Příklad 1

Lineární zobrazení $\varphi : V \to V'$ je zadáno předpisem pro vektor $\vec{x} \in V$.

- Najděte matici A zobrazení φ a obrazy standardní báze prostoru V.
- Najděte $\varphi(\vec{u}), \varphi(\vec{v})$.

1 $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(x_1, x_2) = \underline{(2x_1 + x_2, x_2, x_2 - x_1)}$, $\vec{u} = (2, 3)$, $\vec{v} = (-2, 1)$.

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\varphi(\vec{u}) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

$$\varphi(\vec{v}) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$$

2 $\varphi : \mathbb{R}^3 \to \mathbb{R}^4$, $\varphi(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1)$,
$\vec{u} = (4, -1, 0)$, $\vec{v} = (-3, 0, 5)$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\varphi(\vec{u}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \\ 4 \end{pmatrix}$$

$$\varphi(\vec{v}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 2 \\ -3 \end{pmatrix}$$

Příklad 2

Lineární zobrazení $\varphi : V \to V'$ je zadáno obrazy bázových vektorů V.

- Najděte matici A zobrazení φ.
- Najděte $\varphi(\vec{u}), \varphi(\vec{v})$.

1 $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$,
$\varphi(1, 0, 2) = (1, 3)$, $\varphi(-3, 4, -2) = (2, -1)$, $\varphi(0, 2, 1) = (-3, 5)$,
$\vec{u} = (1, 4, 2)$, $\vec{v} = (-1, 0, 4)$.

→ NEZAPOMENOUT OVĚŘIT, ŽE VEKTORY TVOŘÍ BÁZI.

$$A_S = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

1 $A_S \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 3 $A_S \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

2 $A_S \cdot \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

1 $1 \cdot a_{11} + 0 \cdot a_{12} + 2a_{13} = 1$ $1 \cdot a_{21} + 0 \cdot a_{22} + 2a_{23} = 3$

2 $-3a_{11} + 4a_{12} - 2a_{13} = 2$ $-3 \cdot a_{21} + 4a_{22} - 2a_{23} = -1$

3 $0 \cdot a_{11} + 2a_{12} + 1 \cdot a_{13} = -3$ $0 \cdot a_{21} + 2a_{22} + 1 \cdot a_{23} = 5$

$$\begin{array}{c} \\ +3r_1 \\ \\ \end{array} \left(\begin{array}{ccc|c|c} 1 & 0 & 2 & 1 & 3 \\ -3 & 4 & -2 & 2 & -1 \\ 0 & 2 & 1 & -3 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c|c} 1 & 0 & 2 & 1 & 3 \\ 0 & 4 & 4 & 5 & 8 \\ 0 & 2 & 1 & -3 & 5 \end{array}\right) \sim \begin{array}{c} \\ \\ -2r_2 \end{array} \left(\begin{array}{ccc|c|c} 1 & 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & -3 & 5 \\ 0 & 4 & 4 & 5 & 8 \end{array}\right)$$

a_{21}, a_{22}, a_{23}

a_{11}, a_{12}, a_{13}

$$\sim \left(\begin{array}{ccc|c|c} 1 & 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & -3 & 5 \\ 0 & 0 & 2 & 11 & -2 \end{array}\right)$$

1. řádek A: $2a_{13} = 11 \Rightarrow a_{13} = \frac{11}{2}$

$2a_{12} + \frac{11}{2} = -3 \Rightarrow 2a_{12} = -\frac{17}{2} \Rightarrow a_{12} = -\frac{17}{4}$

$a_{11} + \cancel{2} \cdot \frac{11}{\cancel{2}} = 1 \Rightarrow a_{11} = -10$

2. řádek A: $2a_{23} = -2 \rightarrow a_{23} = -1$

$2a_{22} + (-1) = 5 \Rightarrow a_{22} = 3$

$a_{21} + 2 \cdot (-1) = 3 \Rightarrow a_{21} = 5$

$$A = \begin{pmatrix} -10 & -\frac{17}{4} & \frac{11}{2} \\ 5 & 3 & -1 \end{pmatrix}$$

$$\varphi \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -16 \\ 15 \end{pmatrix}$$

$$\varphi \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 32 \\ -9 \end{pmatrix}$$

Příklad 3

Je dána přímka p a rovina ϱ:
$p = \{[1+t, 2-t, 1-t]; t \in \mathbb{R}\}$
$\varrho : 2x - 3y + z + 1 = 0$

Zjistěte, na jakou množinu bodů se přímka p a rovina ϱ zobrazí pomocí lineárního zobrazení:

