Rotační těleso $\qquad y^2 = x \qquad O_x, \quad x = 2$

$y = \pm\sqrt{x}$

$f(x) = \sqrt{x}$

$$\boxed{S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx}$$

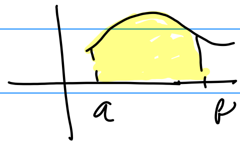
$$f'(x) = \frac{1}{2\sqrt{x}} = (\sqrt{x})' \to (f'(x))^2 = \left(\frac{1}{2\sqrt{x}}\right)^2 = \frac{1}{4x}$$

$$S = 2\pi\int_0^2 \underbrace{\sqrt{x}}_{f(x)} \cdot \underbrace{\sqrt{1+\frac{1}{4x}}}_{\sqrt{1+f'^2}}\,dx = 2\pi\int_0^2 \sqrt{x}\cdot\sqrt{\frac{4x+1}{4x}}\,dx =$$

$$= \cancel{2}\pi\int_0^2 \cancel{\sqrt{x}}\cdot\frac{\sqrt{4x+1}}{\cancel{2}\cancel{\sqrt{x}}} = \pi\int_0^2\sqrt{4x+1}\,dx =$$

$$= \frac{\pi}{4}\int_0^2 \sqrt{4x+1}\,d(4x+1) = \frac{\pi}{4}\int_0^2 (4x+1)^{\frac{1}{2}}\,d(4x+1) \quad \Big| \int t^{\alpha}\,dt, \quad t = 4x+1$$

$$= \frac{\pi}{4}\,\frac{(4x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_0^2 = \frac{\pi}{4}\cdot\frac{2}{3}\cdot\left(9^{3/2}-1\right) = \frac{\pi}{6}\left(3^3-1\right).$$

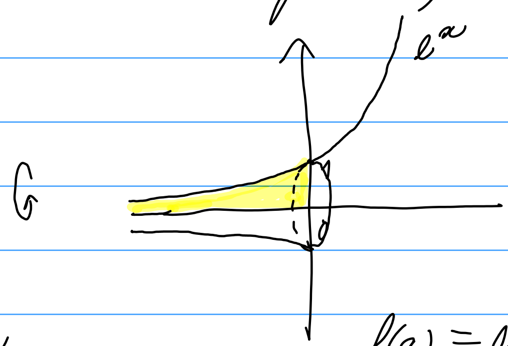


$V = \int_a^b S(x)\,dx$

$$V = \pi\int_a^b f^2(x)\,dx \longrightarrow$$

$$V = \pi\int_0^2 (\sqrt{x})^2\,dx = \pi\int_0^2 x\,dx = \pi\cdot\frac{2^2}{2} = 2\pi$$

Rotační těleso : $y = e^x$, $x<0$ $\circlearrowleft x$



$$\boxed{V = \pi \int_a^b (f(x))^2 \, dx}$$

$f(x) = e^x$, $b = 0$, $a = -\infty$
(integrál nevlastní I. typu – neomezený interval)

$$V = \pi \int_{-\infty}^{0} (e^x)^2 dx = \pi \int_{-\infty}^{0} e^{2x} dx =$$

$$= \pi \lim_{a \to -\infty} \underline{\int_a^0 e^{2x} dx} = \pi \lim_{a \to -\infty} \frac{1}{2} \int_a^0 d(e^{2x}) =$$

$\to 0$ pro $a \to -\infty$

$$= \frac{\pi}{2} \lim_{a \to -\infty} (e^0 - e^{2a}) = \frac{\pi}{2} \lim_{a\to -\infty} (1 - e^{2a}) = \underline{\frac{\pi}{2}}.$$

$$S = 2\pi \int_{-\infty}^{0} e^{2x} \sqrt{1 + (2e^{2x})^2} \, dx =$$

$d(2e^{2x}) = 2 \cdot 2 e^{2x} dx = 4 e^{2x} dx$

$$= 2\pi \cdot \frac{1}{4} \cdot \int_{-\infty}^{0} \sqrt{1 + (2e^{2x})^2} \, d(2e^{2x})$$

$$\int \sqrt{1 + t^2} \, dt$$

↳ integrál nevlastní I. typu (neohraničený interval)

