf{y} = \mathbf{Ax}$ with **A** as in Prob. 12 and **x** and **y** referred to a Cartesian coordinate system.
19. **WRITING PROJECT. Section Summary.** Summarize the main concepts and facts in this section, with illustrative examples of your own.

20. CAS EXPERIMENT. Orthogonal Matrices.

(a) Products. Inverse. Prove that the product of two orthogonal matrices is orthogonal, and so is the inverse of an orthogonal matrix. What does this mean in terms of rotations?

(b) Rotation. Show that (6) is an orthogonal transformation. Verify that it satisfies Theorem 3. Find the inverse transformation.

(c) Powers. Write a program for computing powers $\mathbf{A}^m$ $(m = 1, 2, \cdots)$ of a 2×2 matrix $\mathbf{A}$ and their spectra. Apply it to the matrix in Prob. 9 (call it $\mathbf{A}$). To what rotation does $\mathbf{A}$ correspond? Do the eigenvalues of $\mathbf{A}^m$ have a limit as $m \to \infty$?

(d) Compute the eigenvalues of $(0.9\mathbf{A})^m$, where $\mathbf{A}$ is the matrix in Prob. 9. Plot them as points. What is their limit? Along what kind of curve do these points approach the limit?

(e) Find $\mathbf{A}$ such that $\mathbf{y} = \mathbf{Ax}$ is a counterclockwise rotation through 30° in the plane.

8.4 Eigenbases. Diagonalization. Quadratic Forms

So far we have emphasized properties of eigen*values*. We now turn to general properties of eigen*vectors*. Eigenvectors of an $n \times n$ matrix $\mathbf{A}$ may (or may not!) form a basis for R^n. If we are interested in a transformation $\mathbf{y} = \mathbf{Ax}$, such an **"eigenbasis"** (basis of eigenvectors)—if it exists—is of great advantage because then we can represent any $\mathbf{x}$ in R^n uniquely as a linear combination of the eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$, say,

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

And, denoting the corresponding (not necessarily distinct) eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \cdots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Ax} &= \mathbf{A}(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) \\ &= c_1\mathbf{Ax}_1 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n. \end{aligned} \tag{1}$$

This shows that we have decomposed the complicated action of $\mathbf{A}$ on an arbitrary vector $\mathbf{x}$ into a sum of simple actions (multiplication by scalars) on the eigenvectors of $\mathbf{A}$. This is the point of an eigenbasis.

Now if the n eigenvalues are all different, we do obtain a basis:

THEOREM 1 **Basis of Eigenvectors**

*If an $n \times n$ matrix $\mathbf{A}$ has n **distinct** eigenvalues, then $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ for R^n.*

PROOF All we have to show is that $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are linearly independent. Suppose they are not. Let r be the largest integer such that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is a linearly independent set. Then $r < n$ and the set $\{\mathbf{x}_1, \cdots, \mathbf{x}_r, \mathbf{x}_{r+1}\}$ is linearly dependent. Thus there are scalars $c_1, \cdots, c_{r+1}$, not all zero, such that

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = \mathbf{0} \tag{2}$$

(see Sec. 7.4). Multiplying both sides by $\mathbf{A}$ and using $\mathbf{Ax}_j = \lambda_j\mathbf{x}_j$, we obtain

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = \mathbf{0}. \tag{3}$$

To get rid of the last term, we subtract λ_{r+1} times (2) from this, obtaining

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = \mathbf{0}.$$

Here $c_1(\lambda_1 - \lambda_{r+1}) = 0, \cdots, c_r(\lambda_r - \lambda_{r+1}) = 0$ since $\{x_1, \cdots, x_r\}$ is linearly independent. Hence $c_1 = \cdots = c_r = 0$, since all the eigenvalues are distinct. But with this, (2) reduces to $c_{r+1}\mathbf{x}_{r+1} = \mathbf{0}$, hence $c_{r+1} = 0$, since $\mathbf{x}_{r+1} \neq \mathbf{0}$ (an eigenvector!). This contradicts the fact that not all scalars in (2) are zero. Hence the conclusion of the theorem must hold. ■

EXAMPLE 1 **Eigenbasis. Nondistinct Eigenvalues. Nonexistence**

The matrix $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ has a basis of eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 2$. (See Example 1 in Sec. 8.2.)

Even if not all n eigenvalues are different, a matrix $\mathbf{A}$ may still provide an eigenbasis for R^n. See Example 2 in Sec. 8.1, where $n = 3$.

On the other hand, $\mathbf{A}$ ***may not have enough linearly independent eigenvectors to make up a basis.*** For instance, $\mathbf{A}$ in Example 3 of Sec. 8.1 is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and has only one eigenvector} \quad \begin{bmatrix} k \\ 0 \end{bmatrix} \quad (k \neq 0, \text{ arbitrary}).$$ ■

Actually, eigenbases exist under much more general conditions than those in Theorem 1. An important case is the following.

THEOREM 2 **Symmetric Matrices**

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

For a proof (which is involved) see Ref. [B3], vol. 1, pp. 270–272.

EXAMPLE 2 **Orthonormal Basis of Eigenvectors**

The first matrix in Example 1 is symmetric, and an orthonormal basis of eigenvectors is $[1/\sqrt{2} \quad 1/\sqrt{2}]^\mathsf{T}$, $[1/\sqrt{2} \quad -1/\sqrt{2}]^\mathsf{T}$. ■

Diagonalization of Matrices

Eigenbases also play a role in reducing a matrix $\mathbf{A}$ to a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$. This is done by a "similarity transformation," which is defined as follows (and will have various applications in numerics in Chap. 20).

DEFINITION **Similar Matrices. Similarity Transformation**

An $n \times n$ matrix $\hat{\mathbf{A}}$ is called **similar** to an $n \times n$ matrix $\mathbf{A}$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{4}$$

for some (nonsingular!) $n \times n$ matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called a **similarity transformation.**

The key property of this transformation is that it preserves the eigenvalues of $\mathbf{A}$:

THEOREM 3

Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ *has the same eigenvalues as* $\mathbf{A}$.

