Příklad 1

Jsou dány dvě různé báze α, β vektorového prostoru $\mathbb{R}^3$. Najděte matice přechodu $P_{\beta,\alpha}$, $P_{\alpha,\beta}$ a určete souřadnice vektoru $\vec{u}_\alpha = (1,2,1)$ v bázi β a souřadnice vektoru $\vec{v}_\beta = (-1,0,3)$ v bázi α.

1. $\alpha = ((1,0,1);(2,1,1);(0,0,2))$,
   $\beta = ((0,1,1);(1,0,2);(2,0,2))$.

$$P_{B,\alpha}: \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 & 1 & 2 \end{array}\right) \sim \begin{array}{r} \\ \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 & 1 & 2 \end{array}\right) \sim \begin{array}{r} \\ \\ -2r_2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 2 \end{array}\right) \sim$$

$$\sim \begin{array}{r} \\ +r_3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -4 & 2 \end{array}\right) \sim \begin{array}{r} \\ \\ :(-2) \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & -1 & -4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1/2 & 2 & -1 \end{array}\right)$$

$$P_{B,\alpha} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 1/2 & 2 & -1 \end{pmatrix} \longrightarrow \vec{u}_B = P_{B,\alpha} \cdot \vec{u}_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 1/2 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ \frac{7}{2} \end{pmatrix}_B$$

$$P_{\alpha,B}: \begin{array}{r} \\ \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 2 & 2 \end{array}\right) \sim \begin{array}{r} \\ \\ +r_2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \end{array}\right) \sim \begin{array}{r} -2r_2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 \end{array}\right)$$

$$\sim \begin{array}{r} \\ \\ :2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & 0 \end{array}\right)$$

$$P_{\alpha,B} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1/2 & 0 \end{pmatrix} \longrightarrow \vec{v}_\alpha = P_{\alpha,B} \cdot \vec{v}_B = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix}_\alpha$$

Příklad 2

Lineární zobrazení $\varphi : U \to V$ je zadáno maticí A_S ve standardních bázích U, V. Pro zadané báze α prostoru U a β prostoru V určete matice $A_{S,\alpha}, A_{\beta,S}, A_{\beta,\alpha}$.

1. $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $A_S = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$,
   $\alpha = ((1,2);(-2,1))$, $\beta = ((1,1,1);(1,1,0);(1,2,0))$.

$$\varphi(\vec{u})_S = A_{S,\alpha} \cdot \vec{u}_\alpha$$

$$A_{S,\alpha} = A_S \cdot P_{S,\alpha} \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$A_{S,\alpha} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\varphi(\vec{u})_B = A_{B,S} \cdot \vec{u}_S$$

$$A_{B,S} = P_{B,S} \cdot A_S \cdot \vec{u}_S$$

$$P_{B,S}: \begin{array}{r} \\ -r_1 \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right) \sim \begin{array}{r} -r_3 \\ +r_3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\sim \begin{array}{r} +r_2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \begin{array}{r} \\ :(-1) \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$A_{B,S} = P_{B,S} \cdot A_S = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 5 & 0 \\ -2 & 0 \end{pmatrix}$$

$$A_{B,\alpha} = \underbrace{P_{B,S} \cdot A_S}_{A_{B,S}} \cdot P_{S,\alpha}, \qquad A_S \cdot P_{S,\alpha} = A_{S,\alpha}$$

$$A_{B,\alpha} = \begin{pmatrix} -1 & 1 \\ 5 & 0 \\ -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -10 \\ -2 & 4 \end{pmatrix}$$

$$A_{B,\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -10 \\ -2 & 4 \end{pmatrix}$$

$$\underbrace{\left(\varphi\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_\alpha\right)_B}_{3\ \text{složky}} = A_{B,\alpha} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_\alpha$$

Příklad 3

Lineární transformace $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ je zadána maticí A_S ve standardní bázi prostoru $\mathbb{R}^3$. Pro bázi

$$\alpha = ((1,1,1);(1,1,0);(1,2,0))$$

prostoru $\mathbb{R}^3$ určete matice $A_{S,\alpha}$, $A_{\alpha,S}$, $A_{\alpha,\alpha}$.