Rotační těleso: $y = x \cdot e^{-x}$, $y=0$

$f(x) = x e^{-|x|}$

$f(0)=0$

pro $x \to +\infty$: $\lim\limits_{x\to+\infty} \underset{\to +\infty}{x} \underset{\to 0}{e^{-x}} = 0$ $\quad \infty \cdot 0$

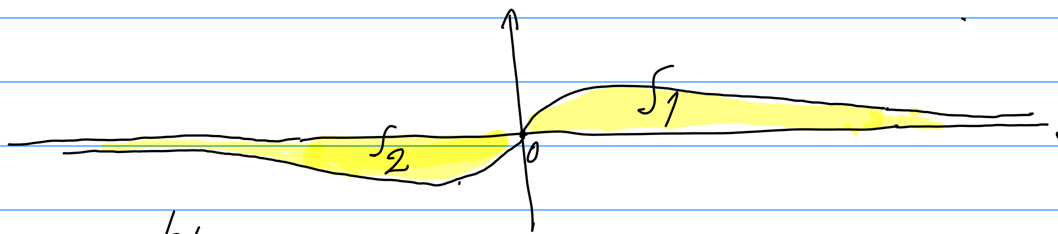
l'Hôpit. pr.

$x e^{-x} = \dfrac{x}{e^x} \overset{\frac{\infty}{\infty}}{} \dfrac{x'}{(e^x)'} = \dfrac{1}{e^x} \to 0$ pro $x \to +\infty$

$\downarrow 0$ potom také

Pro $x \to -\infty$: $\lim\limits_{x\to-\infty} x e^{-|x|} = 0$

$\hookrightarrow$ $y=0$ (osa x) je vodoz. asym.



$f(x) = x e^{-|x|}$

$x>0$: $f(x)>0$

$f(-x) = -x e^{-|-x|} = -x e^{-|x|} = -f(x)$ $\to$ f je lichá

$S_1 = S_2$ (souměrnost)

$S = S_1 + S_2 = 2S_1 = 2 \cdot \int_0^{+\infty} f(x)\,dx = 2\int_0^{+\infty} \underbrace{x e^{-|x|}}_{\text{zde } x \geq 0} dx =$

$= 2\int_0^{+\infty} x e^{-x}\,dx$ – nevlastní integrál I. typu

$\int_0^{+\infty} x e^{-x}\,dx = \lim\limits_{b\to+\infty} \boxed{\int_0^b x e^{-x}\,dx}$

neurč. integrál

$\int x e^{-x}\,dx = ?$

$$\cancel{\left| \begin{matrix} t = x \\ dt = dx \end{matrix}\right.}$$

$$\cancel{\sum_{n=1}^{10} \frac{1}{n} = \sum_{i=1}^{10} \frac{1}{i}}$$

$$\int_0^b x e^{-x}\,dx = \left| \begin{matrix} u = x, & u' = 1 \\ v' = e^{-x}, & v = -e^{-x} \end{matrix} \right|$$

$$= -xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx =$$

$$= -be^{-b} + 0 - \int_0^b d(e^{-x}) = -be^{-b} - (e^{-x})\Big|_0^b =$$

$$= -\underbrace{be^{-b}}_{=\frac{b}{e^b} \to 0} - \underbrace{e^{-b}}_{\to 0} + 1 \xrightarrow[b \to +\infty]{} 1 \Rightarrow \int_0^{+\infty} x e^{-x}\,dx = 1$$

