$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots \qquad \{a_n : n \geq 1\} \text{ posloupnost čísel}$$

nekonečná

nutná podm. neplatí

$$1+2+3+4+\dots = \sum_{n=1}^{+\infty} n = +\infty$$

$$1+1+1+1+\dots = \sum_{n=1}^{\infty} 1 = +\infty$$

nutná podm. platí

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty \qquad \text{harmonická}$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty \qquad \text{zobec. harmonická}$$

$$1-1+1-1+1+\dots = \sum_{n=1}^{+\infty} (-1)^{n+1}$$

$$S_n \begin{cases} S_{2n} = 0 \\ S_{2n+1} \neq 0 \end{cases} \qquad \lim_{n\to+\infty} S_n \text{ neexistuje}$$

$\sum_{n=1}^{\infty} a_n$, $a_n > 0$: ř. sklad. členy

a_n střídá znam. : ř. je alternující

$\sum_{n=1}^{+\infty} a_n$ $\quad a_n$ rostoucí, kladná $\Rightarrow +\infty$

$$\sum_{n=1}^{\infty} |a_n| < \infty \;\Downarrow\; \text{(def.) konv. abs.}$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n, \quad n = 1, 2, \dots$$

$$\sum_{n=1}^{+\infty} a_n \stackrel{def}{=} \lim_{n\to+\infty} S_n$$

(existuje-li tato limita).

existuje konečná limita: konverguje

$\sum a_n = +\infty \; (-\infty)$ diverguje

$\lim_{n\to+\infty} \sum_{i=1}^{n} a_i$ neexistuje : ř. osciluje

Nutná podmínka konvergence:

$$\sum_{n=1}^{\infty} a_n \text{ konverguje} \Rightarrow \lim_{n\to+\infty} a_n = 0$$

$$\sum_{n=1}^{\infty} \frac{n}{n+2} \quad - ? \qquad a_n = \frac{n}{n+2} \qquad \text{pro } n \to +\infty:$$

$a_n \to 1 \neq 0$
(nutná podm. neplatí)

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{+\infty} a_n$$

konvergují / divergují zároveň

$$\sum_{n=3}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} a_n \qquad a_n > 0 \qquad (\exists n_0 : a_n > 0 \text{ pro } n \geq n_0)$$

$$\frac{a_{n+1}}{a_n} \leq q < 1 \quad \text{pro } n \geq n_0 \quad \Rightarrow \text{konverguje}$$

podílové kritérium (d'Alembertovo kr.)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad a_n = \frac{2^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \to q = 0 < 1$$

v limit. tvaru: $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = q < 1 \Rightarrow$ konv.

$$\lim \frac{a_{n+1}}{a_n} = q > 1 \quad \Rightarrow \text{diverguje}$$

$(= 1 : \ ?)$

$a_n > 0 \quad (n \geq n_0) \qquad \sum_{n=1}^{\infty} a_n$

$\lim_{n\to+\infty} \sqrt[n]{a_n} = q < 1 \Rightarrow$ konverguje

$(> 1 \Rightarrow$ diverguje$)$

(Cauchyho kr. v lim. tvaru) – odmocninové kr.

$$\sum_{n=1}^{+\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)} \qquad a_n = \left(\frac{n-1}{n+1}\right)^{n(n-1)}$$

$a_n > 0$ pro $n \geq 2$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} = \left(\frac{n-1}{n+1}\right)^{\frac{\cancel{n}(n-1)}{\cancel{n}}} =$$

$$= \left(\frac{n-1}{n+1}\right)^{n-1} = \left(\frac{n+1-2}{n+1}\right)^{n-1} =$$

$$= \left(1 - \frac{2}{n+1}\right)^{n-1} = \left(1 - \frac{2}{n+1}\right)^{(n+1)\cdot\frac{n-1}{n+1}} =$$

$$= \left(\left(1 + \frac{-2}{n+1}\right)^{n+1}\right)^{\frac{n-1}{n+1} \to 1} \longrightarrow e^{-2} \quad \text{pro } n \to +\infty$$

$\left(1 + \frac{-2}{n+1}\right)^{n+1} \to e^{-2}$

$\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e$

$\lim_{n\to+\infty} \left(1 + \frac{\alpha}{n}\right)^n = e^{\alpha}$

$e^{-2} = \frac{1}{e^2} < 1$

$\Rightarrow$ ř. konverguje (odmocn. kr.)