Furthermore, if $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ *is an eigenvector of* $\hat{\mathbf{A}}$ *corresponding to the same eigenvalue.*

PROOF From $\mathbf{Ax} = \lambda\mathbf{x}$ (λ an eigenvalue, $\mathbf{x} \neq \mathbf{0}$) we get $\mathbf{P}^{-1}\mathbf{Ax} = \lambda\mathbf{P}^{-1}\mathbf{x}$. Now $\mathbf{I} = \mathbf{PP}^{-1}$. By this ***"identity trick"*** the previous equation gives

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{x} = \hat{\mathbf{A}}(\mathbf{P}^{-1}\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x}.$$

Hence λ is an eigenvalue of $\hat{\mathbf{A}}$ and $\mathbf{P}^{-1}\mathbf{x}$ a corresponding eigenvector. Indeed, $\mathbf{P}^{-1}\mathbf{x} = \mathbf{0}$ would give $\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x} = \mathbf{P0} = \mathbf{0}$, contradicting $\mathbf{x} \neq \mathbf{0}$. ■

EXAMPLE 3 **Eigenvalues and Vectors of Similar Matrices**

Let
$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

Then
$$\hat{\mathbf{A}} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Here $\mathbf{P}^{-1}$ was obtained from (4*) in Sec. 7.8 with $\det \mathbf{P} = 1$. We see that $\hat{\mathbf{A}}$ has the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$. The characteristic equation of $\mathbf{A}$ is $(6 - \lambda)(-1 - \lambda) + 12 = \lambda^2 - 5\lambda + 6 = 0$. It has the roots (the eigenvalues of $\mathbf{A}$) $\lambda_1 = 3$, $\lambda_2 = 2$, confirming the first part of Theorem 3.

We confirm the second part. From the first component of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ we have $(6 - \lambda)x_1 - 3x_2 = 0$. For $\lambda = 3$ this gives $3x_1 - 3x_2 = 0$, say, $\mathbf{x}_1 = [1 \quad 1]^\mathsf{T}$. For $\lambda = 2$ it gives $4x_1 - 3x_2 = 0$, say, $\mathbf{x}_2 = [3 \quad 4]^\mathsf{T}$. In Theorem 3 we thus have

$$\mathbf{y}_1 = \mathbf{P}^{-1}\mathbf{x}_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{y}_2 = \mathbf{P}^{-1}\mathbf{x}_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Indeed, these are eigenvectors of the diagonal matrix $\hat{\mathbf{A}}$.

Perhaps we see that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the columns of $\mathbf{P}$. This suggests the general method of transforming a matrix $\mathbf{A}$ to diagonal form $\mathbf{D}$ by using $\mathbf{P} = \mathbf{X}$, the matrix with eigenvectors as columns: ■

THEOREM 4

Diagonalization of a Matrix

If an $n \times n$ matrix $\mathbf{A}$ *has a basis of eigenvectors, then*

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} \tag{5}$$

is diagonal, with the eigenvalues of $\mathbf{A}$ *as the entries on the main diagonal. Here* $\mathbf{X}$ *is the matrix with these eigenvectors as column vectors. Also,*

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \qquad (m = 2, 3, \cdots). \tag{5*}$$

PROOF Let $\mathbf{x}_1, \cdots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of $\mathbf{A}$ be $\lambda_1, \cdots, \lambda_n$, respectively, so that $\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1, \cdots, \mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$. Then $\mathbf{X} = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_n]$ has rank n, by Theorem 3 in Sec. 7.4. Hence $\mathbf{X}^{-1}$ exists by Theorem 1 in Sec. 7.8. We claim that

$$(6) \qquad \mathbf{AX} = \mathbf{A}[\mathbf{x}_1 \ \cdots \ \mathbf{x}_n] = [\mathbf{A}\mathbf{x}_1 \ \cdots \ \mathbf{A}\mathbf{x}_n] = [\lambda_1\mathbf{x}_1 \ \cdots \ \lambda_n\mathbf{x}_n] = \mathbf{XD}$$

where $\mathbf{D}$ is the diagonal matrix as in (5). The fourth equality in (6) follows by direct calculation. (Try it for $n = 2$ and then for general n.) The third equality uses $\mathbf{A}\mathbf{x}_k = \lambda_k\mathbf{x}_k$. The second equality results if we note that the first column of $\mathbf{AX}$ is $\mathbf{A}$ times the first column of $\mathbf{X}$, and so on. For instance, when $n = 2$ and we write $\mathbf{x}_1 = [x_{11} \ \ x_{21}]^\mathsf{T}$, $\mathbf{x}_2 = [x_{12} \ \ x_{22}]^\mathsf{T}$, we have

$$\mathbf{AX} = \mathbf{A}[\mathbf{x}_1 \ \ \mathbf{x}_2] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} \\ a_{21}x_{11} + a_{22}x_{21} \end{bmatrix}}_{\text{Column 1}} \underbrace{\begin{bmatrix} a_{11}x_{12} + a_{12}x_{22} \\ a_{21}x_{12} + a_{22}x_{22} \end{bmatrix}}_{\text{Column 2}} = [\mathbf{A}\mathbf{x}_1 \ \ \mathbf{A}\mathbf{x}_2].$$

If we multiply (6) by $\mathbf{X}^{-1}$ from the left, we obtain (5). Since (5) is a similarity transformation, Theorem 3 implies that $\mathbf{D}$ has the same eigenvalues as $\mathbf{A}$. Equation (5*) follows if we note that

$$\mathbf{D}^2 = \mathbf{DD} = \mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX} = \mathbf{X}^{-1}\mathbf{AAX} = \mathbf{X}^{-1}\mathbf{A}^2\mathbf{X}, \qquad \text{etc.}$$

