Collection of exercises in ELEMENTARY GEOMETRY for Programme Primary school teachers

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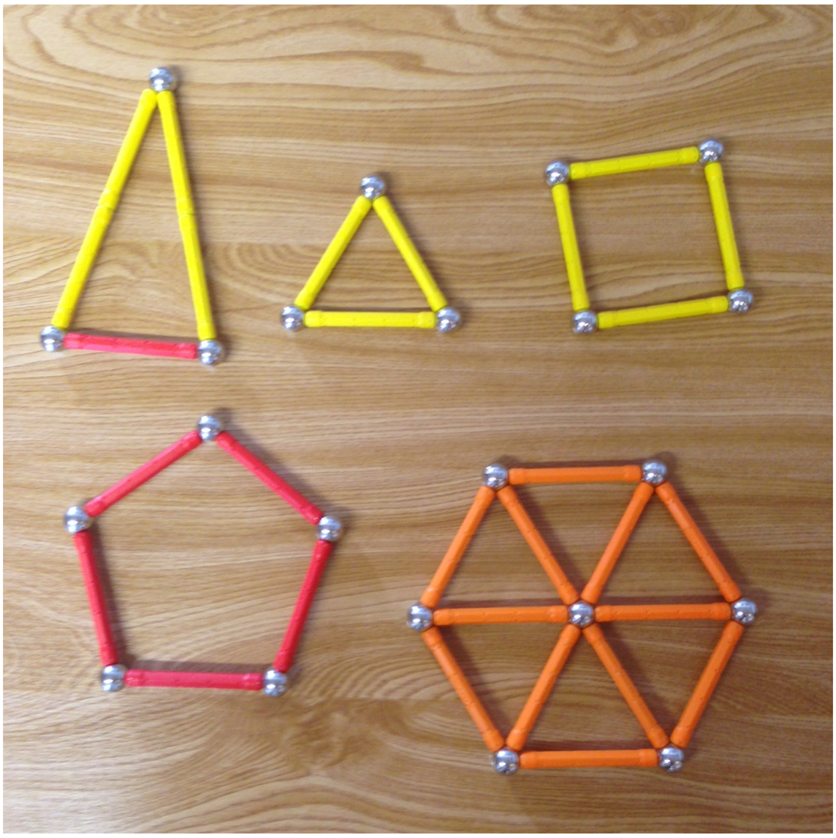

Geometry has two treasures: a pythagorean theorem and a golden section. The first one is worth gold, the second one is more precious stone.

Johannes Kepler (1571 - 1630)[10]

Introduction

This collection of elementary geometry problems was developed as a supporting material to the geometry textbooks for the future elementary school teachers. These texts namely contain only a limited number of exercises and no solved tasks. This booklet offers the students a number of solved tasks as well as another set of exercises. At the same time, it follows the current trend of inter-subject connections and in the provided tasks and examples shows how geometry is related to the other subjects as well as to the world around us.

Many tasks work with the magnetic kit Geomag. If you do not have it, these tasks can be demonstrated using skewers and balls of modeling. For the creation of illustrations, the GeoGebra software was used. It is therefore easy to use the GeoGebra tutorial software directly in the classroom or on a standalone task. There are direct references to selected dynamic applets and stepped constructions for specific constructions.

This text was developed with the support of the project MUNI / FR / 1193/2018, Innovation of four subjects Geometry for Teachers of the 1st level of the elementary school with Geomag kit and Geogebra educational software at the Faculty of Education, Masaryk University in Brno.

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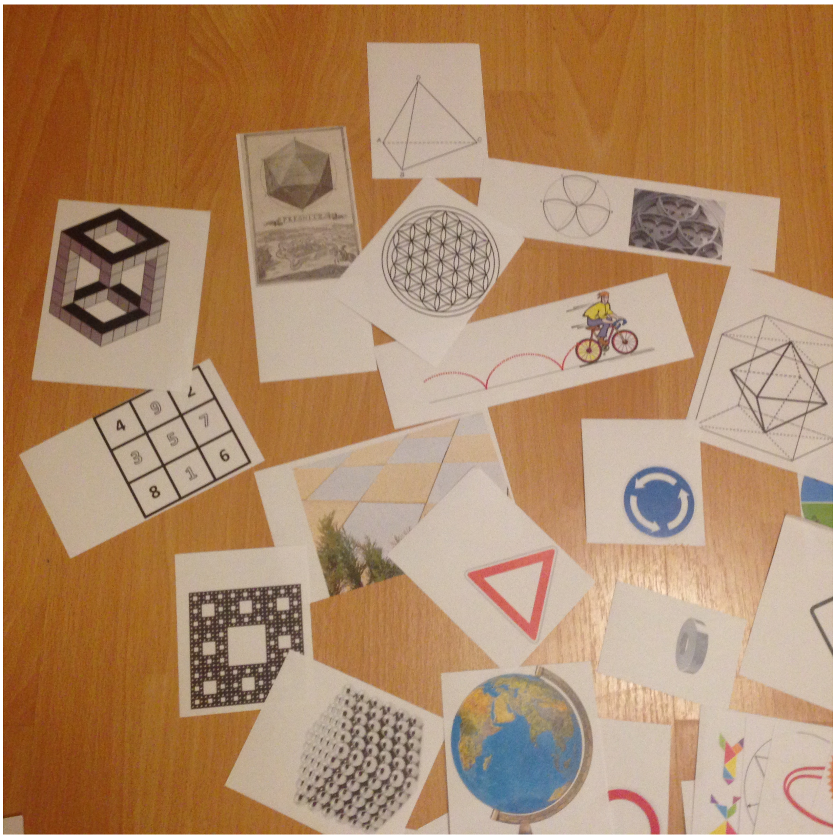
1 Historical development of geometry

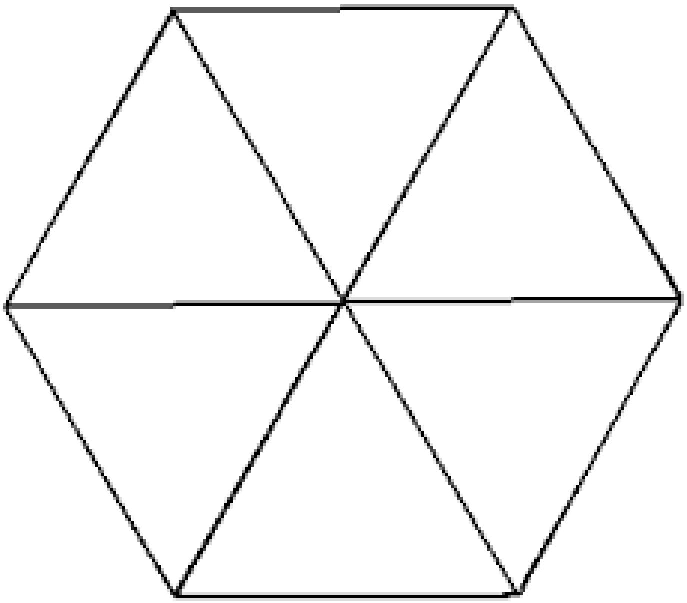
Exercises 1.1. Attachment at the end of the text contains a set of pictures. Sort these images into three groups by assigning them to one of the three basic geometric figures: circle, square, equilateral triangle.



Discuss the pictures in the groups. Why did you assign these pictures to the groups belonging to a circle, square or equilateral triangle. Are there images that could be assigned to two or even to all three groups?

An example of such a discussion:

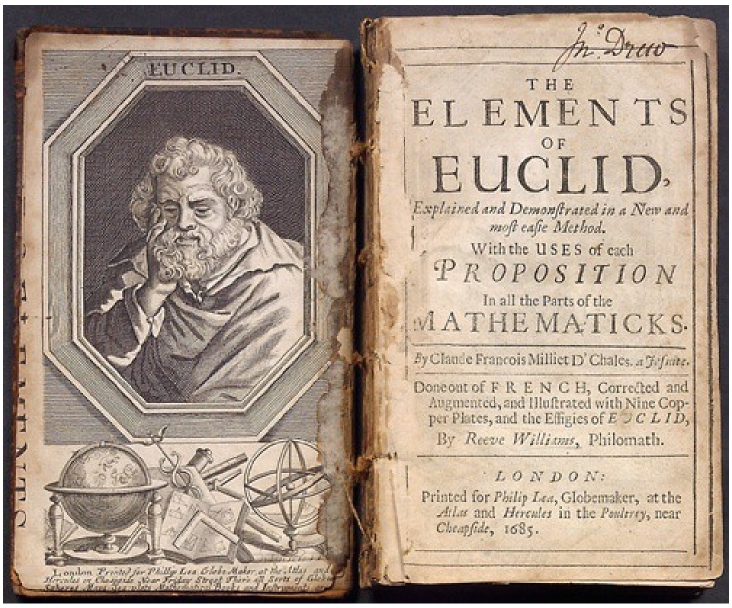
This picture can be seen as a regular hexagon, so it can be assigned to an equilateral triangle, since a regular hexagon consists of six equal equilateral triangles. A regular hexagon is a regular polygon, ie it can be copied and inscribed a circle, so we can assign it to a circle. This image can also be seen as a wireframe cube. If appropriate, explain this view to classmates who can not see it.



A regular hexagon or cube?

Exercises 1.2. How did geometry begin to form in the distant past? (Formulate the answer in several sentences.)

Exercises 1.3. What do you know about the books we call Euclid's *Elements*?



Exercises 1.4. Correctly associate branches of geometry with the names of important mathematicians, who worked in them:

- René Descartes (1596 – 1650),
- Johann Carl Friedrich Gauss (1777 – 1855),
- Georg Friedrich Bernhard Riemann (1826 – 1866),
- David Hilbert (1862 – 1943).

a) German mathematician and physicist. He was interested in geometry, mathematical analysis, number theory, astronomy, electrostatics, geodesy and optics. He strongly influenced most of these fields of knowledge. He stood at the birth of non-Euclidean geometry.

b) His work La Géométrie is often considered the beginning of analytic geometry as a science.

c) German mathematician, who in his work *Foundations of Geometry* constructed discipline currently called Euclidean geometry, he created the system of axioms of Euclidean geometry.

d) German mathematician, who contributed significantly to the development of mathematical analysis and differential geometry. Algebraic geometry and complex surface theory were also developed on the basic of his ideas, which became the core of differential geometry on manifolds and topology.

Solution: René Descartes (b), Johann Carl Friedrich Gauss (a), Georg Friedrich Bernhard Riemann (d), David Hilbert (c).

Exercises 1.5. Explain the difference between an axiom and a mathematical theorem. Give an example of an axiom and a mathematical theorem.

2 Basic geometric formations and their properties

Example 2.1. Which geometric formations can arise as the intersection of two half-lines that are lie on the same line? Show and describe.

Solution: Point, line segment, line, half line.

Example 2.2. Investigate all possible relative positions of three different lines lying in one plane. Show and describe.

Řešení: Let's denote the lines a, b, c. Then the following situations can occur:

a) all lines are mutually parallel, ie. $a \cap b = \emptyset \wedge b \cap c = \emptyset$,

b) two lines are parallel and the third line is parallel to them, e.g.

$$a \cap b = \emptyset \wedge a \cap c = X \wedge B \cap C = Y$$

c) all lines are mutually parallel and pass through a single common point, $a \cap b \cap C = P$,

d) all lines are mutually parallel and intersect at different points.

Exercises 2.3. Draw a line AB. On the line AB mark:

a) C, so that A is between C and B,

b) D, so that B is between A and D,

c) a point P that does not lie on the AB line but lies on the AD line.

Exercises 2.4. Draw line KL. Select D between KL, and mark:

a) R, so that K is between R and L,

b) S, so that L is between K and S,

c) T, so that S is between L, T.

Decide which statement is true:

1) $S \in \mapsto KL$,

2) $\mapsto RS \cap \mapsto KL = KL$,

3) $\mapsto RD \cap ST = \emptyset$,

4) $R \in \leftrightarrow KL$,

Exercises 2.5. There is a line p and a point A that does not lie on it. Draw:

a) a point M that belongs to the $\mapsto pA$,

b) a point P that lies in both halves defined by p,

c) a point N that lies in the half-plane opposite to $\mapsto pA$.

Exercises 2.6. Three different points A, B, C are given.

a) How many line segments, half-lines, and lines are determined by these points? How do these numbers depend on the position of the points given?

b) Which point sets can be the intersection of two of these line segments (half-lines, lines)?

Show and discuss.

Exercises 2.7. Let point R lies between P, Q. From half-lines PR, PQ, RP, RQ, QR, QP choose pairs of half-lines that: a) coincide, b) are opposite, c) one is part of the other, d) their intersection is a line segment.

Exercises 2.8. Determine what shapes may arise as an intersection of:

a) a line segment and a half-plane,

b) a half-line and a half-plane,

c) a line and a half-plane.

For all cases, consider the situation in a single plane. Show and describe.

Exercises 2.9. There are n straight lines in the plane, of which no two intersect and no three meet at the same point. How many significant intersections of these lines are there?