1. $A_S = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix}$

$$\left[\varphi\begin{pmatrix}1\\0\\-1\end{pmatrix}_\alpha\right]_\alpha = ?$$

$A_{S,\alpha}$

$$A_{\alpha,\alpha} = P_{\alpha,S}\cdot A_S \cdot P_{S,\alpha}$$

$A_{\alpha,S}$

$$P_{S,\alpha} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_{\alpha,S}: \begin{array}{c} \\ -r_1 \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\sim \begin{array}{c} -r_3 \\ +r_3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \begin{array}{c} +r_2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \begin{array}{c} \\ \cdot(-1) \\ \\ \end{array}$$

$$P_{\alpha,S} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$A_{S,\alpha} = A_S \cdot P_{S,\alpha} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 1 & 0 \\ 9 & 5 & 6 \end{pmatrix}$$

$$A_{\alpha,S} = P_{\alpha,S} \cdot A_S = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 4 \\ -4 & 2 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

$$A_{\alpha,\alpha} = \boxed{P_{\alpha,S}\cdot A_S \cdot P_{S,\alpha}} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 & 3 \\ 3 & 1 & 0 \\ 9 & 5 & 6 \end{pmatrix} = \boxed{\begin{pmatrix} 9 & 5 & 6 \\ -4 & -2 & 0 \\ -1 & -1 & -3 \end{pmatrix}}$$

$$\varphi\begin{pmatrix}1\\0\\-1\end{pmatrix}_\alpha = \boxed{\begin{pmatrix} 9 & 5 & 6 \\ -4 & 2 & 0 \\ -1 & -1 & -3 \end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix}} = \begin{pmatrix}3\\-4\\2\end{pmatrix}_\alpha$$

Příklad 4

Lineární transformace φ vektorového prostoru $\mathbb{R}^2$ je dána maticí A ve standardní bázi. Nalezněte vlastní čísla a jim odpovídající vlastní vektory lineární transformace φ.

a) $A = \begin{pmatrix} 2 & 6 \\ 6 & -3 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 6 \\ 6 & -3-\lambda \end{vmatrix} = (2-\lambda)\cdot(-3-\lambda) - 6\cdot 6$$

$$= -6 + 3\lambda - 2\lambda + \lambda^2 - 36$$

$$= \lambda^2 + \lambda - 42 = 0$$

$$(\lambda+7)\cdot(\lambda-6) = 0$$

$$\lambda_1 = -7: \left(\begin{array}{cc|c} 2-(-7) & 6 & 0 \\ 6 & -3-(-7) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 9 & 6 & 0 \\ 6 & 4 & 0 \end{array}\right)\begin{array}{c} :3 \\ :2 \end{array} \sim \begin{array}{c} \begin{array}{cc} v_{11} & v_{12} \end{array} \\ \left(\begin{array}{cc|c} 3 & 2 & 0 \\ \cancel{3} & \cancel{2} & \cancel{0} \end{array}\right) \end{array}$$

$v_{12} = t$

$3v_{11} + 2\cdot t = 0$

$v_{11} = -\frac{2}{3}t$

$$\vec{v}_1 = t\cdot\left(-\frac{2}{3}, 1\right) = \left(-\frac{2}{3}, 1\right)$$

$$\lambda_2 = 6: \left(\begin{array}{cc|c} 2-6 & 6 & 0 \\ 6 & -3-6 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -4 & 6 & 0 \\ 6 & -9 & 0 \end{array}\right)\begin{array}{c} :(-2) \\ :3 \end{array} \sim$$

$$\sim \begin{array}{c} \begin{array}{cc} v_{21} & v_{22} \end{array} \\ \left(\begin{array}{cc|c} 2 & -3 & 0 \\ \cancel{2} & \cancel{-3} & \cancel{0} \end{array}\right) \end{array} \quad 2v_{21} - 3t = 0 \Rightarrow v_{21} = \frac{3}{2}t$$

$v_{22} = t$

$$\vec{v}_2 = t\cdot\left(\frac{3}{2}, 1\right) \Rightarrow \vec{v}_2 = \left(\frac{3}{2}, 1\right)$$

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