EXAMPLE 4 **Diagonalization**

Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}.$$

Solution. The characteristic determinant gives the characteristic equation $-\lambda^3 - \lambda^2 + 12\lambda = 0$. The roots (eigenvalues of $\mathbf{A}$) are $\lambda_1 = 3$, $\lambda_2 = -4$, $\lambda_3 = 0$. By the Gauss elimination applied to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1, \lambda_2, \lambda_3$ we find eigenvectors and then $\mathbf{X}^{-1}$ by the Gauss–Jordan elimination (Sec. 7.8, Example 1). The results are

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \qquad \mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Calculating $\mathbf{AX}$ and multiplying by $\mathbf{X}^{-1}$ from the left, we thus obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Quadratic Forms. Transformation to Principal Axes

By definition, a **quadratic form** Q in the components $x_1, \cdots, x_n$ of a vector $\mathbf{x}$ is a sum of n^2 terms, namely,

$$
\textbf{(7)} \qquad
\begin{aligned}
Q = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_jx_k \\
= \quad & a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\
& + a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2.
\end{aligned}
$$

$\mathbf{A} = [a_{jk}]$ is called the **coefficient matrix** of the form. We may assume that $\mathbf{A}$ is ***symmetric,*** because we can take off-diagonal terms together in pairs and write the result as a sum of two equal terms; see the following example.

EXAMPLE 5 Quadratic Form. Symmetric Coefficient Matrix

Let

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2.$$

Here $4 + 6 = 10 = 5 + 5$. From the corresponding ***symmetric*** matrix $\mathbf{C} = [c_{jk}]$, where $c_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$, thus $c_{11} = 3$, $c_{12} = c_{21} = 5$, $c_{22} = 2$, we get the same result; indeed,

$$\mathbf{x}^\mathsf{T}\mathbf{C}\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2. \quad \blacksquare$$

Quadratic forms occur in physics and geometry, for instance, in connection with conic sections (ellipses $x_1^2/a^2 + x_2^2/b^2 = 1$, etc.) and quadratic surfaces (cones, etc.). Their transformation to principal axes is an important practical task related to the diagonalization of matrices, as follows.

By Theorem 2 the *symmetric* coefficient matrix $\mathbf{A}$ of (7) has an orthonormal basis of eigenvectors. Hence if we take these as column vectors, we obtain a matrix $\mathbf{X}$ that is orthogonal, so that $\mathbf{X}^{-1} = \mathbf{X}^\mathsf{T}$. From (5) we thus have $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^\mathsf{T}$. Substitution into (7) gives

$$(8) \qquad Q = \mathbf{x}^\mathsf{T}\mathbf{X}\mathbf{D}\mathbf{X}^\mathsf{T}\mathbf{x}.$$

If we set $\mathbf{X}^\mathsf{T}\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^\mathsf{T} = \mathbf{X}^{-1}$, we get

$$(9) \qquad \mathbf{x} = \mathbf{X}\mathbf{y}.$$

Furthermore, in (8) we have $\mathbf{x}^\mathsf{T}\mathbf{X} = (\mathbf{X}^\mathsf{T}\mathbf{x})^\mathsf{T} = \mathbf{y}^\mathsf{T}$ and $\mathbf{X}^\mathsf{T}\mathbf{x} = \mathbf{y}$, so that Q becomes simply

$$\textbf{(10)} \qquad Q = \mathbf{y}^\mathsf{T}\mathbf{D}\mathbf{y} = \lambda_1y_1^2 + \lambda_2y_2^2 + \cdots + \lambda_ny_n^2.$$

This proves the following basic theorem.

THEOREM 5

Principal Axes Theorem

The substitution (9) *transforms a quadratic form*

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_jx_k \qquad (a_{kj} = a_{jk})$$

to the principal axes form or **canonical form** (10), *where* $\lambda_1, \cdots, \lambda_n$ *are the* (*not necessarily distinct*) *eigenvalues of the* (*symmetric!*) *matrix* **A**, *and* **X** *is an orthogonal matrix with corresponding eigenvectors* $\mathbf{x}_1, \cdots, \mathbf{x}_n$, *respectively, as column vectors.*

EXAMPLE 6

Transformation to Principal Axes. Conic Sections

Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128.$$

Solution. We have $Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This gives the characteristic equation $(17 - \lambda)^2 - 15^2 = 0$. It has the roots $\lambda_1 = 2$, $\lambda_2 = 32$. Hence (10) becomes

$$Q = 2y_1^2 + 32y_2^2.$$

We see that $Q = 128$ represents the ellipse $2y_1^2 + 32y_2^2 = 128$, that is,

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1.$$

If we want to know the direction of the principal axes in the x_1x_2-coordinates, we have to determine normalized eigenvectors from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1 = 2$ and $\lambda = \lambda_2 = 32$ and then use (9). We get

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

hence

$$\mathbf{x} = \mathbf{X}\mathbf{y} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \begin{aligned} x_1 &= y_1/\sqrt{2} - y_2/\sqrt{2} \\ x_2 &= y_1/\sqrt{2} + y_2/\sqrt{2}. \end{aligned}$$

This is a 45° rotation. Our results agree with those in Sec. 8.2, Example 1, except for the notations. See also Fig. 158 in that example. ■

PROBLEM SET 8.4

1–9 DIAGONALIZATION OF MATRICES

Find an eigenbasis (a basis of eigenvectors) and diagonalize. (Show the details.)

1. $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 16 \\ 4 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 2 \\ -5 & -4 \end{bmatrix}$

5. $\begin{bmatrix} 1.0 & 6.0 \\ 1.5 & 1.0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 7 \\ 6 & -9 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -6 & -6 & 10 \\ -5 & -5 & 5 \\ -9 & -9 & 13 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 10 & -15 \\ -18 & 39 & 9 \\ -24 & 40 & -15 \end{bmatrix}$

10. **(Orthonormal basis)** Illustrate Theorem 2 with further examples.

11. **(No basis)** Find further 2×2 and 3×3 matrices without eigenbases.

12. **PROJECT. Similarity of Matrices.** Similarity is basic, for instance in designing numeric methods.