Example 2.10. How many different lines are determined by n points that lie in one plane and no three lying on one straight line?

Solution: For a single point, the task is meaningless. Let's outline the situation for some finite number of points: for two points there will be one straight line, for three points just three lines, four points will determine six lines, five points will be ten lines, etc. Now we can do the following: from each point we lead a line to $(n-1)$ points, but in this way I count them each line twice. The result is:

$$\frac{n(n-1)}{2}.$$

Exercises 2.11. In the plane there are n lines, two of which intersect and no three of them meet the same point. How many intersections there are?

Exercises 2.12. Determine what shapes may arise as an intersection of two half-planes. Consider the situation in a single plane.

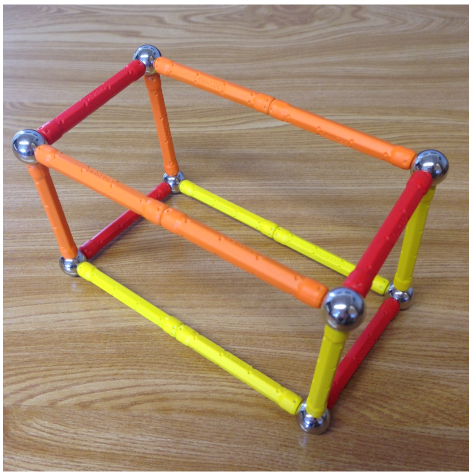
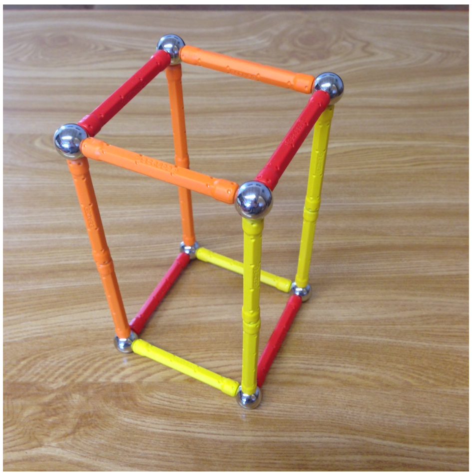
Exercises 2.13. Select points A, B inside one half-plane, which is determined by the line p. Inside the opposite half-plane, select C, D so that the lines AB and CD are parallel to the line p. On line AB select M, on line CD select N. How must the points M, N be choosen so that the line segment MN contains a point of line p lying between M and N?

Example 2.14. Construct a cuboid of $ABCDEFGH$ (using GeoMag or using skewers and plasticine).
A) Determine all incident lines with cuboid edges that are with BC:

- parallel,
- intersecting,
- skew.

B) Using the points of the cuboid, you list three planes that form a bundle of planes and write down the intersection of these three planes.

Řešení:

- parallel: $\leftrightarrow AD$, $\leftrightarrow EF$, $\leftrightarrow HG$
- intersecting: $\leftrightarrow AB$, $\leftrightarrow EB$, $\leftrightarrow DC$, $\leftrightarrow CF$
- skew: $\leftrightarrow EH$, $\leftrightarrow FG$, $\leftrightarrow AH$, $\leftrightarrow DG$

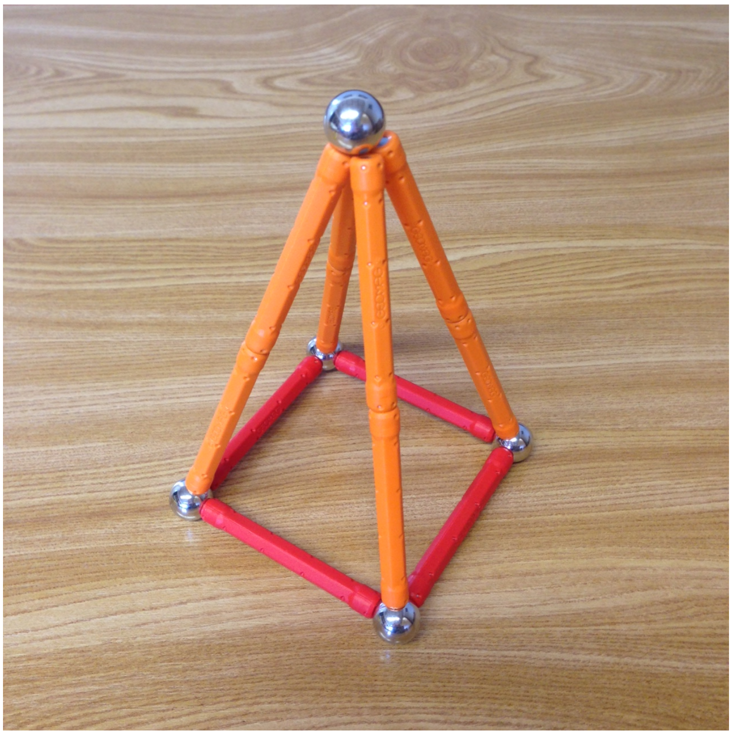
A bundle of planes consists of planes $\leftrightarrow ABC$, $\leftrightarrow ABE$ a $\leftrightarrow AF$:

$$\leftrightarrow ABC \cap \leftrightarrow ABE \cap \leftrightarrow ABF = \leftrightarrow AB.$$

Example 2.15. Construct a regular tetrahedral pyramid $ABCDV$ (using GeoMag kit or skewers and plasticine).
A) Determine all straight lines specified by A, B, C, D, V that are:

- parallel to BC,
- intersecting to BC,
- skew to BC.

B) Using the pyramid points A, B, C, D, V, give an example of the three planes that make up the bunch of planes and write the intersection of the three planes.
Solution:



- parallel: $\leftrightarrow AD$,
- intersecting: $\leftrightarrow AB$, $\leftrightarrow BV$, $\leftrightarrow CV$, $\leftrightarrow CD$,
- skew $\leftrightarrow AV$, $\leftrightarrow DV$.

A bunch of planes is made up of planes $\leftrightarrow ABC$, $\leftrightarrow ABV$ a $\leftrightarrow BCV$:

$$\leftrightarrow ABC \cap \leftrightarrow ABV \cap \leftrightarrow BCV = \leftrightarrow \{B\}.$$

3 Convex and non-convex set, convex and non-convex angle

Exercises 3.1. How can we find out whether a geometrical figure is convex or non-convex? Sort geometric shapes into convex and non-convex: a line segment, line, circle, triangle, quadrilateral, pentagon, circle with hole.

Exercises 3.2. Look around and try to see the angles determined by the edges of the board or the edges of the bench, parts of the window frame, but also the angles formed, for example, the legs of the chair and the floor. Mark such angles in the illustration.

Exercises 3.3. Draw the lines $\mapsto SC$ and $\mapsto SD$. Mark with a red arc the convex angle $\sphericalangle CSD$ and with a blue one non-convex angle $\measuredangle CSD$. Mark the point E of angle $\sphericalangle CSD$ and the point F of angle $\measuredangle CSD$. Can you determine point H, which is the point of the angle $\sphericalangle CSD$ as well as of the angle $\measuredangle CSD$?

Exercises 3.4. Draw the angle $\sphericalangle ADB$. Mark point H in it. Draw the angle $\sphericalangle ADH$. Write down all convex angles.

Exercises 3.5. Draw three lines with a common S origin. Mark one of the points A, B, C on each of the lines. Mark the curves in these angles and write them down.

Exercises 3.6. Sketch two **convex** planar formations such that their

a) union is a convex set,

b) union is a non-convex set,

c) intersection is a convex set,

d) intersection is a non-convex set.

Exercises 3.7. Sketch two **non-convex** planar formations such that their

a) union is a convex set,

b) union is a non-convex set,

c) intersection is a convex set,

d) intersection is a non-convex set.

Exercises 3.8. Sketch the following and determine whether it is a convex or non-convex set:

a) a triangle ABC without its vertices,

b) triangle KLM without one inner point of one side,

c) union of the inside of any triangle and two different points of its perimeter,

d) difference of a convex angle AVB and its arm VA,

e) difference of square $ABCD$ and the union of its two sides,

f) union of the inside of square $ABCD$ and its two sides,

g) circle.

Example 3.9. Investigate all geometric shapes that may arise as an intersection of two triangles. Show and describe.

Solution: The intersection f two triangles may be:

A) a point, e. g. $\triangle ABC \cap \triangle EFD = \{D\}$,

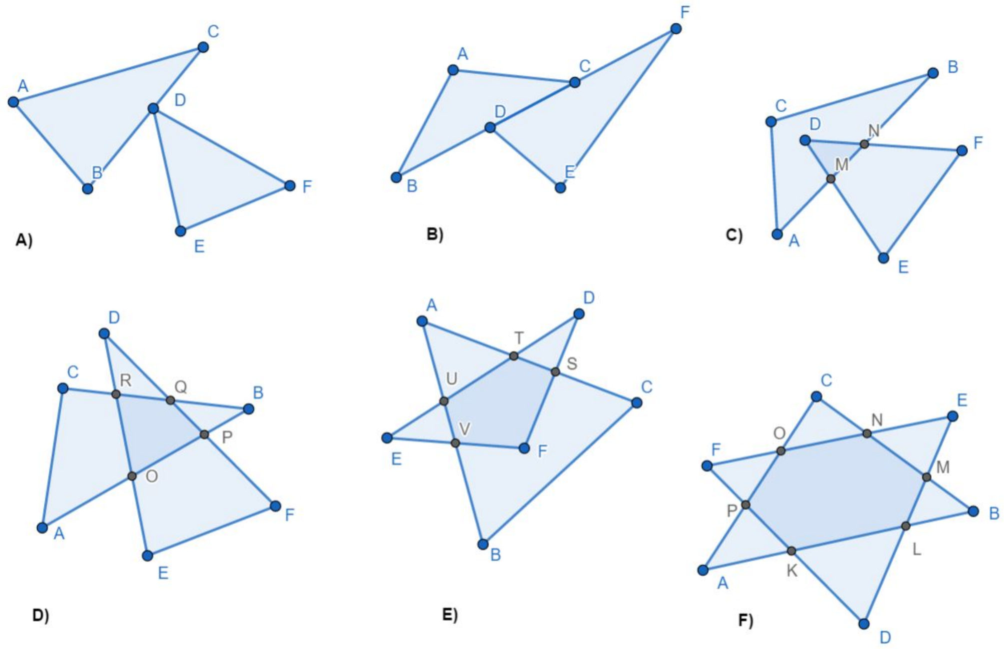
B) a line segment, e.g. $\triangle ABC \cap \triangle EFD = DC$,

C) a triangle, e. g. $\triangle ABC \cap \triangle EFD = \triangle DMN$,

D) a quadrilateral, e.g. $\triangle ABC \cap \triangle EFD =$ quadrilateral $OPQR$,

E) a pentagon, e.g. $\triangle ABC \cap \triangle EFD =$ pentagon $FSTUV$,

F) hexagon, eg. $\triangle ABC \cap \triangle EFD =$ hexagon $KLMNOP$.



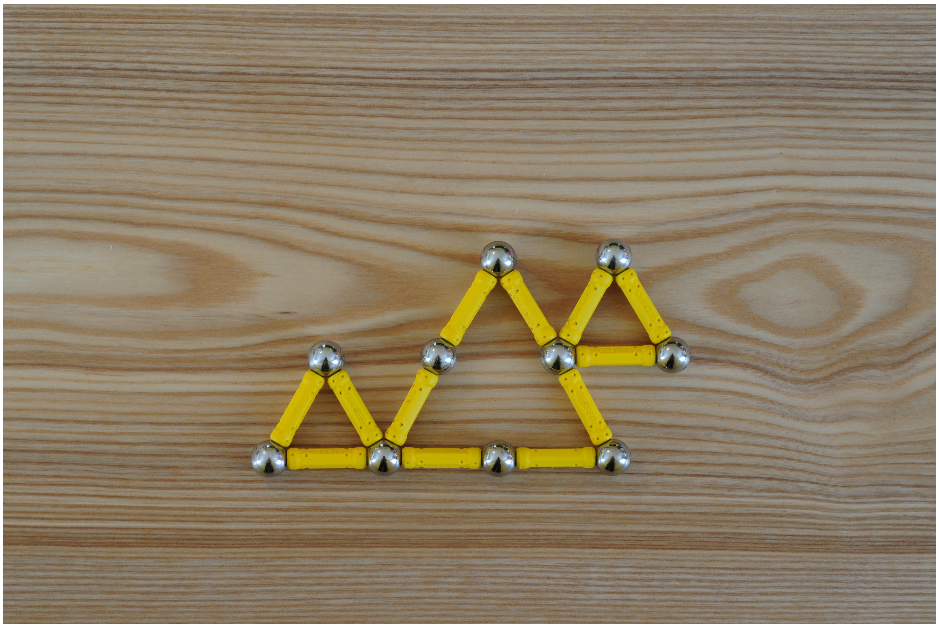
Exercises 3.10. Choose pairs of convex angles (neither full nor zero ones). Investigate which geometrical shapes can arise as the intersections of these angles. Draw and describe all cases.