- $\varphi_1 : \mathbb{R}^3 \to \mathbb{R}^3$, které je zadáno maticí
$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & -3 \end{pmatrix}$

$$\varphi(p) = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1+t \\ 2-t \\ 1-t \end{pmatrix} = \begin{pmatrix} 3\cdot(1+t) + 2\cdot(2-t) + 1\cdot(1-t) \\ 1\cdot(1+t) + \cancel{0\cdot(2-t)} + 2\cdot(1-t) \\ 1\cdot(1+t) + 2\cdot(2-t) - 3\cdot(1-t) \end{pmatrix} = \begin{pmatrix} 8 \\ 3-t \\ 2+2t \end{pmatrix}$$

$$\varphi(p): \begin{aligned} x &= 8 \\ y &= 3-t \\ z &= 2+2t, \ t \in \mathbb{R} \end{aligned}$$

$\varrho : 2x - 3y + z + 1 = 0 \quad \to A[-1,0,1], B[1,1,0], C[0,0,-1]$
$\qquad \overrightarrow{AB} = (2,1,-1), \ \overrightarrow{AC} = (1,0,-2)$

$$\varrho: \begin{aligned} x &= -1 + 2r + s \\ y &= r \\ z &= 1 - r - 2s, \ r,s \in \mathbb{R} \end{aligned}$$

$$\varphi_1(\varrho) = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -1+2r+s \\ r \\ 1-r-2s \end{pmatrix} = \begin{pmatrix} 3\cdot(-1+2r+s) + 2r + 1 - r - 2s \\ -1+2r+s \quad + 2\cdot(1-r-2s) \\ -1+2r+s+2r-3\cdot(1-r-2s) \end{pmatrix} = \boxed{\begin{pmatrix} -2+7r+s \\ 1 \quad -3s \\ -4+7r+7s \end{pmatrix}}$$

$$\left.\varphi_1(\varrho): \begin{aligned} x &= -2 + 7r + s \\ y &= 1 \quad -3s \\ z &= -4 + 7r + 7s \end{aligned}\right\} \begin{aligned} \vec{u} &= (7,0,7) \\ \vec{v} &= (1,-3,7) \\ \vec{u} \times \vec{v} &= (21,-42,-21) = 21\cdot(1,-2,-1) \end{aligned}$$

$\varphi_1(\varrho): x - 2y - z + d = 0$
$\quad [-2; 1; -4]: -2 - 2\cdot 1 - (-4) + d = 0 \Rightarrow d = 0$
$\varphi_1(\varrho): x - 2y - z = 0$

Příklad 4

Nalezněte jádro a obor hodnot lineárního zobrazení φ a určete jejich dimenze.

1. $\varphi : \mathbb{R}^3 \to \mathbb{R}^4, \ \varphi(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1)$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad h(A): \begin{array}{r} \\ \\ -r_1 \\ -r_1 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{array}{r} \\ \\ +r_2 \\ +r_2 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ \cancel{0} & \cancel{0} & \cancel{1} \end{pmatrix}$$

$h(A) = 3$

$\dim \operatorname{Im} \varphi = h(A) = 3$
$\dim \ker \varphi = n - h(A) = \dim \mathbb{R}^3 - h(A) = 3 - 3 = 0$
$\ker \varphi = \{(0,0,0)\} \to \varphi$ je prosté
$\operatorname{Im} \varphi = \langle(1,0,1,1), (1,1,0,0), (0,1,1,0)\rangle$
$\quad \to \dim \mathbb{R}^4 = 4$, ale $\dim \operatorname{Im} \varphi = 3 \Rightarrow \varphi$ není surj.

2. $\varphi : \mathbb{R}^3 \to \mathbb{R}^2, \ \varphi(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \to h(A) = 2$

$\dim \operatorname{Im} \varphi = h(A) = 2$
$\dim \ker \varphi = n - h(A) = \dim \mathbb{R}^3 - h(A) = 3 - 2 = 1$

$A \cdot \vec{x} = \vec{o}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\left.\begin{aligned} x_3 &= t \\ x_2 + t &= 0 \Rightarrow x_2 = -t \\ x_1 + (-t) &= 0 \Rightarrow x_3 = t \end{aligned}\right\} \begin{aligned} K &= \{(t,-t,t), t \in \mathbb{R}\} \\ &= \{t\cdot(1,-1,1), t \in \mathbb{R}\} \end{aligned}$$

$\ker \varphi = \langle(1,-1,1)\rangle$
$\operatorname{Im} \varphi = \langle(1,0),(1,1)\rangle$