(a) Trace. By definition, the **trace** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the sum of the diagonal entries,

$$\text{trace } \mathbf{A} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Show that the trace equals the sum of the eigenvalues, each counted as often as its algebraic multiplicity indicates. Illustrate this with the matrices in Probs. 1, 3, 5, 7, 9.

(b) Trace of product. Let $\mathbf{B} = [b_{jk}]$ be $n \times n$. Show that similar matrices have equal traces, by first proving

$$\text{trace } \mathbf{AB} = \sum_{i=1}^{n} \sum_{l=1}^{n} a_{il} b_{li} = \text{trace } \mathbf{BA}.$$

(c) Find a relationship between $\hat{\mathbf{A}}$ in (4) and $\tilde{\mathbf{A}} = \mathbf{PAP}^{-1}$.

(d) Diagonalization. What can you do in (5) if you want to change the order of the eigenvalues in $\mathbf{D}$, for instance, interchange $d_{11} = \lambda_1$ and $d_{22} = \lambda_2$?

13–18 SIMILAR MATRICES HAVE EQUAL SPECTRA

Verify this for $\mathbf{A}$ and $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$. Find eigenvectors $\mathbf{y}$ of $\hat{\mathbf{A}}$. Show that $\mathbf{x} = \mathbf{Py}$ are eigenvectors of $\mathbf{A}$. (Show the details of your work.)

13. $\mathbf{A} = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

14. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

15. $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$

16. $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 7 & -2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 8 & 2 \\ 1 & 4 \end{bmatrix}$

17. $\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 4 & 0 & 6 \\ 0 & 2 & 0 \\ 6 & 0 & 10 \end{bmatrix}$

18. $\mathbf{A} = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19–28 TRANSFORMATION TO PRINCIPAL AXES. CONIC SECTIONS

What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes. Express $\mathbf{x}^\mathsf{T} = [x_1 \quad x_2]$ in terms of the new coordinate vector $\mathbf{y}^\mathsf{T} = [y_1 \quad y_2]$, as in Example 6.

19. $x_1^2 + 24x_1x_2 - 6x_2^2 = 5$

20. $3x_1^2 + 4\sqrt{3}x_1x_2 + 7x_2^2 = 9$

21. $3x_1^2 - 8x_1x_2 - 3x_2^2 = 0$

22. $6x_1^2 + 16x_1x_2 - 6x_2^2 = 20$

23. $4x_1^2 + 2\sqrt{3}x_1x_2 + 2x_2^2 = 10$

24. $7x_1^2 - 24x_1x_2 = 144$

25. $x_1^2 - 12x_1x_2 + x_2^2 = 35$

26. $3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$

27. $12x_1^2 + 32x_1x_2 + 12x_2^2 = 112$

28. $6.5x_1^2 + 5.0x_1x_2 + 6.5x_2^2 = 36$

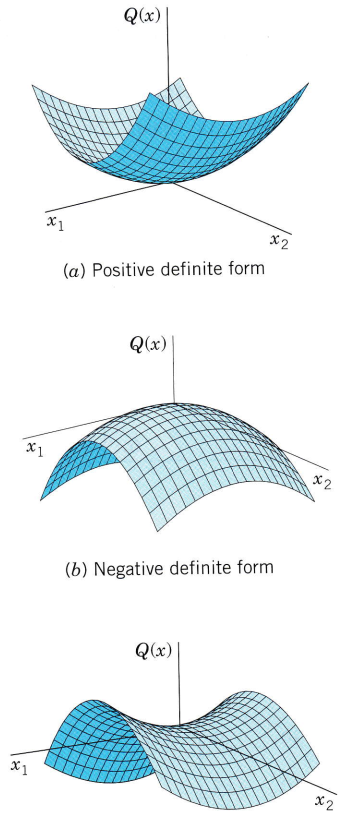
29. **(Definiteness)** A quadratic form $Q(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix $\mathbf{A}$ are called **(a) positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, **(b) negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$, **(c) indefinite** if $Q(\mathbf{x})$ takes both positive and negative values. (See Fig. 160.) [$Q(\mathbf{x})$ and $\mathbf{A}$ are called *positive semidefinite* (*negative semidefinite*) if $Q(\mathbf{x}) \geqq 0$ ($Q(\mathbf{x}) \leqq 0$) for all $\mathbf{x}$.] A necessary and sufficient condition for positive definiteness is that all the **"principal minors"** are positive (see Ref. [B3], vol. 1, p. 306), that is,

$$a_{11} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0, \qquad \cdots, \qquad \det \mathbf{A} > 0.$$

Show that the form in Prob. 23 is positive definite, whereas that in Prob. 19 is indefinite.

30. **(Definiteness)** Show that necessary and sufficient for (a), (b), (c) in Prob. 29 is that the eigenvalues of $\mathbf{A}$ are (a) all positive, (b) all negative, (c) both positive and negative. *Hint.* Use Theorem 5.



(*a*) Positive definite form

(*b*) Negative definite form

(*c*) Indefinite form

Fig. 160. Quadratic forms in two variables

8.5 Complex Matrices and Forms. *Optional*

The three classes of real matrices in Sec. 8.3 have complex counterparts that are of practical interest in certain applications, mainly because of their spectra (see Theorem 1 in this section), for instance, in quantum mechanics. To define these classes, we need the following standard

> **Notations**
>
> $\overline{\mathbf{A}} = [\bar{a}_{jk}]$ is obtained from $\mathbf{A} = [a_{jk}]$ by replacing each entry $a_{jk} = \alpha + i\beta$ (α, β real) with its complex conjugate $\bar{a}_{jk} = \alpha - i\beta$. Also, $\overline{\mathbf{A}}^\mathsf{T} = [\bar{a}_{kj}]$ is the transpose of $\overline{\mathbf{A}}$, hence the conjugate transpose of $\mathbf{A}$.