Exercises 3.11. Select different non-parallel lines p, q and mark their intersection V. Select P on the line p, Q on the line q. Define each pair of vertical and adjacent angles determined by the intersecting lines p and q using the half-planes pQ, qP and the half-planes opposite to them. Use symbolic notation.

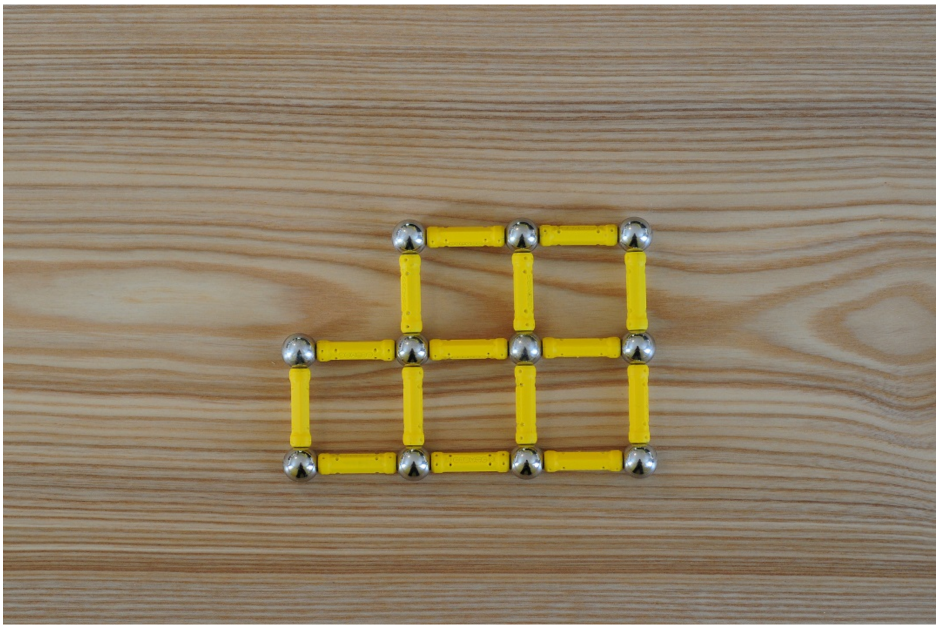
Exercises 3.12. From the Geomag kit, model and then draw: a) an isosceles triangle, b) an equilateral triangle, (c) a square, (d) a regular pentagon, (e) a regular hexagon.

Example 3.13. Model the Geomag kit with the following specifications, and then practice your imagination to solve them:

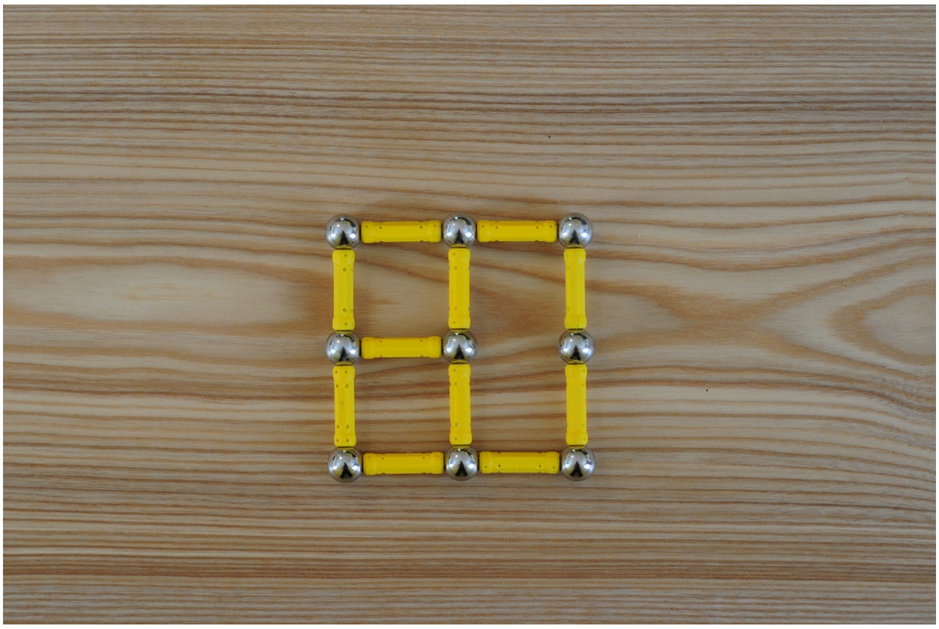
A) Move 3 equal lines (yellow sticks) to form 2 large and one small triangle. (The task has two solutions).



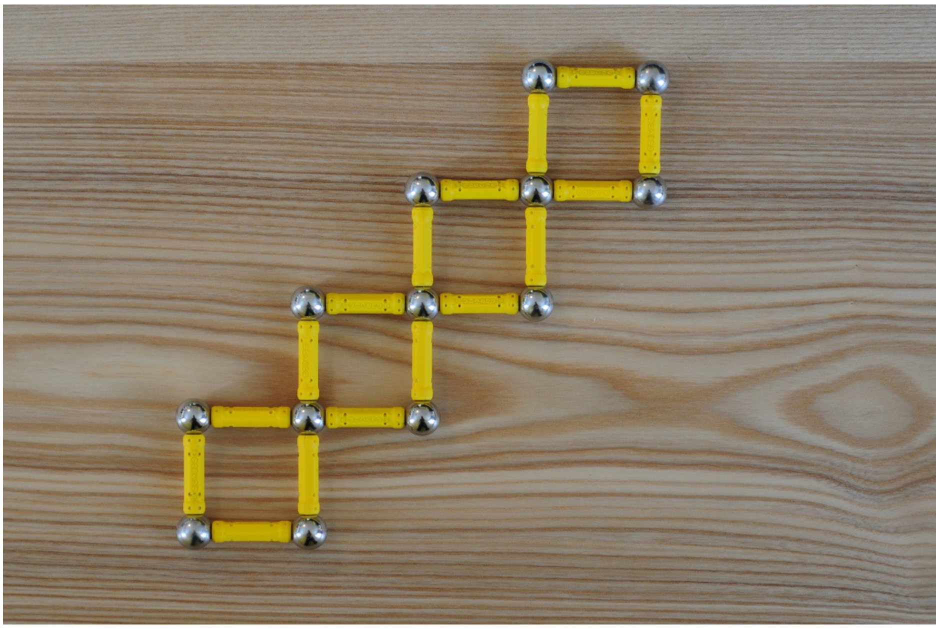
B) Remove 3 equal lines (yellow sticks) to form 3 squares.



C) Remove one line (yellow sticks) to get 2 squares. (The task has two solutions.)



D) Move 4 identical lines (yellow sticks) to create four squares again, but not all of the same size.



Solution: See the end of the text.

Exercises 3.14. Draw a regular hexagon $ABCDEF$ with the center S and mark the pairs of angles in the picture:

a) adjacent (not supplementary),

b) supplementary,

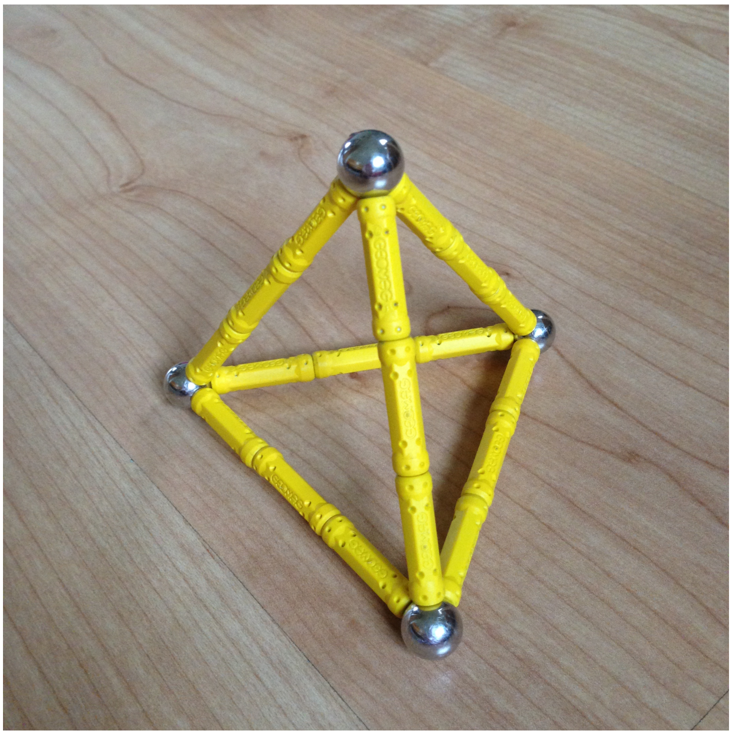
c) vertical,

d) corresponding,

e) alternate exterior angels,

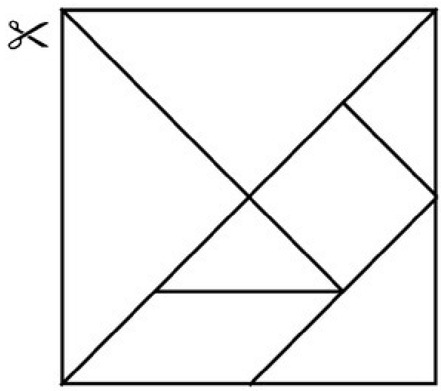
f) alternate interior angels.

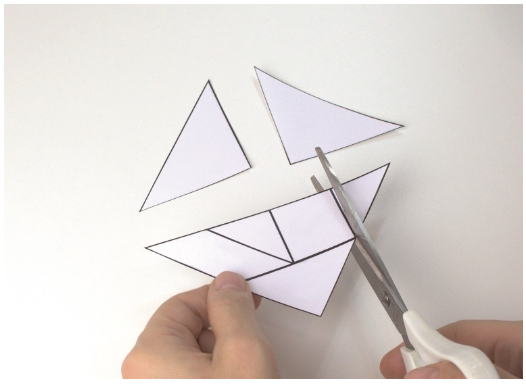
Example 3.15. Model a regular tetrahedron $ABCD$ and then display it. Determine its intersection with the $EFGH$ halfspace if A is between E and C, B is between F and C, and G between D and C.



4 Circle, round, triangle, quadrilateral, regular polygon

Exercises 4.1. Tangram is the oldest known puzzle in the world, it comes from ancient China. It is a square divided in a thoughtful way into seven parts, from which various geometric figures, objects, animals and human figures can be assembled. Make your own tangram from a square of paper according to the pictures below:





Then build, using all seven parts:

a) a triangle,

b) a parallelogram,

c) a trapezoid.

Chinese mathematicians who have investigated the tangram found that a whole series of convex polygons can be constructed from the seven pieces of tangram:

a) 1 triangle,

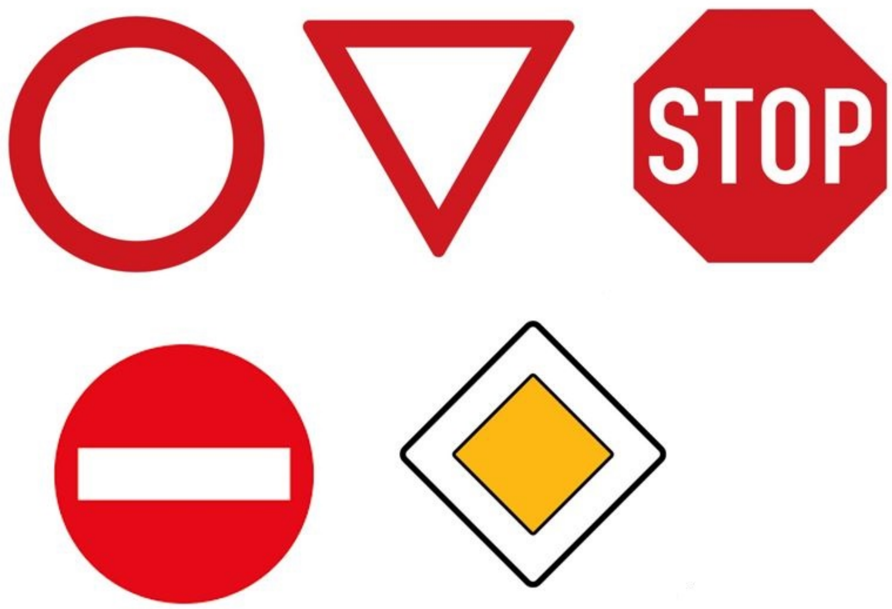
b) 6 quadrilaterals,

c) 2 pentagons,

d) 4 hexagons.

So if you have mastered the geometric shapes a) - d), you can try this somewhat more difficult task.

Example 4.2. The picture shows a few well-known road signs. Answer the following questions:



1) Which geometric shapes are depicted in the picture? (A single-point set is not considered to be a shape.)

2) With a ruler and compass construct all geometric shapes from task 1).