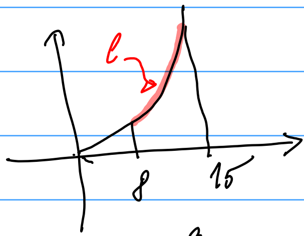
Так $\int = 2 \cdot \int_0^{+\infty} x e^{-x}\,dx = 2.$

Křivka: $y = \frac{2}{3}x\sqrt{x}$, $8 < x < 15$

$$\boxed{l = \int_a^b \sqrt{1+f'(x)^2}\,dx}$$

$f(x) = \frac{2}{3}x\sqrt{x} = \frac{2}{3}x^{\frac{3}{2}}$, $\frac{3}{2} > 1$

$$l = \int_8^{15}\sqrt{1+x}\,dx = \quad \Big| \; d(x+1) = dx \text{ konst.!}$$



$f'(x) = \cancel{\frac{2}{3}}\cdot\cancel{\frac{3}{2}}x^{\frac{3}{2}-1} = x^{\frac{1}{2}} = \sqrt{x}$

$$= \int_8^{15}\sqrt{1+x}\,d(x+1) \to \int \sqrt{t}\,dt = \int t^{1/2}\,dt$$

$$= \int_8^{15}(x+1)^{\frac{1}{2}}\,d(x+1) = \frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_8^{15}$$

$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1}$ $(\alpha \neq -1)$

$$= \frac{2}{3}(x+1)^{3/2}\Big|_8^{15} = \frac{2}{3}\left[16^{3/2} - 9^{3/2}\right] = \frac{2}{3}\left[4^3 - 3^3\right].$$

Subst.: $x+1 = t$, $dt = dx$, meze: $x = 8 \to t = 9$, $x = 15 \to t = 16$

$$\int_9^{16}\sqrt{t}\,dt = \ldots$$

$$\int_{-\pi}^{\frac{\pi}{2}} \cos^3 x \underbrace{\sin x\,dx}_{-d(\cos x)} = ?$$

$(\cos x)' = -\sin x$, $d(\cos x) = -\sin x\,dx$

$\int t^3\,dt$

$$= -\int_{-\pi}^{\frac{\pi}{2}} \cos^3 x\,d(\cos x) \quad \Big| \; t = \cos x, \; dt = -\sin x\,dx$$

$x = -\pi$: $t = \cos(-\pi) = -1$

$x = \frac{\pi}{2}$: $t = \cos\frac{\pi}{2} = 0$

$$= -\int_{-1}^{0} t^3\,dt = -\frac{t^4}{4}\Big|_{-1}^{0} = -0 + \frac{1}{4} = \frac{1}{4}.$$

$$\int \cos^3 x \sin^2 x\,dx = ? \quad \Big|$$

p.p.

$u = \sin^2 x$, $u' = 2\sin x\cos x$

$v' = \cos^3 x$

$v' = \cos^3 x$, $\quad v = \int \cos^3 x\, dx \overset{?}{=} \int \cos^2 x \cos x\, dx =$

$= \int (1-\sin^2 x)\cos x\, dx = \underbrace{\int \cos x\, dx}_{\text{tabulka}} - \int \sin^2 x \overset{d(\sin x)}{\boxed{\cos x\, dx}}$

$(\sin x)' = \cos x$

$d(\sin x) = \cos x\, dx$

$\int s^2\, ds$

Mocninná řada

$\sum_{n=1}^{\infty} \frac{(x+1)^n}{(n+2)4^n} \to$ řada mocninná

$\sum a_n (x-x_0)^n$

$a_n = \frac{1}{(n+2)4^n}$ $\quad x_0 = -1$

$\sum_{n=1}^{\infty} a_n$ – řada číselná

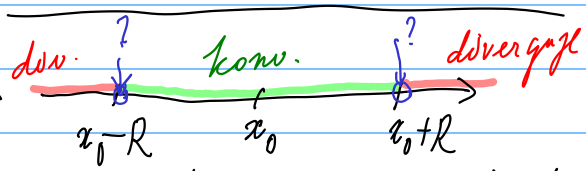
$\sum \frac{1}{n}$ – řada harmonická (diverguje, $=+\infty$)