EXAMPLE 1 Notations

If $\mathbf{A} = \begin{bmatrix} 3+4i & 1-i \\ 6 & 2-5i \end{bmatrix}$, then $\overline{\mathbf{A}} = \begin{bmatrix} 3-4i & 1+i \\ 6 & 2+5i \end{bmatrix}$ and $\overline{\mathbf{A}}^\mathsf{T} = \begin{bmatrix} 3-4i & 6 \\ 1+i & 2+5i \end{bmatrix}$. ■

DEFINITION

Hermitian, Skew-Hermitian, and Unitary Matrices

A square matrix $\mathbf{A} = [a_{kj}]$ is called

Hermitian	if	$\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A},$	that is,	$\bar{a}_{kj} = a_{jk}$
skew-Hermitian	if	$\overline{\mathbf{A}}^{\mathsf{T}} = -\mathbf{A},$	that is,	$\bar{a}_{kj} = -a_{jk}$
unitary	if	$\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}.$		

The first two classes are named after Hermite (see footnote 13 in Problem Set 5.8).

From the definitions we see the following. If $\mathbf{A}$ is Hermitian, the entries on the main diagonal must satisfy $\bar{a}_{jj} = a_{jj}$; that is, they are real. Similarly, if $\mathbf{A}$ is skew-Hermitian, then $\bar{a}_{jj} = -a_{jj}$. If we set $a_{jj} = \alpha + i\beta$, this becomes $\alpha - i\beta = -(\alpha + i\beta)$. Hence $\alpha = 0$, so that a_{jj} must be pure imaginary or 0.

EXAMPLE 2 **Hermitian, Skew-Hermitian, and Unitary Matrices**

$$\mathbf{A} = \begin{bmatrix} 4 & 1 - 3i \\ 1 + 3i & 7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3i & 2 + i \\ -2 + i & -i \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$$

are Hermitian, skew-Hermitian, and unitary matrices, respectively, as you may verify by using the definitions. ■

If a Hermitian matrix is real, then $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} = \mathbf{A}.$ Hence a real Hermitian matrix is a symmetric matrix (Sec. 8.3.).

Similarly, if a skew-Hermitian matrix is real, then $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} = -\mathbf{A}.$ Hence a real skew-Hermitian matrix is a skew-symmetric matrix.

Finally, if a unitary matrix is real, then $\overline{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}$. Hence a real unitary matrix is an orthogonal matrix.

This shows that *Hermitian, skew-Hermitian, and unitary matrices generalize symmetric, skew-symmetric, and orthogonal matrices, respectively.*

Eigenvalues

It is quite remarkable that the matrices under consideration have spectra (sets of eigenvalues; see Sec. 8.1) that can be characterized in a general way as follows (see Fig. 161).

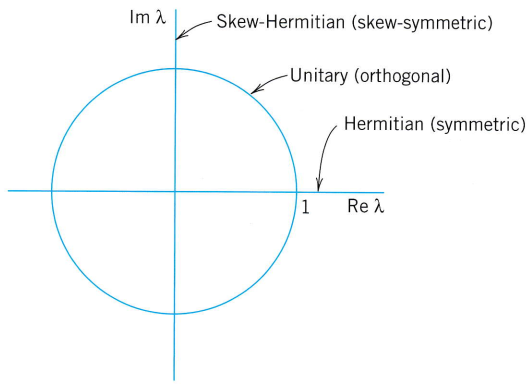


Fig. 161. Location of the eigenvalues of Hermitian, skew-Hermitian, and unitary matrices in the complex λ-plane

THEOREM 1

Eigenvalues

(a) *The eigenvalues of a Hermitian matrix (and thus of a symmetric matrix) are real.*

(b) *The eigenvalues of a skew-Hermitian matrix (and thus of a skew-symmetric matrix) are pure imaginary or zero.*

(c) *The eigenvalues of a unitary matrix (and thus of an orthogonal matrix) have absolute value* 1.

EXAMPLE 3 **Illustration of Theorem 1**

For the matrices in Example 2 we find by direct calculation

Matrix		Characteristic Equation	Eigenvalues
A	Hermitian	$\lambda^2 - 11\lambda + 18 = 0$	$9, \quad 2$
B	Skew-Hermitian	$\lambda^2 - 2i\lambda + 8 = 0$	$4i, \quad -2i$
C	Unitary	$\lambda^2 - i\lambda - 1 = 0$	$\frac{1}{2}\sqrt{3} + \frac{1}{2}i, \quad -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

and $|\pm\frac{1}{2}\sqrt{3} + \frac{1}{2}i|^2 = \frac{3}{4} + \frac{1}{4} = 1.$ ■

PROOF We prove Theorem 1. Let λ be an eigenvalue and $\mathbf{x}$ an eigenvector of $\mathbf{A}$. Multiply $\mathbf{Ax} = \lambda\mathbf{x}$ from the left by $\bar{\mathbf{x}}^{\mathsf{T}}$, thus $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax} = \lambda\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}$, and divide by $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x} = \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n = |x_1|^2 + \cdots + |x_n|^2$, which is real and not 0 because $\mathbf{x} \neq \mathbf{0}$. This gives

$$\text{(1)} \qquad \lambda = \frac{\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax}}{\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}} \,.$$

(a) If $\mathbf{A}$ is Hermitian, $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}$ or $\mathbf{A}^{\mathsf{T}} = \bar{\mathbf{A}}$ and we show that then the numerator in (1) is real, which makes λ real. $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax}$ is a scalar; hence taking the transpose has no effect. Thus

$$\text{(2)} \qquad \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax} = (\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax})^{\mathsf{T}} = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\bar{\mathbf{x}} = \mathbf{x}^{\mathsf{T}}\bar{\mathbf{A}}\bar{\mathbf{x}} = (\overline{\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax}}).$$

Hence, $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax}$ equals its complex conjugate, so that it must be real. ($a + ib = a - ib$ implies $b = 0$.)

(b) If $\mathbf{A}$ is skew-Hermitian, $\mathbf{A}^{\mathsf{T}} = -\bar{\mathbf{A}}$ and instead of (2) we obtain

$$\text{(3)} \qquad \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax} = -(\overline{\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax}})$$

so that $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{Ax}$ equals minus its complex conjugate and is pure imaginary or 0. ($a + ib = -(a - ib)$ implies $a = 0$.)