3) Use the ruler and compass to construct the centers of both circles on the first sign. What is the relationship between the two circles?

4) What is the area of the triangle that forms the second tag if its side is 900 mm? Try to solve the problem in several ways (recall Heron's formula).

5) The first sign has a diameter of 700 mm. The side of the triangle on the second sign is 900 mm. For which of these signs do we need more sheet metal?

6) Take pictures of the signs you meet on your way to school and formulate similar questions.

Solution: 1) Line, circle, round, equilateral triangle, square, rectangle, regular octagon. 3) These are concentric circles that have a common center. This center can be found, for example, using two different arbitrary chords, taking advantage of the fact that the axis of each line that is a chord of a circle is a line passing through the center of the circle.

Example 4.3. Construct a square if its side is AB. Select claims that are not true:

a) All angles are equal in a square.

b) There is exactly one right angle in a square.

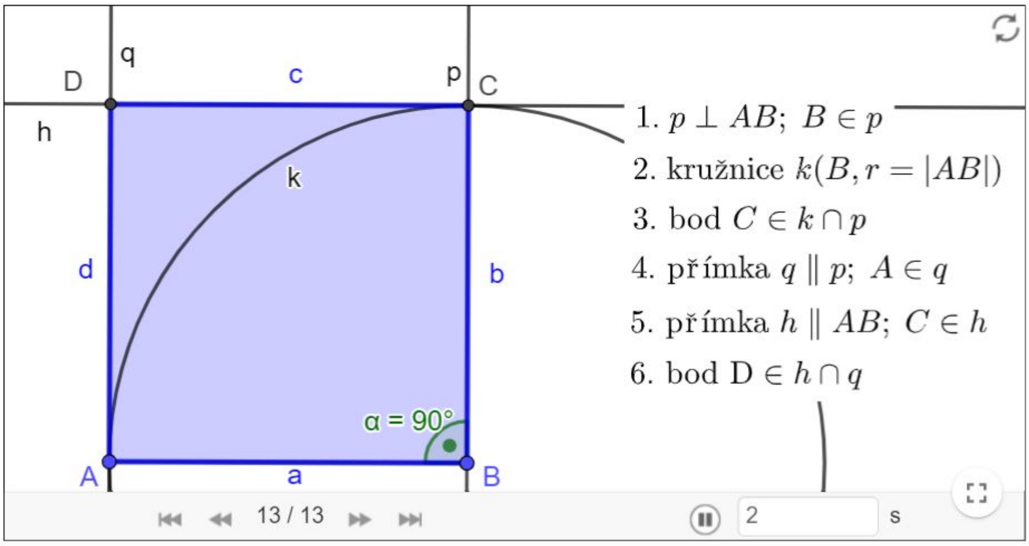
c) In a square two sides must be horizontal.

d) In a square, adjacent sides must be perpendicular to each other.

e) The angle between the diagonal and the adjacent side of the square is $45°$.

f) Diagonals in a square form an angle of $60°$.

Solution:



False statements are statements b), c), f).

Exercises 4.4. If in an isosceles triangle ABC the angle at base AB equals three times the angle at the vertex C and if the angle $\sphericalangle BAC$ at the base is divided into three equal angles (so that M, N are those points of BC for which $\sphericalangle NAB \cong \sphericalangle MAN \cong \sphericalangle CAM$), then $AB \cong AN \cong BM$, $AM \cong CM$. Prove.

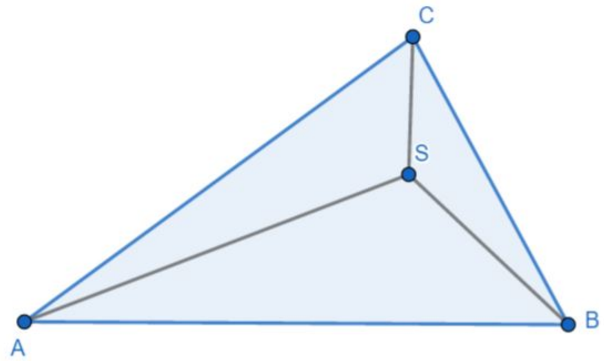
Exercises 4.5. A point A lying outside the circle $k(S, r)$ leads the secant CD so that $AC < AD$ and $|AC| = r$. Prove that

$$\sphericalangle ASC = \frac{1}{3}\sphericalangle BSD,$$

where the point B is the point of intersection of the line AS with the circle k such that S lies between the points A and B.

Exercises 4.6. Inside the ABC triangle, select a point S. Prove that the sum of the lines SA, SB, SC is greater than a half of the sum of the sides of the triangle, i.e.

$$SA + SB + SC > \frac{1}{2}(AB + BC + CA). \tag{1}$$



Solution: Point S is the inner point of the triangle ABC, so there are three other triangles for which the triangle inequality holds:

for triangle ABS: $AS + BS > AB$,

for triangle ACS: $AS + CS > AC$,

for triangle BCS: $BS + CS > BC$.

Adding the right and left sides of the inequalities we get:

$$2 \cdot AS + 2 \cdot BS + 2 \cdot CS > AB + BC + AC, \tag{2}$$

thus proving inequality (1).

Example 4.7. Prove that for the sum of the centroids t_a, t_b, t_c of triangle ABC, the relation is:

$$\frac{1}{2}(a + b + c) < t_a + t_b + t_c < a + b + c. \tag{3}$$

Solution: First we prove the inequality

$$\frac{1}{2}(a + b + c) < t_a + t_b + t_c. \tag{4}$$

Let's denote A_1 center of BC, B_1 center of AC and C_1 center of AB triangle ABC. From the triangular inequality, it follows for triangle ABA_1: $t_a + \frac{a}{2} > c$,

for triangle ACC_1: $t_c + \frac{c}{2} > b$,

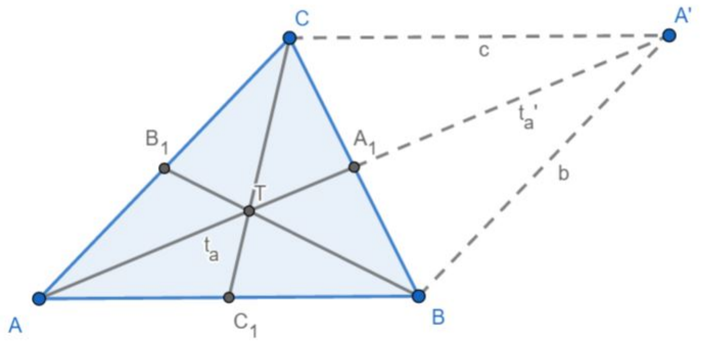
for triangleBCB_1: $t_b + \frac{b}{2} > a$.

Adding the right and left sides of the inequalities we get:

$$t_a + t_b + t_c + \frac{1}{2}(a + b + c) > a + b + c, \tag{5}$$

i.e.

$$t_a + t_b + t_c > \frac{1}{2}(a + b + c). \tag{6}$$



Let us prove the inequality

$$t_a + t_b + t_c < a + b + c. \tag{7}$$

Let the points A_1, B_1, C_1 be again the centers of the sides BC, AC and AB of the triangle. Let's construct A' so that A_1 is the center line of AA'. The quadrilateral $ABA'C$ is a parallelogram, its diagonals halve each other. So $AC \cong BA'$ holds. The triangular inequality for triangle ABA' implies:

$$2t_a < b + c. \tag{8}$$

Similarly, constructing B' and C' so that B_1 is the center of the line BB' and C_1 is the center of the line CC':

$$2t_b < a + c \tag{9}$$

and

$$2t_c < a + b. \tag{10}$$

Adding the right and left sides of the three inequalities we get:

$$2t_a + 2t_b + 2t_c < 2a + 2b + 2c, \tag{11}$$

that is, we proved inequality (7).

Example 4.8. Prove that the sum of the lines that connect the inner point P of the triangle to the endpoints of one side is less than the sum of the remaining two sides of the triangle.

Proof: For example, according to the task assignment,

$$AP + BP < AC + BC \tag{12}$$

We can now prove the argument (12). Since the P point belongs to the inside of the ABC triangle, there must be a X point on the side of BC and on the AP half-line after P. For triangles ACX and BPX we express a triangular inequality

for triangle ACX: $AX < AC + CX$,

for triangle BPX: $BP < XB + PX$.

After adding both inequalities we get:

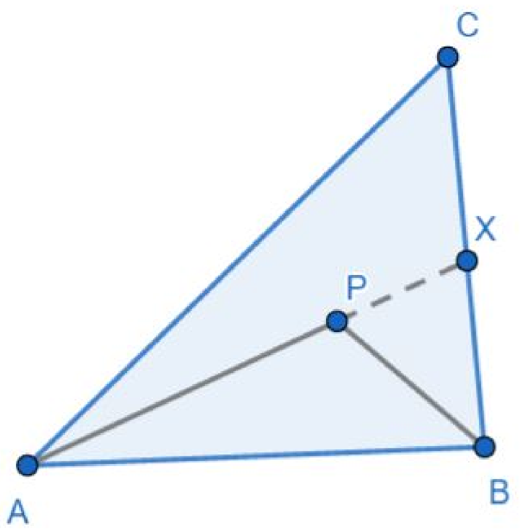
$$AX + BP < AC + CX + BX + PX. \tag{13}$$

Expressing the line AX as the sum of the lines $AP + PX$ and considering that $CX + BX = BC$ we get:

$$(AP + BP) + PX < AC + (CX + XB) + PX,$$

$$(AP + BP) + PX < (AC + BC) + PX$$

and thus inequality (12) is proven.



Exercises 4.9. The straight line o is the axis of the line segment AB. The point X is any inner point of the half-plane oA. Prove that: $AX < BX$.

Exercises 4.10. Point U is the inner point of the triangle ABC.
Prove that the following applies: $\sphericalangle AUB > \sphericalangle ACB$, $\sphericalangle BUC > \sphericalangle BAC$ and $\sphericalangle AUC > \sphericalangle ABC$.

Exercises 4.11. If the point X lies on the axis of the given convex angle AVB, then it has the same distances from its arms. Prove.

Exercises 4.12. If the line of gravity of the triangle coincides with its corresponding height, the triangle is isosceles. Prove.

Exercises 4.13. In the triangle ABC it holds that $\sphericalangle BAC = \alpha = 50°$, $\sphericalangle ABC = \beta = 60°$, $\sphericalangle ABC$ crosses AC v point D. Sort the lines AB, BC, CD, AD, AC, BD by size.

Exercises 4.14. Determine the size of the inner angles of the triangle $A_1B_1C_1$, whose vertices are the intersections of the axes of the outer angles of the triangle ABC.

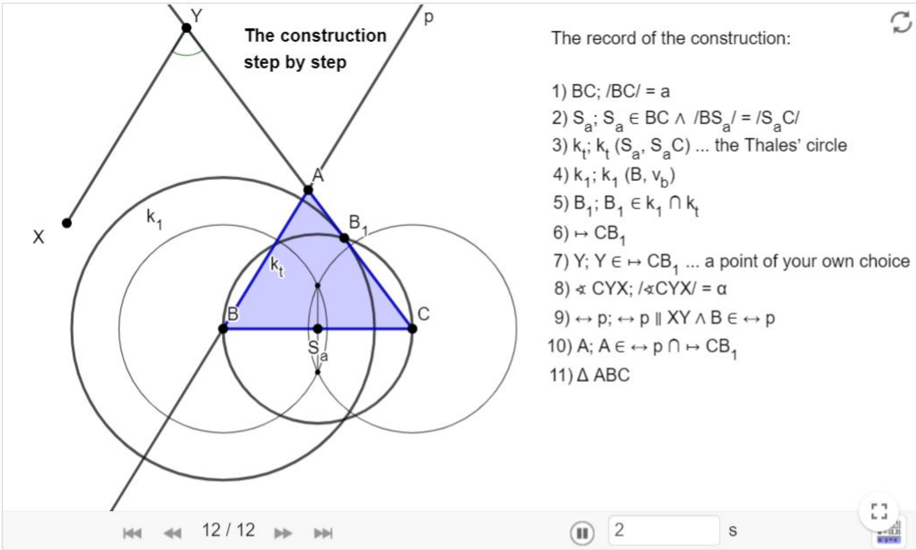
Exercises 4.15. An isosceles triangle ABC is given and a point D which is the center of its base AB. Through point D, lead a line perpendicular to the arms AC, BC triangle ABC. Their plumb points are labeled M, N. Prove that $\triangle DMC \cong \triangle DNC$.