$\sum_{n=1}^{\infty} f_n(x)$ – funkční řada

obor konvergence $=$

$= \{x :$ řada konverguje$\}$

Mocninná řada: $f_n(x) = a_n(x-x_0)^n$



$|x-x_0| < R \Rightarrow$ řada konverguje absolutně (tj. konvergují i řada z abs. hodnot)

$|x-x_0| > R \Rightarrow$ řada diverguje

R poloměr konvergence

$R=0$ – řada konv. pouze pro $x=x_0$ (triviální př.)

$R>0$ – viz. obrázek

$R=+\infty \Rightarrow (-\infty, \infty)$ – interv. konv.

$$\frac{1}{R} = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| \quad \text{(existuje-li limita)}$$

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{(n+2)4^n} = \sum_{n=1}^{\infty} f_n(x), \quad \text{kde} \quad f_n(x) = \frac{(x+1)^n}{(n+2)4^n}$$

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{(x+1)^{n+1}}{(n+3)\,4^{n+1}}\cdot\frac{(n+2)\,4^n}{(x+1)^n}\right| \;(=)$$

(=) používáme podílové kritérium pro řadu $\sum |f_n(x)|$,

$$(=)\; |x+1|\cdot\frac{n+2}{n+3}\cdot\frac{1}{4} \xrightarrow[n\to+\infty]{} |x+1|\cdot\frac{1}{4}$$

Podílové krit.

$\sum u_n$, $u_n \geq 0$, číselná řada

$\lim\limits_{n\to+\infty}\frac{u_{n+1}}{u_n} = q$

$q<1 \to$ konv.

$q>1 \to$ div.

$\boxed{\lim\limits_{n\to+\infty}\frac{n+2}{n+3} = 1}$

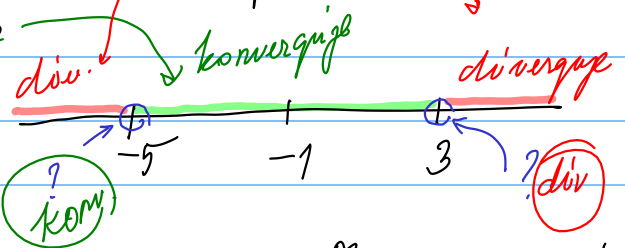
Konverguje, je-li $|x+1|\cdot\frac{1}{4} < 1$

diverguje: $|x+1|\cdot\frac{1}{4} > 1$.

Konvergence: $|x+1| < 4$.

$-4 < x+1 < 4$

$-5 < x < 3$



$x = 3$: $\sum\limits_{n=1}^{\infty}\frac{(x+1)^n}{(n+2)\,4^n}$

$$\sum_{n=1}^{\infty}\frac{\cancel{4^n}}{(n+2)\cancel{4^n}} = \sum_{n=1}^{\infty}\frac{1}{n+2} \to \text{diverguje}$$

$\sum\limits_{n=1}^{\infty}\frac{1}{n}$ řada harm. diverguje.

$x = -5$: $\sum\limits_{n=1}^{\infty}\frac{(-4)^n}{(n+2)\,4^n} = \sum\frac{(-1)^n\cancel{4^n}}{(n+2)\cancel{4^n}} = \sum\frac{(-1)^n}{n+2}$ — konverguje

podle Leibnizova krit.:

$\sum(-1)^n b_n$, $0 < b_n \searrow 0$

$\Rightarrow \sum(-1)^n b_n$ konverg. Leibn. kr.

$0 < b_n = \frac{1}{n+2} \searrow 0$

V bodě $x=-5$ řada konverguje (neabsolutně, tj. relativně, neboť řada z abs. hodnot je: $\sum\frac{1}{n+2}$ — diverguje.)