(c) Let $\mathbf{A}$ be unitary. We take $\mathbf{Ax} = \lambda\mathbf{x}$ and its conjugate transpose

$$(\overline{\mathbf{Ax}})^{\mathsf{T}} = (\bar{\lambda}\bar{\mathbf{x}})^{\mathsf{T}} = \bar{\lambda}\bar{\mathbf{x}}^{\mathsf{T}}$$

and multiply the two left sides and the two right sides,

$$(\overline{\mathbf{Ax}})^{\mathsf{T}}\mathbf{Ax} = \bar{\lambda}\lambda\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x} = |\lambda|^2\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}.$$

But $\mathbf{A}$ is unitary, $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}$, so that on the left we obtain

$$(\overline{\mathbf{A}\mathbf{x}})^{\mathsf{T}}\mathbf{A}\mathbf{x} = \bar{\mathbf{x}}^{\mathsf{T}}\bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{I}\,\mathbf{x} = \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}.$$

Together, $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x} = |\lambda|^2\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}$. We now divide by $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}$ ($\neq 0$) to get $|\lambda|^2 = 1$. Hence $|\lambda| = 1$.
This proves Theorem 1 as well as Theorems 1 and 5 in Sec. 8.3. ■

Key properties of orthogonal matrices (invariance of the inner product, orthonormality of rows and columns; see Sec. 8.3) generalize to unitary matrices in a remarkable way.

To see this, instead of R^n we now use the **complex vector space** C^n of all complex vectors with n complex numbers as components, and complex numbers as scalars. For such complex vectors the **inner product** is defined by (note the overbar for the complex conjugate)

$$\mathbf{a} \bullet \mathbf{b} = \bar{\mathbf{a}}^{\mathsf{T}}\mathbf{b}. \tag{4}$$

The **length** or **norm** of such a complex vector is a *real* number defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{\bar{\mathbf{a}}^{\mathsf{T}}\mathbf{a}} = \sqrt{\bar{a}_1 a_1 + \cdots + \bar{a}_n a_n} = \sqrt{|a_1|^2 + \cdots + |a_n|^2}. \tag{5}$$

THEOREM 2

Invariance of Inner Product

A **unitary transformation,** *that is,* $\mathbf{y} = \mathbf{A}\mathbf{x}$ *with a unitary matrix* $\mathbf{A}$, *preserves the value of the inner product* (4), *hence also the norm* (5).

PROOF The proof is the same as that of Theorem 2 in Sec. 8.3, which the theorem generalizes. In the analog of (9), Sec. 8.3, we now have bars,

$$\mathbf{u} \bullet \mathbf{v} = \bar{\mathbf{u}}^{\mathsf{T}}\mathbf{v} = (\overline{\mathbf{A}\mathbf{a}})^{\mathsf{T}}\mathbf{A}\mathbf{b} = \bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A}\mathbf{b} = \bar{\mathbf{a}}^{\mathsf{T}}\mathbf{I}\,\mathbf{b} = \bar{\mathbf{a}}^{\mathsf{T}}\mathbf{b} = \mathbf{a} \bullet \mathbf{b}.$$ ■

The complex analog of an orthonormal systems of real vectors (see Sec. 8.3) is defined as follows.

DEFINITION

Unitary System

A *unitary system* is a set of complex vectors satisfying the relationships

$$\mathbf{a}_j \bullet \mathbf{a}_k = \bar{\mathbf{a}}_j^{\mathsf{T}}\mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases} \tag{6}$$

Theorem 3 in Sec. 8.3 extends to complex as follows.

THEOREM 3

Unitary Systems of Column and Row Vectors

A complex square matrix is unitary if and only if its column vectors (and also its row vectors) form a unitary system.

PROOF The proof is the same as that of Theorem 3 in Sec. 8.3, except for the bars required in $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}$ and in (4) and (6) of the present section. ■

THEOREM 4

Determinant of a Unitary Matrix

Let $\mathbf{A}$ *be a unitary matrix. Then its determinant has absolute value one, that is,* $|\det \mathbf{A}| = 1$.

PROOF Similarly as in Sec. 8.3 we obtain

$$1 = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A}\bar{\mathbf{A}}^{\mathsf{T}}) = \det \mathbf{A} \det \bar{\mathbf{A}}^{\mathsf{T}} = \det \mathbf{A} \det \bar{\mathbf{A}}$$
$$= \det \mathbf{A}\, \overline{\det \mathbf{A}} = |\det \mathbf{A}|^2.$$

Hence $|\det \mathbf{A}| = 1$ (where $\det \mathbf{A}$ may now be complex). ■

EXAMPLE 4 **Unitary Matrix Illustrating Theorems 1c and 2–4**

For the vectors $\mathbf{a}^{\mathsf{T}} = [2 \quad -i]$ and $\mathbf{b}^{\mathsf{T}} = [1 + i \quad 4i]$ we get $\bar{\mathbf{a}}^{\mathsf{T}} = [2 \quad i]^{\mathsf{T}}$ and $\bar{\mathbf{a}}^{\mathsf{T}}\mathbf{b} = 2(1 + i) - 4 = -2 + 2i$ and with

$$\mathbf{A} = \begin{bmatrix} 0.8i & 0.6 \\ 0.6 & 0.8i \end{bmatrix} \quad \text{also} \quad \mathbf{Aa} = \begin{bmatrix} i \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Ab} = \begin{bmatrix} -0.8 + 3.2i \\ -2.6 + 0.6i \end{bmatrix},$$

as one can readily verify. This gives $(\overline{\mathbf{Aa}})^{\mathsf{T}}\mathbf{Ab} = -2 + 2i$, illustrating Theorem 2. The matrix is unitary. Its columns form a unitary system,

$$\bar{\mathbf{a}}_1^{\mathsf{T}}\mathbf{a}_1 = -0.8i \cdot 0.8i + 0.6^2 = 1, \qquad \bar{\mathbf{a}}_1^{\mathsf{T}}\mathbf{a}_2 = -0.8i \cdot 0.6 + 0.6 \cdot 0.8i = 0,$$
$$\bar{\mathbf{a}}_2^{\mathsf{T}}\mathbf{a}_2 = 0.6^2 + (-0.8i)0.8i = 1$$

and so do its rows. Also, $\det \mathbf{A} = -1$. The eigenvalues are $0.6 + 0.8i$ and $-0.6 + 0.8i$, with eigenvectors $[1 \quad 1]^{\mathsf{T}}$ and $[1 \quad -1]^{\mathsf{T}}$, respectively. ■

Theorem 2 in Sec. 8.4 on the existence of an eigenbasis extends to complex matrices as follows.