Exercises 4.16. Construct a triangle ABC if three independent data are given:

a)	c, b, t_c	b)	α, c, t_c	c)	a, v_a, b
d)	a, α, v_b	e)	b, c, v_a	f)	α, v_b, r_v
g)	b, γ, v_c	h)	γ, v_a, v_b	i)	c, v_a, v_b
j)	a, v_a, v_b	k)	γ, v_a, v_c	l)	r_o, v_c, t_c
m)	a, b, t_c	n)	α, β, r_v	o)	α, β, r_o
p)	b, β, v_b	q)	a, β, r_v	r)	c, t_a, t_b
s)	b, β, t_a	t)	a, t_a, t_b	u)	a, v_a, t_b
v)	t_a, t_b, t_c	w)	t_a, t_b, γ	z)	t_a, v_a, v_b

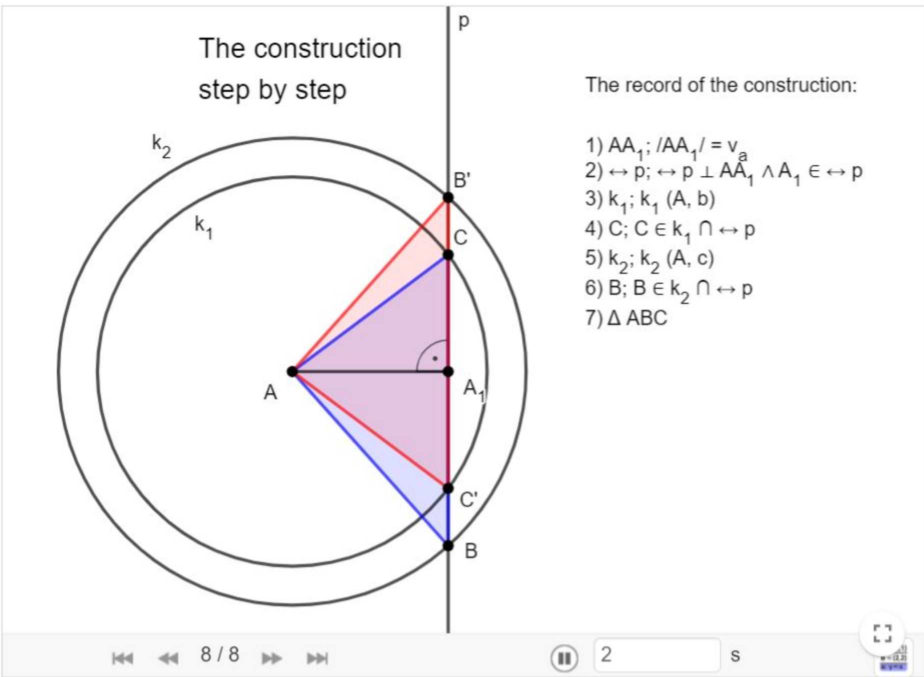
where r_o is the radius of the circle discribed to the triangle ABC and r_v is the radius of the circle inscribed to the triangle ABC. Solutions to some of these tasks can be found in the following examples.

Example 4.17. Construct a triangle ABC if a, α, v_b are given.



The construction step by step:
https://www.geogebra.org/m/u7e5f3qn#material/raxdkacg

Example 4.18. Construct a triangle ABC if b, c, v_a are given.



The construction step by step:
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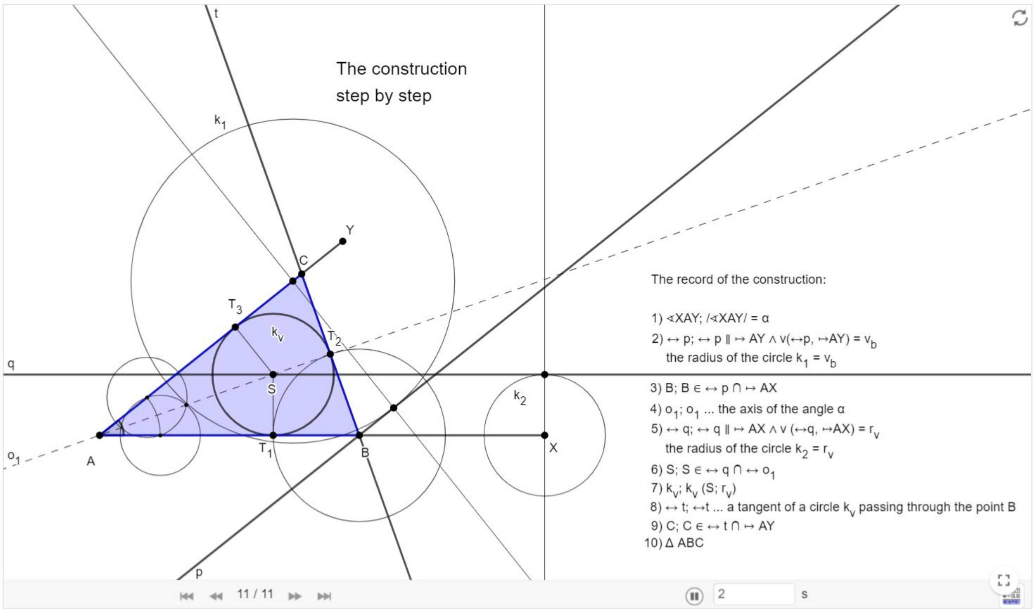
Example 4.19. Construct a triangle ABC if α, v_b, r_v are given.



The construction step by step:
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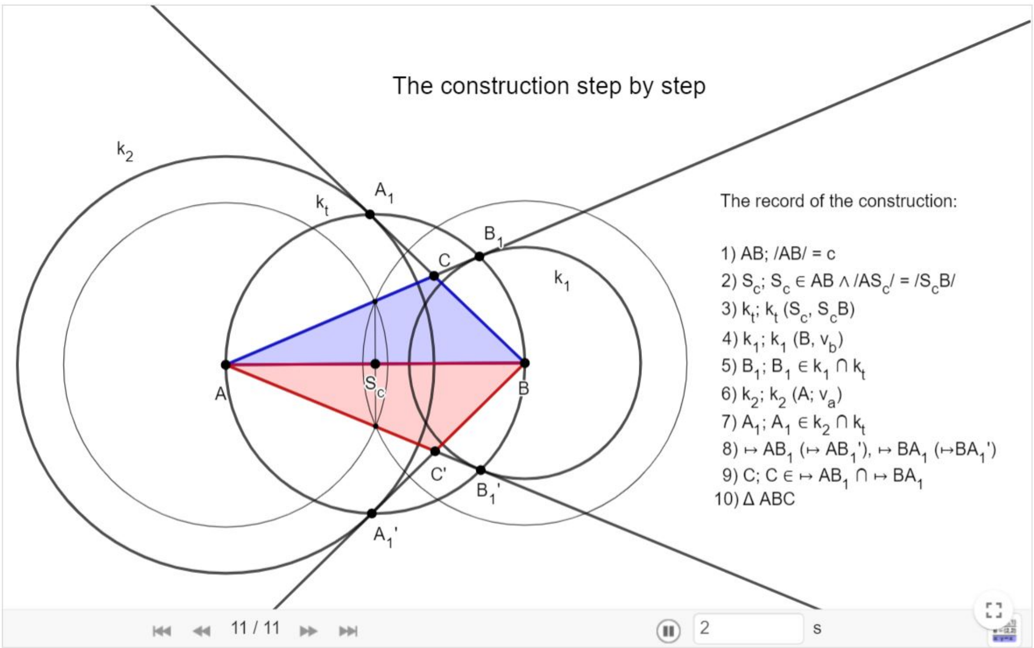
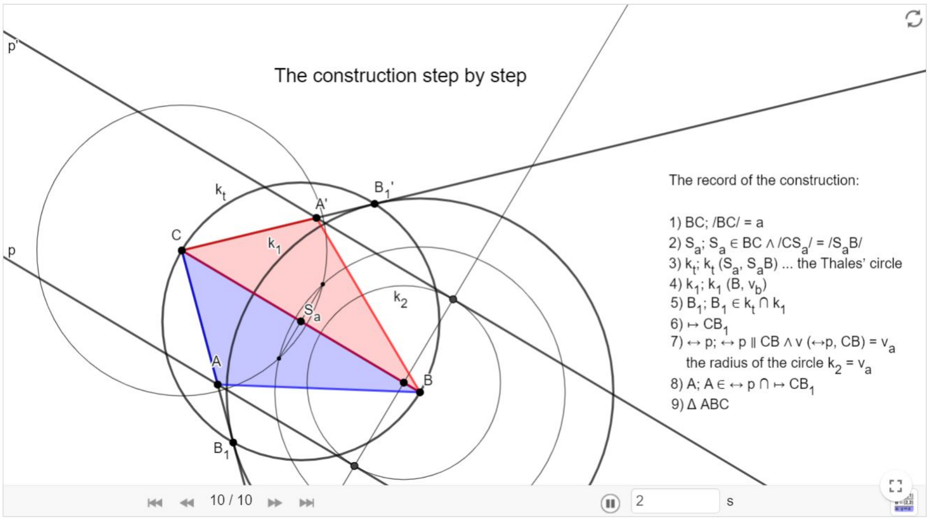
Example 4.20. Construct a triangle ABC if c, v_a, v_b are given.



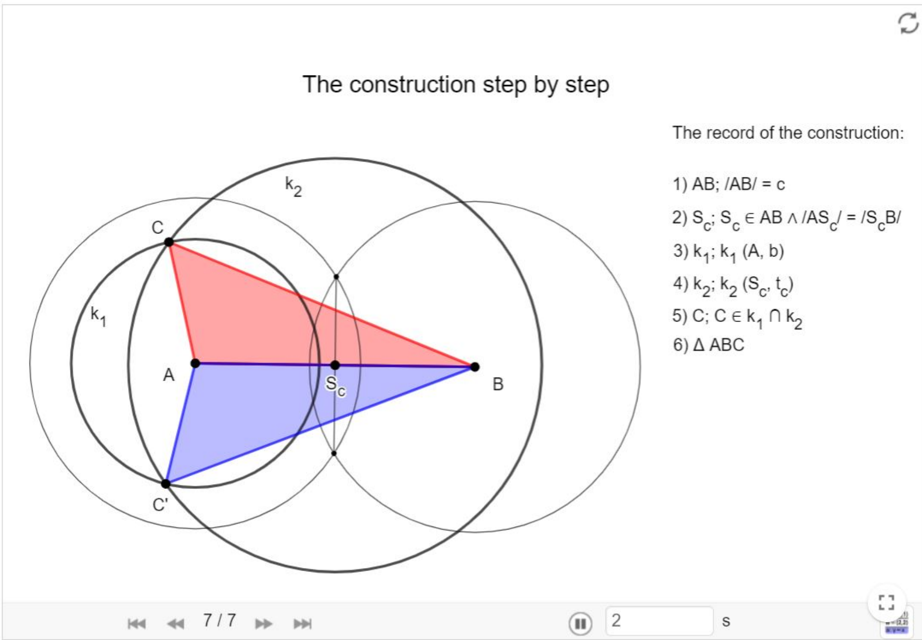
The construction step by step:
`https://www.geogebra.org/m/u7e5f3qn#material/yzcd6acd`

Example 4.21. Construct a triangle ABC if a, v_a, v_b are given.



The construction step by step:
`https://www.geogebra.org/m/u7e5f3qn#material/rkdepcxf`

Example 4.22. Construct a triangle ABC if b, c, t_c are given.



The construction step by step:
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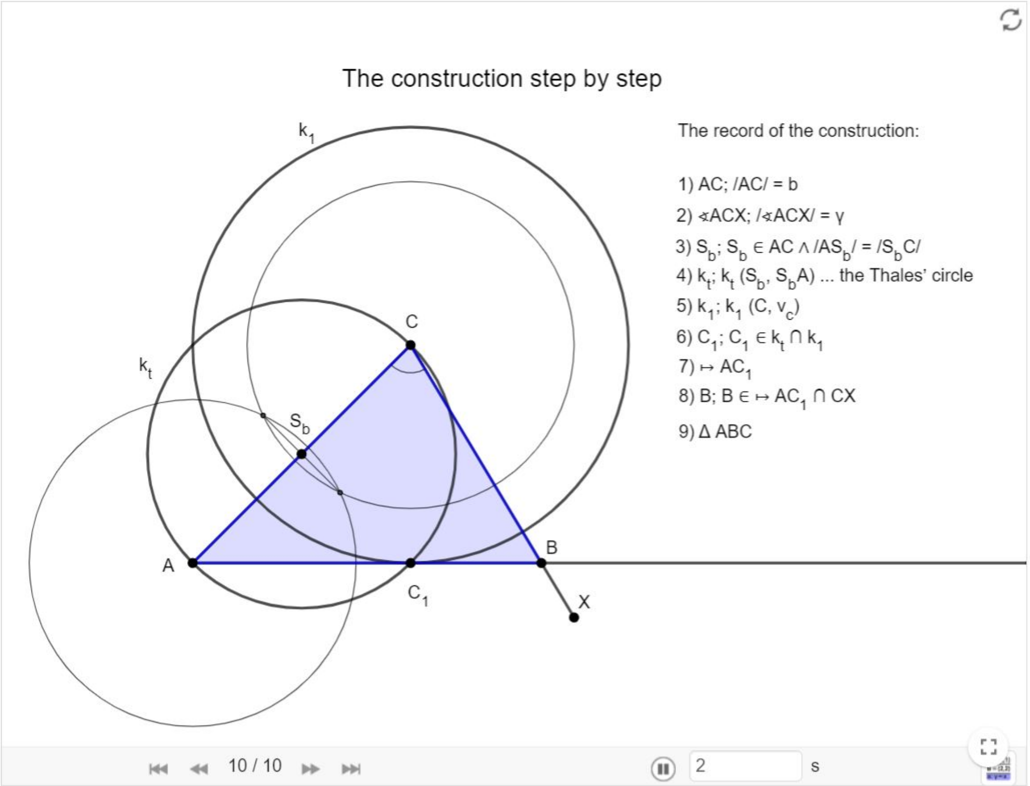
Example 4.23. Construct a triangle ABC if b, γ, v_c are given.