$u(x)^{v(x)} \to$ přes ln.

$u(x) \to L_1, \; x \to x_0$

$v(x) \to L_2, \; x \to x_0$

$u(x)^{v(x)} \to L_1^{L_2}$

$u(x)^{v(x)} = e^{\ln\left(u(x)^{v(x)}\right)} =$

$= e^{\boxed{v(x) \ln u(x)}}$

$$\sum_{n=1}^{\infty} \frac{3^n n!}{n^n} \qquad a_n = \frac{3^n n!}{n^n} > 0$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} = 3\cancel{(n+1)} \frac{n^n}{(n+1)^n \cancel{(n+1)}} =$$

$$= 3\left(\frac{n}{n+1}\right)^n = 3\left(\frac{n+1-1}{n+1}\right)^n = 3\left(1 - \frac{1}{n+1}\right)^n =$$

$$= 3\left(1 + \frac{-1}{n+1}\right)^{(n+1)\cdot\frac{n}{n+1} \to 1} \xrightarrow[n\to\infty]{} 3 \cdot e^{-1} = \frac{3}{e} > 1 \Rightarrow \text{diverguje (podíl.)}$$

$\left(1 + \frac{-1}{n+1}\right)^{n+1} \to e^{-1}$

$e \approx 2{,}8$

Zobecn. Cauchyho (odmocn.):

$$\limsup_{n\to+\infty} \sqrt[n]{a_n} = q < 1$$

horní lim.

$\sqrt{2} \approx 1{,}4 \Rightarrow a_n > 0$

$$\sum_{n=1}^{\infty} \frac{n^3\left(\sqrt{2}+(-1)^n\right)^n}{3^n} \qquad a_n = \frac{n^3\left(\sqrt{2}+\boxed{(-1)^n}\right)^n}{3^n}$$

$$\boxed{a_{2k} = \frac{(2k)^3\left(\sqrt{2}+1\right)^{2k}}{3^{2k}}}$$

$$\limsup (-1)^n = 1$$

$$a_{2k-1} = \frac{(2k-1)^3\left(\sqrt{2}-1\right)^{2k-1}}{3^{2k-1}}$$

$$\limsup_{n\to+\infty} \sqrt[n]{a_n} = \lim_{k\to+\infty} \sqrt[2k]{a_{2k}} = \lim_{k\to+\infty} \sqrt[2k]{\frac{(2k)^3\left(\sqrt{2}+1\right)^{2k}}{3^{2k}}} =$$

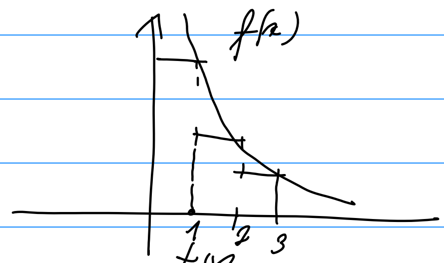
$$= \lim_{k\to+\infty} \frac{\boxed{(2k)^{\frac{3}{2k}}}\left(\sqrt{2}+1\right)}{3} = \frac{\overset{\approx 2{,}4}{\sqrt{2}+1}}{3} < 1$$

$\lim n^{\frac{1}{n}} = 1$

$\Rightarrow$ konverg. (zobecn. podíl.)

Integrální kr.

$$\sum_{n=1}^{\infty} a_n \qquad a_n = f(n)$$



$\sum a_n$ konverguje / diverguje $\Leftrightarrow$ $\int_1^{+\infty} f(x)\,dx$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad f(x) = \frac{1}{x}$$

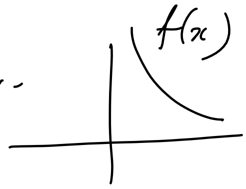
$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to+\infty} (\ln b - \ln 1) =$$

$$= +\infty$$

zobecn. harm. ř.