THEOREM 5

Basis of Eigenvectors

A Hermitian, skew-Hermitian, or unitary matrix has a basis of eigenvectors for C^n *that is a unitary system.*

For a proof see Ref. [B3], vol. 1, pp. 270–272 and p. 244 (Definition 2).

EXAMPLE 5 **Unitary Eigenbases**

The matrices **A, B, C** in Example 2 have the following unitary systems of eigenvectors, as you should verify.

$$\mathbf{A}: \quad \frac{1}{\sqrt{35}}[1 - 3i \quad 5]^{\mathsf{T}} \quad (\lambda = 9), \qquad \frac{1}{\sqrt{14}}[1 - 3i \quad -2]^{\mathsf{T}} \quad (\lambda = 2)$$

$$\mathbf{B}: \quad \frac{1}{\sqrt{30}}[1 - 2i \quad -5]^{\mathsf{T}} \quad (\lambda = -2i), \qquad \frac{1}{\sqrt{30}}[5 \quad 1 + 2i]^{\mathsf{T}} \quad (\lambda = 4i)$$

$$\mathbf{C}: \quad \frac{1}{\sqrt{2}}[1 \quad 1]^{\mathsf{T}} \quad (\lambda = \tfrac{1}{2}(i + \sqrt{3})), \qquad \frac{1}{\sqrt{2}}[1 \quad -1]^{\mathsf{T}} \quad (\lambda = \tfrac{1}{2}(i - \sqrt{3})).$$

■

Hermitian and Skew-Hermitian Forms

The concept of a quadratic form (Sec. 8.4) can be extended to complex. We call the numerator $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x}$ in (1) a **form** in the components $x_1, \cdots, x_n$ of **x**, which may now be complex. This form is again a sum of n^2 terms

$$
\begin{aligned}
\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x} &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\bar{x}_j x_k \\
&= \quad a_{11}\bar{x}_1 x_1 + \cdots + a_{1n}\bar{x}_1 x_n \\
&\quad + a_{21}\bar{x}_2 x_1 + \cdots + a_{2n}\bar{x}_2 x_n \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + a_{n1}\bar{x}_n x_1 + \cdots + a_{nn}\bar{x}_n x_n.
\end{aligned} \tag{7}
$$

A is called its **coefficient matrix.** The form is called a **Hermitian** or **skew-Hermitian form** if **A** is Hermitian or skew-Hermitian, respectively. *The value of a Hermitian form is real, and that of a skew-Hermitian form is pure imaginary or zero.* This can be seen directly from (2) and (3) and accounts for the importance of these forms in physics. Note that (2) and (3) are valid for any vectors because in the proof of (2) and (3) we did not use that **x** is an eigenvector but only that $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}$ is real and not 0.

EXAMPLE 6 Hermitian Form

For **A** in Example 2 and, say, $\mathbf{x} = [1 + i \quad 5i]^{\mathsf{T}}$ we get

$$\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x} = [1 - i \quad -5i]\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}\begin{bmatrix} 1+i \\ 5i \end{bmatrix} = [1 - i \quad -5i]\begin{bmatrix} 4(1+i) + (1-3i)\cdot 5i \\ (1+3i)(1+i) + 7\cdot 5i \end{bmatrix} = 223. \quad \blacksquare$$

Clearly, if **A** and **x** in (4) are real, then (7) reduces to a quadratic form, as discussed in the last section.

PROBLEM SET 8.5

1. **(Verification)** Verify the statements in Examples 2 and 3.
2. **(Product)** Show $(\overline{\mathbf{BA}})^{\mathsf{T}} = -\mathbf{AB}$ for **A** and **B** in Example 2. For any $n \times n$ Hermitian **A** and skew-Hermitian **B**.
3. Show that $(\overline{\mathbf{ABC}})^{\mathsf{T}} = -\mathbf{C}^{-1}\mathbf{BA}$ for any $n \times n$ Hermitian **A**, skew-Hermitian **B**, and unitary **C**.
4. **(Eigenvectors)** Find eigenvectors of **A, B, C** in Examples 2 and 3.

5–11 EIGENVALUES AND EIGENVECTORS

Are the matrices in Probs. 5–11 Hermitian? Skew-Hermitian? Unitary? Find their eigenvalues (thereby verifying Theorem 1) and eigenvectors.

5. $\begin{bmatrix} 4 & i \\ -i & 2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

8. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

9. $\begin{bmatrix} 5i & 0 & 0 \\ 0 & 0 & 5i \\ 0 & 5i & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}$

12. PROJECT. Complex Matrices

(a) Decomposition. Show that any square matrix may be written as the sum of a Hermitian and a skew-Hermitian matrix. Give examples.

(b) Normal matrix. This important concept denotes a matrix that commutes with its conjugate transpose, $\mathbf{A}\bar{\mathbf{A}}^{\mathsf{T}} = \bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A}$. Prove that Hermitian, skew-Hermitian, and unitary matrices are normal. Give corresponding examples of your own.

(c) Normality criterion. Prove that $\mathbf{A}$ is normal if and only if the Hermitian and skew-Hermitian matrices in (a) commute.

(d) Find a simple matrix that is not normal. Find a normal matrix that is not Hermitian, skew-Hermitian, or unitary.