The construction step by step:
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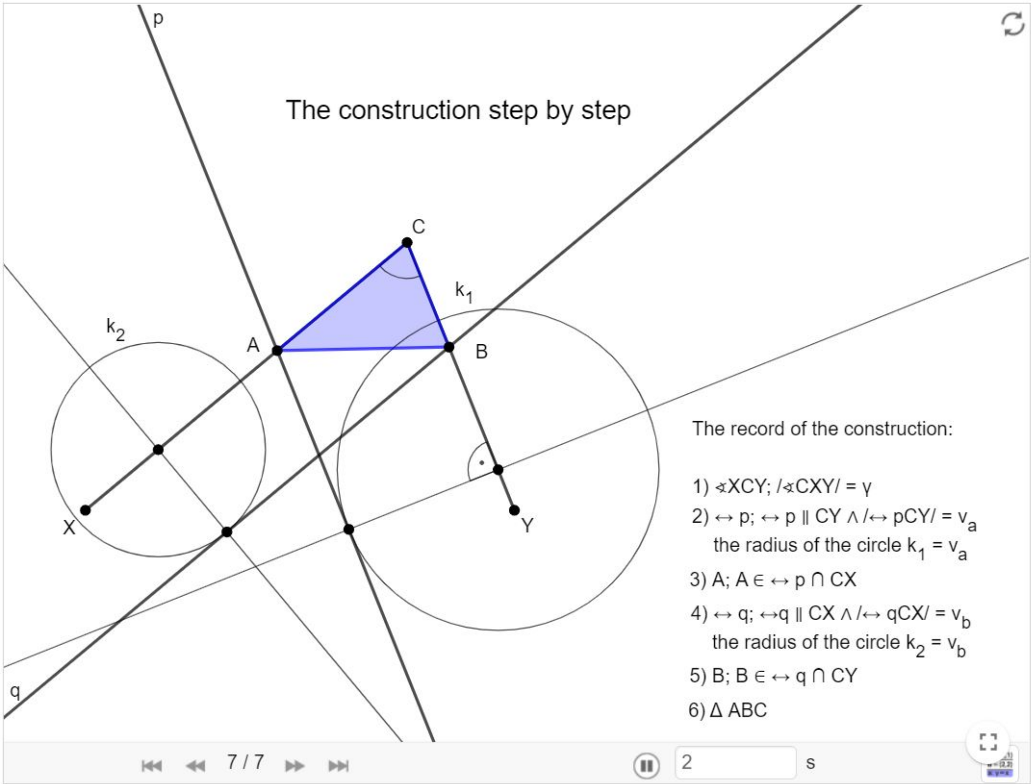
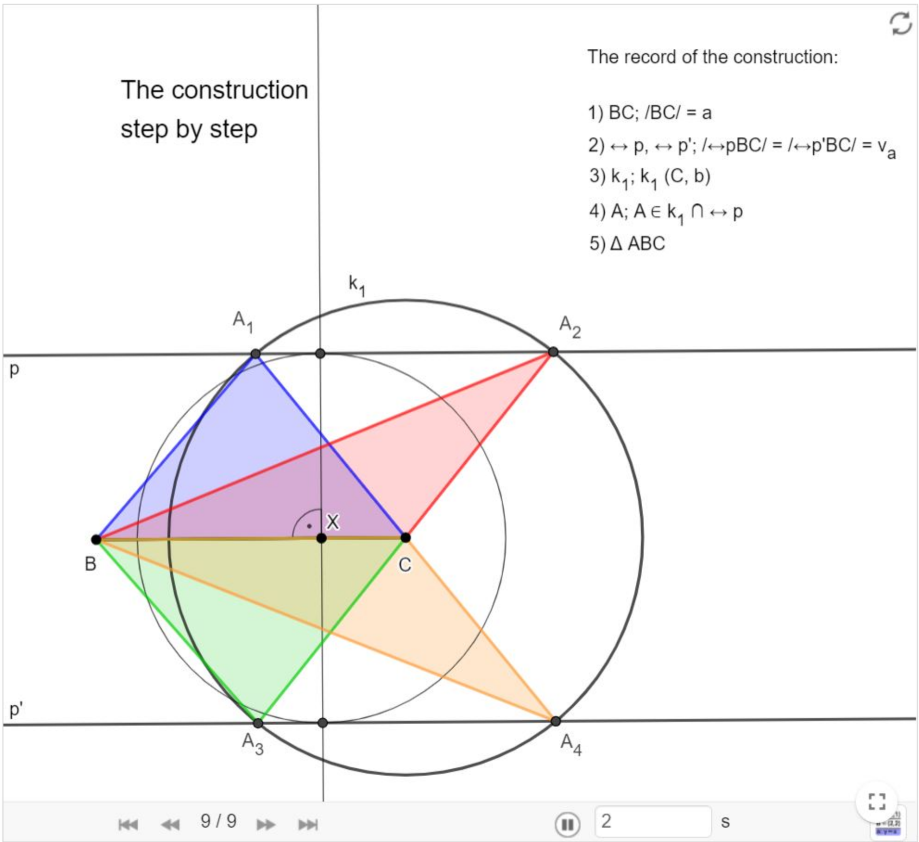
Example 4.24. Construct a triangle ABC if γ, v_a, v_b are given.



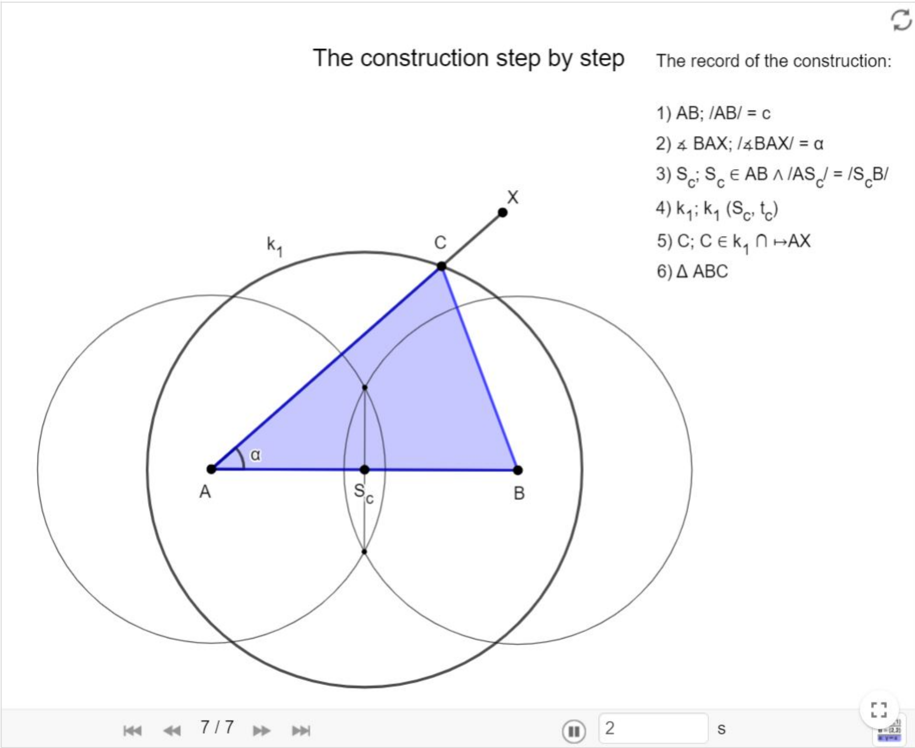
The construction step by step:
https://www.geogebra.org/m/u7e5f3qn#material/sn3wvaed

Example 4.25. Construct a triangle ABC if a, v_a, b are given.



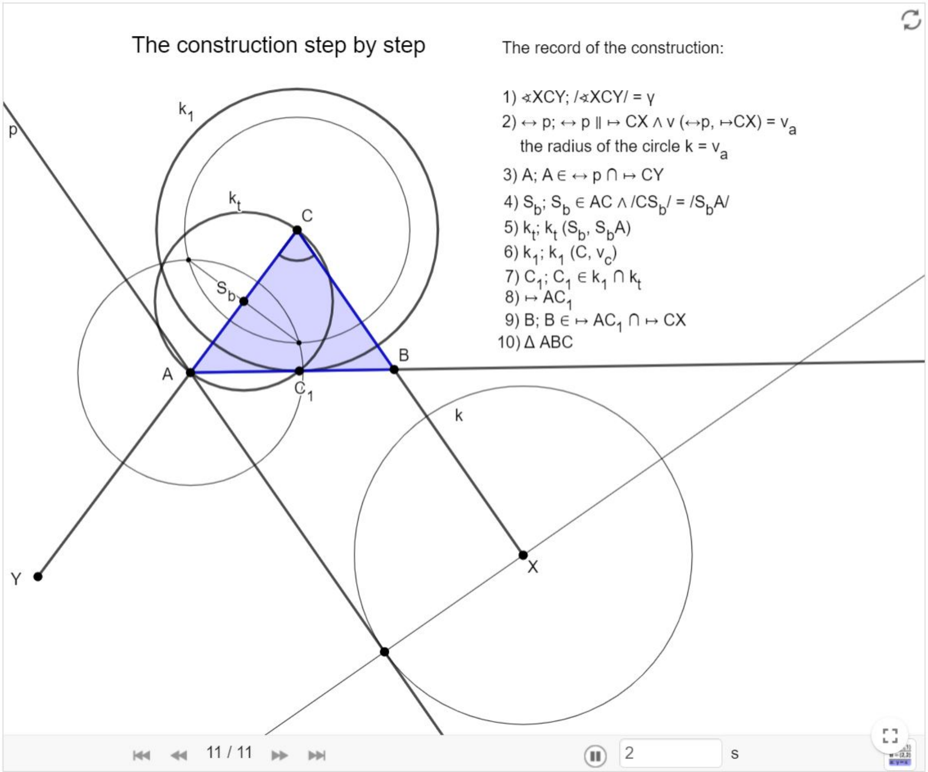
The construction step by step:
`https://www.geogebra.org/m/u7e5f3qn#material/ntwfvxns`

Example 4.26. Construct a triangle ABC if α, c, t_c are given.



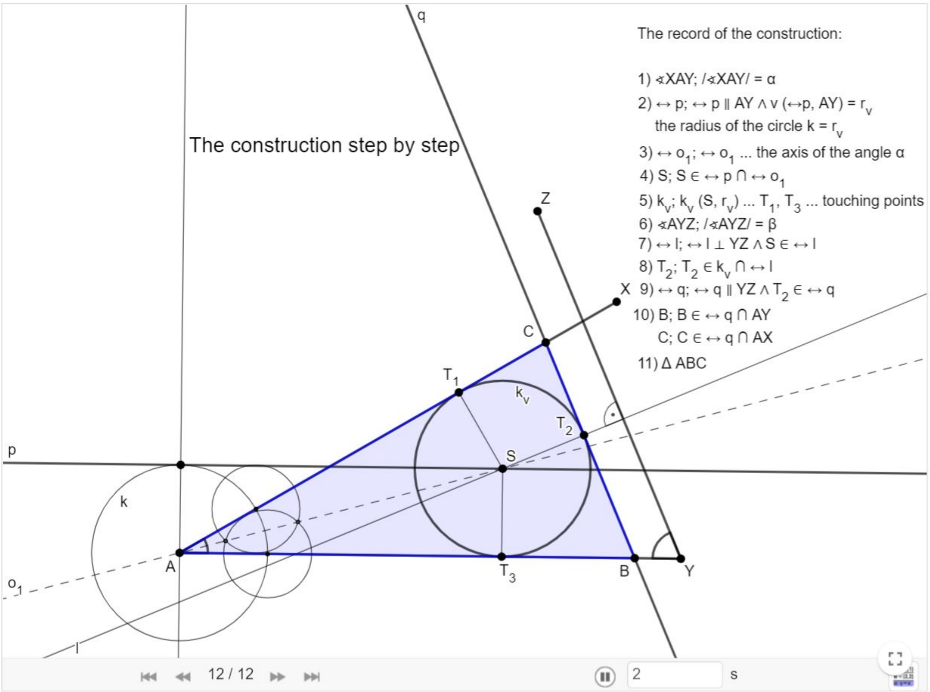
The construction step by step:
`https://www.geogebra.org/m/u7e5f3qn#material/trhbazkf`

Example 4.27. Construct a triangle ABC if γ, v_a, v_c are given.