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

$\alpha > 1$ kon.



$$f(x) = \frac{1}{x^{\alpha}}$$

$$\int_1^{+\infty} \frac{dx}{x^{\alpha}} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^{\alpha}} = \lim_{b \to +\infty} \left. \frac{x^{-\alpha+1}}{-\alpha+1} \right|_1^b$$

$$= \lim_{b \to +\infty} (\underbrace{b^{1-\alpha}}_{\to 0} - 1) \cdot \frac{1}{1-\alpha} = \frac{1}{\alpha - 1} < \infty$$

$$\sum_{n=1}^{\infty} f_n(x) \qquad x \in E \qquad f_n : E \to R \quad n = 1, 2 \ldots$$

funkční

- konvergence bodová ($\forall x \in E$: ř. kon.)
- konverg. stejnoměrná ($s_n(x) \rightrightarrows s(x)$ na E)

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad f_n(x) = a_n (x - x_0)^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$x_0 = 0, \quad a_n = \frac{1}{n!}$$

$x \in E$, $E = ?$

R poloměr konv. mocn. ř.:

$$\frac{1}{R} = \limsup_{n \to +\infty} \sqrt[n]{|a_n|}$$

$E \supset (-R, R)$

$$|x - x_0| < R$$
$$\downarrow$$
kon.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x+1)^{n+1} \qquad \sum f_n(x), \quad f_n(x) = \frac{(-1)^{n+1}}{n}(x+1)^{n+1} =$$

$$= a_{n+1}\boxed{(x-x_0)^{n+1}}$$

$$a_n = \frac{(-1)^n}{n-1} \qquad x_0 = -1$$

mocninná

$$\sqrt[n]{|f_n(x)|} = \sqrt[n]{\frac{|x+1|^{n+1}}{n}} = \frac{\sqrt[n]{|x+1|^n \cdot |x+1|}}{\sqrt[n]{n}} = \frac{|x+1| \cdot \sqrt[n]{|x+1|}}{\sqrt[n]{n}}$$

$\sqrt[n]{|x+1|} \to 1$, $\sqrt[n]{n} \to 1$

pro $n \to +\infty$:

$$\lim_{n\to+\infty} \sqrt[n]{|f_n(x)|} = |x+1|$$

Cauchyho odmocn. kr.:

$|x+1| < 1 \Rightarrow$ konverguje $\qquad$ ($|x+1| > 1 \Rightarrow$ $\Rightarrow$ diverg.)

$\Updownarrow$

$-1 < x+1 < 1$

$-2 < x < 0 \qquad x \in (-2, 0)$

$x = -2$: $\quad f_n(-2) = \frac{(-1)^{n+1}(-1)^{n+1}}{n} = \frac{1}{n}$

$$\sum_{n=1}^{\infty} f_n(-2) = \sum_{n=1}^{\infty} \frac{1}{n} \text{ – harmon., diverguje}$$

$x = 0$: $\quad f_n(0) = \frac{(-1)^{n+1}}{n}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Leibniz. k.

konverguje.

$(-2, 0\rangle$ obor konv.

$\sum \frac{(-1)^{n+1}}{n}$ konv.

$\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n}$ diverg.

$\sum (-1)^n b_n \quad 0 < b_n \searrow 0$

$\Downarrow$ konverguje

Leibn. kr.

$\to$ řady konv. relativně (neabsolutně).

$\sum a_n \,, \sum b_n \qquad 0 \leq a_n \leq b_n$

1. $\sum b_n$ konv. $\Rightarrow \sum a_n$ konv.

2. $\sum a_n$ divers. $\Rightarrow \sum b_n$ div.

$$\sum_{n=1}^{\infty} \frac{1}{n^3+60n^2+1} \quad \text{konv.}$$

$$\frac{1}{n^3+\boxed{60n^2+1}} < \frac{1}{n^3}$$

$$\frac{1}{n(n-1)} < \frac{1}{(n-1)^2} \qquad \sum \frac{1}{n(n-1)} \text{ konv.}$$

$\sum a_n, \ \sum b_n \qquad a_n > 0, \ b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0 \Rightarrow \quad \sum a_n \text{ konv./div.} \Leftrightarrow \sum b_n \text{ konv./div.}$$