(e) Unitary matrices. Prove that the product of two unitary $n \times n$ matrices and the inverse of a unitary matrix are unitary. Give examples.

(f) Powers of unitary matrices in applications may sometimes be very simple. Show that $\mathbf{C}^{12} = \mathbf{I}$ in Example 2. Find further examples.

13–15 COMPLEX FORMS

Is the given matrix (call it $\mathbf{A}$) Hermitian or skew-Hermitian? Find $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x}$. (Show all the details.) a, b, c, k are real.

13. $\begin{bmatrix} 0 & -3i \\ -3i & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4+i \\ 3-i \end{bmatrix}$

14. $\begin{bmatrix} a & b+ic \\ b-ic & k \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 1+i \\ 1-i & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} i \\ 2i \end{bmatrix}$

16. (Pauli spin matrices) Find the eigenvalues and eigenvectors of the so-called *Pauli spin matrices* and show that $\mathbf{S}_x\mathbf{S}_y = i\mathbf{S}_z$, $\mathbf{S}_y\mathbf{S}_x = -i\mathbf{S}_z$, $\mathbf{S}_x^2 = \mathbf{S}_y^2 = \mathbf{S}_z^2 = \mathbf{I}$, where

$$\mathbf{S}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{S}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\mathbf{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

CHAPTER 8 REVIEW QUESTIONS AND PROBLEMS

1. In solving an eigenvalue problem, what is given and what is sought?
2. Do there exist square matrices without eigenvalues? Eigenvectors corresponding to more than one eigenvalue of a given matrix?
3. What is the defect? Why is it important? Give examples.
4. Can a complex matrix have real eigenvalues? Real eigenvectors? Give reasons.
5. What is diagonalization of a matrix? Transformation of a form to principal axes?
6. What is an eigenbasis? When does it exist? Why is it important?
7. Does a 3×3 matrix always have a real eigenvalue?
8. Give a few typical applications in which eigenvalue problems occur.

9–14 DIAGONALIZATION

Find an eigenbasis and diagonalize. (Show the details.)

9. $\begin{bmatrix} 101 & 72 \\ -144 & -103 \end{bmatrix}$

10. $\begin{bmatrix} 14.4 & -11.2 \\ -11.2 & 102.6 \end{bmatrix}$

11. $\begin{bmatrix} -14 & 10 \\ -10 & 11 \end{bmatrix}$

12. $\begin{bmatrix} 15 & 4 & -4 \\ 6 & 10 & 8 \\ -12 & -2 & -7 \end{bmatrix}, \quad \lambda = 18$

13. $\begin{bmatrix} 5 & \frac{8}{3} & -\frac{2}{3} \\ 2 & \frac{2}{3} & \frac{4}{3} \\ -4 & -\frac{4}{3} & -\frac{8}{3} \end{bmatrix}$

14. $\begin{bmatrix} -6 & 11 & 3 \\ 4 & 1 & 3 \\ -4 & 10 & 8 \end{bmatrix}, \quad \lambda = 2$

15–17 SIMILARITY

Verify that $\mathbf{A}$ and $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$ have the same spectrum. Here, $\mathbf{A}, \mathbf{P}$ are:

15. $\begin{bmatrix} 3.8 & 2.4 \\ 2.4 & 0.2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} -22 & 20 & 10 \\ -4 & 20 & -8 \\ 28 & -14 & 29 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 2 & 8 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 2 & -2 \\ 3 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$

Transformation to Canonical Form. Reduce the quadratic form to principal axes.

18. $11.56x_1^2 + 20.16x_1x_2 + 17.44x_2^2 = 100$

19. $1.09x_1^2 - 0.06x_1x_2 + 1.01x_2^2 = 1$

20. $14x_1^2 + 24x_1x_2 - 4x_2^2 = 20$

SUMMARY OF CHAPTER 8

Linear Algebra: Matrix Eigenvalue Problems

The practical importance of matrix eigenvalue problems can hardly be overrated. The problems are defined by the vector equation

$$\mathbf{Ax} = \lambda\mathbf{x}. \tag{1}$$

$\mathbf{A}$ is a given square matrix. All matrices in this chapter are ***square.*** λ is a scalar. To *solve* the problem (1) means to determine values of λ, called **eigenvalues** (or **characteristic values**) of $\mathbf{A}$, such that (1) has a nontrivial solution $\mathbf{x}$ (that is, $\mathbf{x} \neq \mathbf{0}$), called an **eigenvector** of $\mathbf{A}$ corresponding to that λ. An $n \times n$ matrix has at least one and at most n numerically different eigenvalues. These are the solutions of the **characteristic equation** (Sec. 8.1)

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{2}$$

$D(\lambda)$ is called the **characteristic determinant** of $\mathbf{A}$. By expanding it we get the **characteristic polynomial** of $\mathbf{A}$, which is of degree n in λ. Some typical applications are shown in Sec. 8.2.

Section 8.3 is devoted to eigenvalue problems for **symmetric** ($\mathbf{A}^\mathsf{T} = \mathbf{A}$), **skew-symmetric** ($\mathbf{A}^\mathsf{T} = -\mathbf{A}$), and **orthogonal matrices** ($\mathbf{A}^\mathsf{T} = \mathbf{A}^{-1}$). Section 8.4 concerns the diagonalization of matrices and the transformation of quadratic forms to principal axes and its relation to eigenvalues.

Section 8.5 extends Sec. 8.3 to the complex analogs of those real matrices, called **Hermitian** ($\bar{\mathbf{A}}^\mathsf{T} = \mathbf{A}$), **skew-Hermitian** ($\bar{\mathbf{A}}^\mathsf{T} = -\mathbf{A}$), and **unitary matrices** ($\bar{\mathbf{A}}^\mathsf{T} = \mathbf{A}^{-1}$). All the eigenvalues of a Hermitian matrix (and a symmetric one) are real. For a skew-Hermitian (and a skew-symmetric) matrix they are pure imaginary or zero. For a unitary (and an orthogonal) matrix they have absolute value 1.