The construction step by step:
https://www.geogebra.org/m/u7e5f3qn#material/njnjbvh9

Example 4.28. Construct a triangle ABC if α, β, r_v, are given, where r_v is the radius of the circle inscribed to triangle ABC.



The construction step by step:
`https://www.geogebra.org/m/u7e5f3qn#material/w547a5au`

Exercises 4.29. Construct a triangle ABC if the following are given:

a)	$a+b, \gamma, v_a$	b)	$a-b, \gamma, c$	c)	$a+b+c, \alpha, \beta$
d)	$a, b, \alpha-\beta$	e)	$a+b+c, \alpha, v_c$		

Exercises 4.30. The line AB is given.

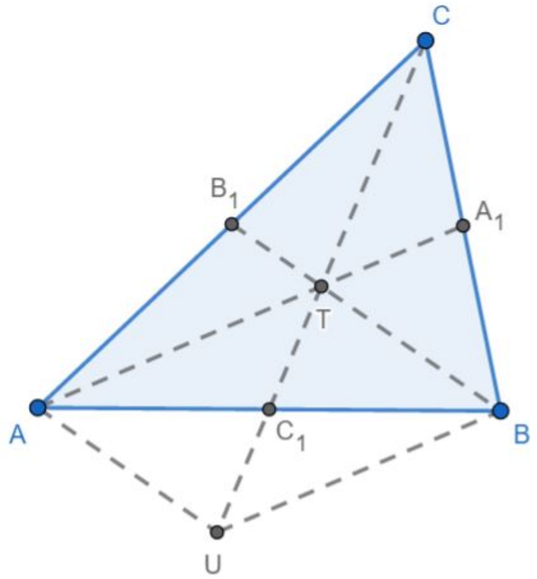
a) Construct the set of all vertices of the convex angle $\sphericalangle ACB = \gamma$, whose arms pass through the endpoints of the line segment AB.

b) Construct $\triangle ABC$ if $|AB| = 6$, $\gamma = 60°$, $v_C = 4$.

Exercises 4.31. Construct a triangle ABC if t_a, t_b, t_c are given.

Example 4.32. Prove the theorem about the centroids of the triangle: The centroids of each triangle intersect at one point, called the centroid of the triangle. the center of gravity divides each center of gravity into two lines, the one containing the vertex of the triangle is twice the other.

Proof: The triangle ABC is given, the points A_1, B_1, C_1 are the centers of its sides BC, AC and AB, the lines AA_1, BB_1 and CC_1 are its center of gravity. In this triangle, we consider the AA_1 and BB_1 lines that intersect at T. We prove that CC_1 goes through T.



Let's construct a line CT and a point U on it so that the point T is the center of the line CU, i.e. $CT \cong TU$. In the triangle AUC, the line B_1T is the middle bar and therefore $B_1T \parallel AU$. Since the points B_1, T, B lie on one straight line, it is i $BT \parallel AU$. Analogously in the BUC triangle, the line A_1T is the middle rung and therefore $A_1T \parallel BU$, and hence $AT \parallel BU$. Hence the quadrilateral $ATBU$ has two parallel sides parallel, i.e. it is a parallelogram and its diagonals AB and TU are bisected. Hence, the center of the AB side, C_1, lies on the line CT. This proves that CC_1 goes through T. Therefore, the centroids of the triangle ABC intersect at one point. This point always belongs to the inside of the given triangle.

The properties of the middle rungs B_1T and A_1T of the triangles AUC and BUC and the properties of the parallelogram $AUBT$ further imply:

for triangle AUC: $B_1T = \frac{1}{2}AU$, $AU \cong BT$, i.e.. $B_1T = \frac{1}{2}BT$, for triangle BUC: $A_1T = \frac{1}{2}BU$, $BU \cong AT$, i.e. $A_1T = \frac{1}{2}AT$.

This proves that the center of gravity T divides each of the AA_1, BB_1 lines into two parts, the one containing the vertex of the triangle is twice as long as the other. By repeating the considerations in choosing another pair of lines, we obtain further relation, which imply the correctness of the statement of the second part of the theorem.

Exercises 4.33. Prove that two triangles are identical when they match in two sides and in the center of gravity to one of them.

Hint: Prove the identity of triangles by using triangles, which are created by dividing the given triangle by the centroid.

Exercises 4.34. Above the sides of the acute triangle ABC there are equilateral triangles ABH and ACK. Prove line segments CH and BK are equal.

Hint: The assertion follows from the equality of triangles ACH and AKB.

Exercises 4.35. The triangle ABC is given. Through its peaks, lines parallel with the opposite sides are drawn. Prove that the intersections of these lines determine the triangle, which is a union of four triangles identical to the triangle ABC.

Hint: Use the theorems of triangles identity and properties of pairs of angles between parallel lines.

Exercises 4.36. The largest side of the convex quadrilateral $ABCD$ is AB, the smallest CD. Prove that $\sphericalangle ABC < \sphericalangle ADC$.

Hint: The diagonal BD divides the quadrilateral $ABCD$ into two triangles. The assumption results in inequalities, the sum of which gives us the assertion.

Exercises 4.37. On the diagonal AC of the square $ABCD$, the point E is given such that $AE \cong AB$. The perpendicular to the line AC through the point E crosses the side BC at the point F. Prove that $BF \cong EF$.

Hint: Prove that triangle ECF is isosceles and triangle AFE is identical to triangle AFB.

Example 4.38. The illustration shows seven different quadrilaterals. Assign them their names and then add their properties (some properties may belong to more than one quadrilateral):
square, rectangle, rhombus, (generic) parallelogram, non-convex quadrilateral, deltoid, trapezoid.

a) The opposite sides are always the same.

b) At least two internal angles are always right.

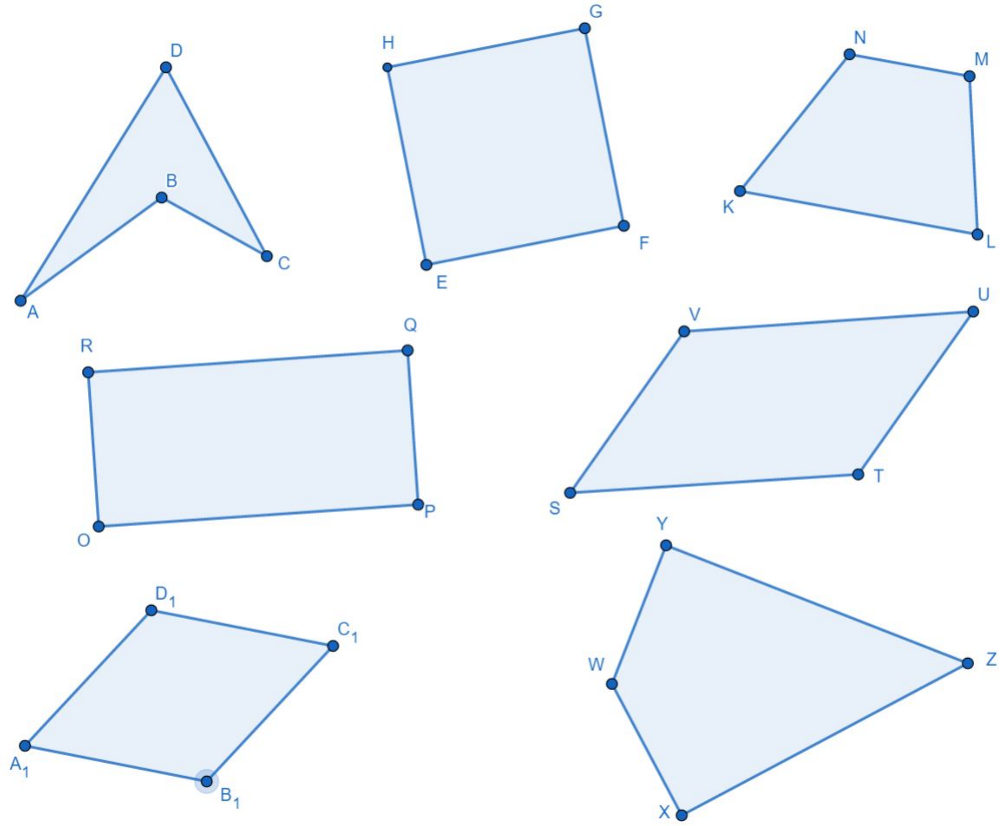
c) The diagonals are halved.

d) The diagonals are identical.

e) You can circle it.

f) You can write a circle.

g) Just one pair of the sides are parallel lines.

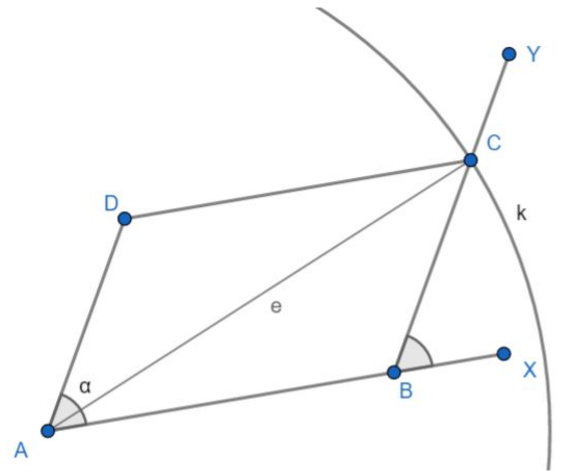


Solution:

- square $EFGH$, a), b), c), d), e), f),
- rectangle $OPQR$, a), b), c), d), e),
- rhombus $A_1B_1C_1D_1$, a), c), f),
- (general) parallelogram $STUV$, a), c),
- non-convex quadrilateral $ABCD$,
- deltoid $XZYW$, b), e), f),
- trapezoid $KLMN$, g).

Example 4.39. Construct a parallelogram $ABCD$, given the side a, the angle DAC, $|\sphericalangle DAC| = \alpha$, and the size of the diagonal $e = |AC|$.

Solution: Select the line AB, $|AB| = a$. The point C lies at a distance e from the point A, i.e. on the circle $k(A, e)$. In addition, the ray BC also forms an angle α with the side AB. For D, $CD \parallel AB$ and $AD \parallel BC$ apply.



The construction step by step:

1. AB, $|AB| = a$
2. k, $k(A, e)$
3. X, $X \in \mapsto AB$
4. $\sphericalangle XBY$, $|\sphericalangle XBY| = \alpha$

5. C, $C \in k \cap \mapsto BY$

6. D, $CD \parallel AB \wedge AD \parallel BC$

7. parallelogram $ABCD$

Conclusion: The problem has one solution in the given half-plane.

Exercises 4.40. Construct a parallelogram $ABCD$, given the size of its diagonals e, f and the size of the height v_a.

Exercises 4.41. Construct a parallelogram $PQRS$, given its diagonal PR, the angle angle RPQ and the distance of the parallel sides PQ and RS.

Exercises 4.42. Construct a rhombus $ABCD$ if $e = |AC|$ is given and the angle DAB is α.

Exercises 4.43. Construct a rectangle $KLMN$ if given $|KL| = 6$ and the angle KSL is $120°$, where S is the intersection of the diagonals.

Exercises 4.44. Construct a trapezoid $ABCD$ if all sides of a, b, c, d are given.

Exercises 4.45. Construct a trapezoid $ABCD$, $AB \parallel CD$ if the diagonal sizes are e, f, the angle size $DAB = \alpha$, and the angle size $AEB = \omega$, where E is intersection of diagonals.

Exercises 4.46. Construct trapezoid $ABCD$ if given: side size AB, side size BC, size of both diagonals AC, BD and angle size $AEB = \omega$, where E is the intersection of diagonals .

Exercises 4.47. A quadrilateral which can be described and inscribed by a circle, ie a quadrilateral which is both chord and tangent *two-centered*. Can you identify at least one non-square two-centered quadrilateral?

Exercises 4.48. Construct a circle k if its tangent t is given with a touch point T and another tangent q.

Exercises 4.49. Construct the k circle that touches the m circle at that point T a

a) is centered on the given line p,

b) goes through M,

c) it touches a given line q.

Exercises 4.50. A circle k is given and two different points K, L outside. Construct the rhombus $KLMN$ so that one of its vertices lies on a circle k.

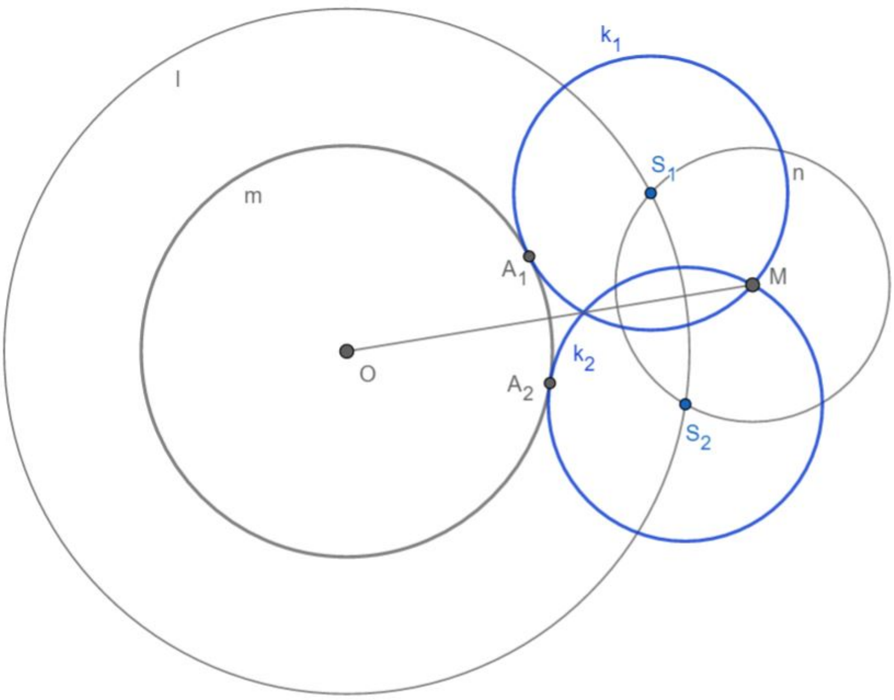
Exercises 4.51. Construct a circle that passes through the A point and directly touches the t line at T.

Exercises 4.52. Construct a circle that is centered on the m circle and touches the two

- parallel lines a, b,
- non-parallel lines c, d.

Example 4.53. Construct a circle with a radius of $r = 2$ cm that touches the outside of the circle $m(0.3\ \text{ cm })$ and goes through that point M, $|SM| = 6$ cm.

Solution: We construct a circle m, $m(0, 3\,\text{cm}\ )$ and a point M, $|SM| = 6$ cm. The center S of the searched circle k is at a distance of 2 cm from the point M, ie on the circle $n(M, 2\,\text{cm}\ )$. Also, the distance of the center S from the touch points, eg A, of the searched circle k with the given circle m is 2cm. The set of all such points will be on a circle with a radius 2cm larger than the radius of the given circle m, eg on the circle $l(0, 5\,\text{cm}\ )$. The searched center S lies at the intersection of the circle n and l.



The construction step by step:

1. m, $m(O, 3\,\text{cm})$; M, $|SM| = 6$ cm
2. n, $n(M, 2\,\text{cm})$
3. l, $l(O, 5\,\text{cm})$
4. S, $S \in n \cap l$
5. k, $k(S, 2\,\text{cm})$

Conclusion: The problem has two solutions in the plane.

Exercises 4.54. Construct a circle that touches the two concentric circles k_1, k_2 and goes through the point P, which is the inner point of the annulus specified by the circles k_1, k_2.

Exercises 4.55. There are two concentric circles $k_1(S, r_1)$, $k_2(S, r_2)$. Investigate the set of centers of all circles that touch k_1, k_2.

Exercises 4.56. Investigate the set of centers of all circles that

a) have a given radius of r and go through two different points A, B;

b) have a given radius r and touch a given line p;

c) touch two given parallels a, b;

d) touch two given divergences a, b;

e) touch a given line p at a given point A;

f) touch a given circle k at the given point A;

g) have a given radius of r and have $k(S, r_1)$ external touch.

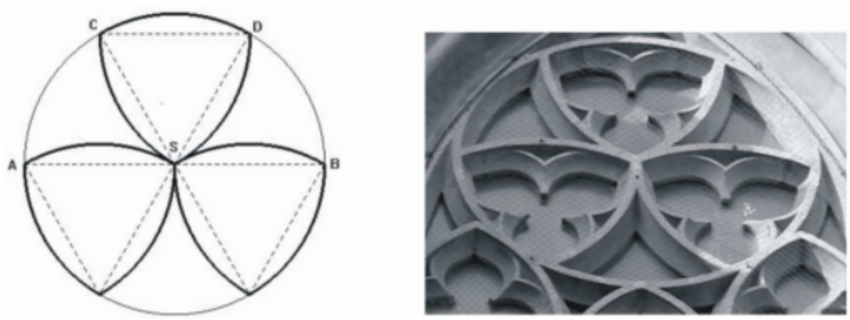
Model these tasks in GeoGebra.

Exercises 4.57. A circle $k(S, r)$ is given, followed by a point A. Investigate the set of centers of all chords of the k circle that pass through A. Model the task in GeoGebra.

Exercises 4.58. A circle $k(S, r)$ is given, and on it a point N that belongs to the outer region of that circle. Investigate the set of centers of all the chords of the k circle that lie on the cuts passing through N.
Model the task in GeoGebra.

Exercises 4.59. Construct a regular a) octagon, b) dvanáctiúhelník, c) šestnáctiúhelník.

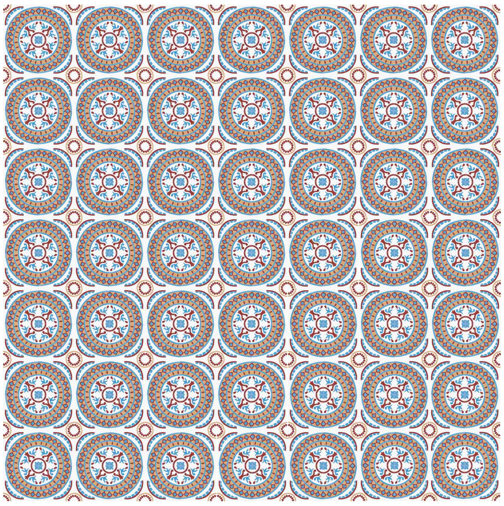
Example 4.60. Construct a grid of curves that was used to create a Gothic window.

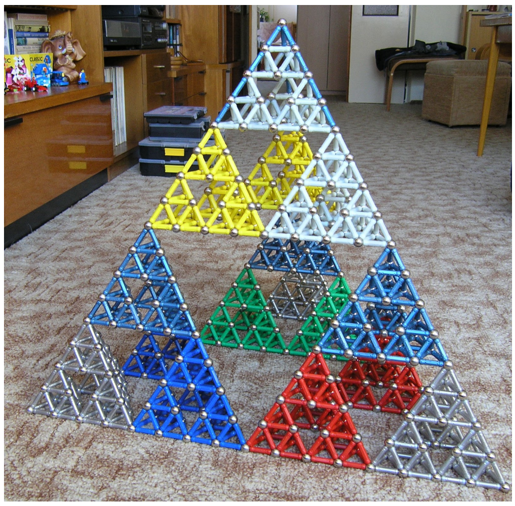


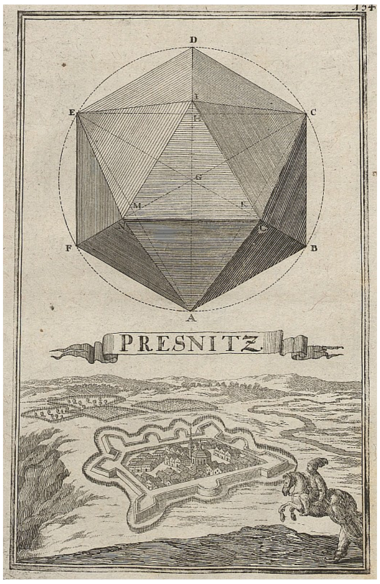
Attachments

Pictures to exercise 1.1

4	9	2
3	5	7
8	1	6



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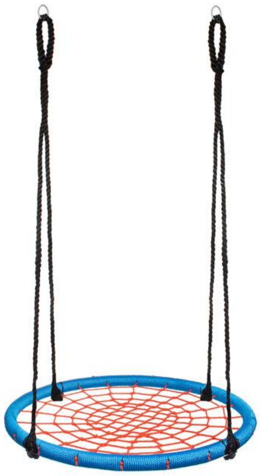

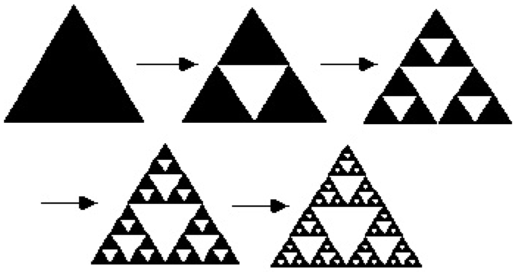

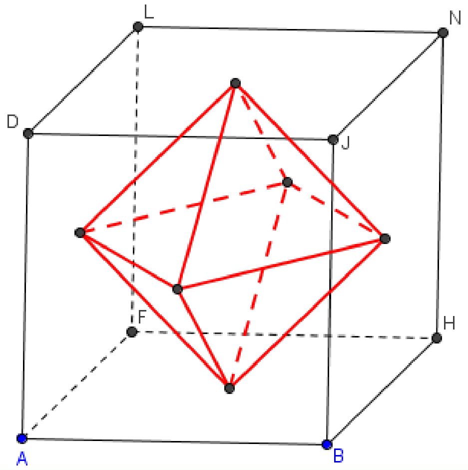
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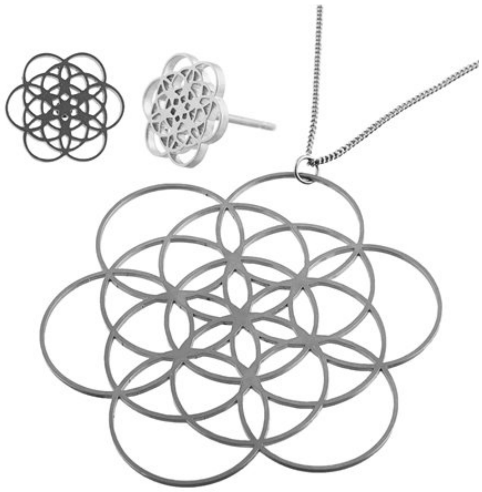

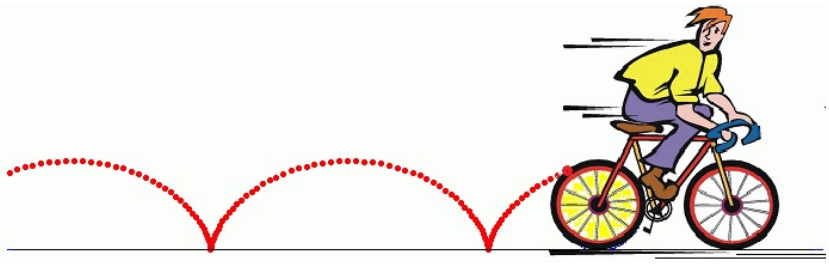

Solving puzzles from the example 3.13

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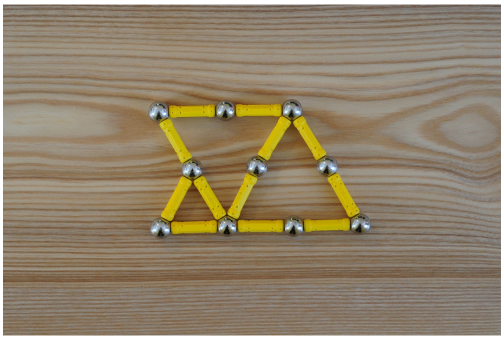
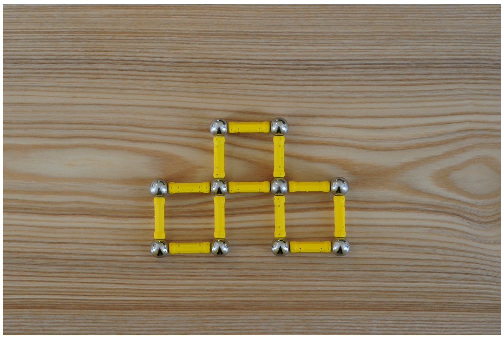
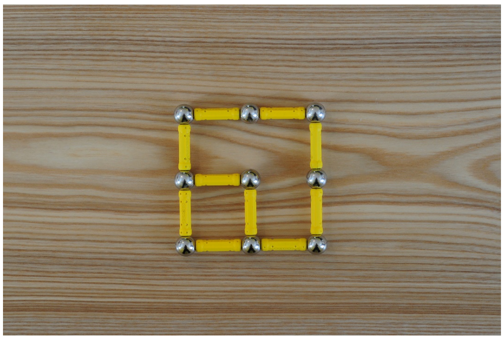


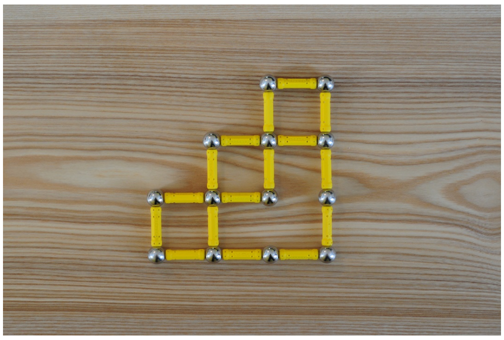
B)



C)



D)



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[1] Francová, M., Lvovská, L., *Texty k základům elementární geometrie pro studium učitelství 1. stupně základní školy*, skriptum PedF MU, Brno 2014.

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Contents