$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$ $\{a_n : n \geq 1\}$ posloupnost čísel

⊖ $1+2+3+\dots = \sum_{n=1}^{+\infty} n = +\infty$ div.

⊖ $1+1+1+\dots = \sum_{n=1}^{+\infty} 1 = +\infty$ div.

⊕ $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ div. ← harmonická ř.

⊕ $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{n^2}+\dots = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$ konv. ← zobecn. harm. ř.

⊖ $1-1+1-1+\dots = \sum_{n=1}^{\infty} (-1)^{n+1}$ osc. $s_{2k} = 0$
$s_{2k+1} = 1$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} a_i}$$

(existuje-li limita)

$\sum_{i=1}^{n} a_i = a_1 + \dots + a_n = s_n$ částečné součty

- limita existuje a je konečná — konvergují
- limita existuje a je nevlastní (tj. $\pm\infty$) — divergují
- limita neexistuje. — osciluje

Nutná podmínka pro konvergenci č. ř.:

$$\sum_{n=1}^{\infty} a_n \text{ konverguje} \Rightarrow \boxed{\lim_{n\to+\infty} a_n = 0}$$

[$s_n = a_1 + \dots + a_n$, $s_{n+1} = a_1 + \dots + a_n + a_{n+1} =$
$= s_n + a_{n+1}$.

$$\underset{\substack{\downarrow \\ s}}{s_{n+1}} - \underset{\substack{\downarrow \\ s}}{s_n} = a_{n+1} \to 0 \text{ , } n\to\infty$$]

$\sum_{n=1}^{+\infty} a_n$ řada s kl. čl. : $a_n > 0$ pro $n \geq n_0$

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{N} a_n + \underbrace{\sum_{n=N+1}^{+\infty} a_n}$$

$a_n > 0$ $\qquad \sum_{n=1}^{+\infty} a_n$

$\frac{a_{n+1}}{a_n} \le q < 1 \Rightarrow$ ř. konv.

(d'Alembertovo)

$\left\{ \begin{array}{l} \lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = q < 1 \Rightarrow \text{ř. konv.} \\ \qquad '' \quad = q > 1 \Rightarrow \text{ř. diverguje} \end{array} \right.$

(podílové krit. v limitním tvaru)

($q = 1$ — ?)

$\sum_{n=1}^{+\infty} \frac{1}{n!}$ $\qquad a_n = \frac{1}{n!} > 0$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{n!}{(n+1)!} = \frac{\cancel{n!}}{\cancel{n!}(n+1)} = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0 < 1$$

$\Rightarrow$ podíl. kr. konverg.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$

$\sum_{n=1}^{\infty} 2^n$ $\qquad a_n = 2^n \to +\infty$ $\qquad a_n \not\to 0$ $\Rightarrow$ neplatí ani nutná podm. (diverguje).

$\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^n$

$$0 < a_n = \left(\frac{n}{n+2}\right)^n = \left(\frac{n+2-2}{n+2}\right)^n = \left(1 - \frac{2}{n+2}\right)^n =$$

$$= \left[\left(1 + \frac{-2}{n+1}\right)^{n+1}\right]^{\cdot \frac{n}{n+1}} \to e^{-2}, n\to\infty$$

$\lim a_n \neq 0$.

neplatí nutná podm. konv.

$a_n > 0$ $\qquad \sum a_n$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = q$$

$q < 1$: konv.

$q > 1$: div.

odmocninové (Cauchyho) krit.

$$\sum_{n=1}^{+\infty} \left(\frac{n-1}{n+1}\right)^{\underbrace{n(n-1)}_{}} \qquad a_n = \left(\frac{n-1}{n+1}\right)^{n(n-1)} \qquad n \geq 2$$

$a_n > 0$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} = \left(\frac{n-1}{n+1}\right)^{n\cdot(n-1)\cdot\frac{1}{n}} = \left(\frac{n-1}{n+1}\right)^{n-1} =$$

$$= \left(\frac{n+1-2}{n+1}\right)^{n-1} = \left(1-\frac{2}{n+1}\right)^{n-1} =$$

$$= \underset{e^{-2}}{\boxed{\left(1+\frac{-2}{n+1}\right)^{n+1}}}^{\overset{\to 1}{\frac{n-1}{n+1}}} \longrightarrow e^{-2}, \; n \to +\infty$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}\cdot a} = e^a$$

$$u^v = e^{\ln u^v} = e^{v \ln u}$$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \frac{1}{e^2} < 1 \qquad e > 1$$

$\Rightarrow$ Cauchyho odm. kr. konv.

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \qquad c > 0$$

$$a_n = \frac{c^n}{n!} > 0$$

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{(n+1)!} \cdot \frac{n!}{c^n} = c \cdot \frac{1}{n+1} \to 0, \; n \to +\infty$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$$

$\Rightarrow$ podle podíl. kr. konv.

$$\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n} \qquad a_n = \frac{3^n n!}{n^n} > 0$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} = 3\cancel{(n+1)} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{\cancel{n+1}} =$$

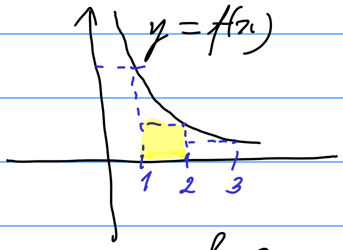
$$= 3\left(\frac{n}{n+1}\right)^n = 3\left(\frac{n+1-1}{n+1}\right)^n = 3\left(1-\frac{1}{n+1}\right)^n =$$

$$= 3\left(1+\frac{-1}{n+1}\right)^{n+1 \cdot \frac{n}{n+1}} \rightarrow 3e^{-1}, \quad n \to +\infty$$

$$e \approx 2{,}8 < 3 \qquad 3e^{-1} > 1$$

podíl. kr. ⇒ ř. diverguje

$$\sum_{n=1}^{\infty} a_n \qquad a_n > 0$$



Integrální kr.

$$a_n = f(n) \qquad f \geq 0$$
$$f \searrow 0$$

$$\sum_{n=1}^{+\infty} a_n \text{ konverg./diverguje} \Leftrightarrow \int_1^{+\infty} f(x)\,dx \text{ konv./div.}$$

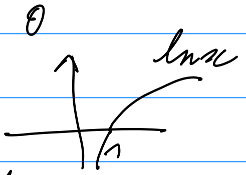
$$\sum_{n=1}^{+\infty} \frac{1}{n} \quad \text{harm.} \qquad \frac{1}{n} = f(n), \qquad f(x) = \frac{1}{x}$$

div.

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} (\ln b - \underbrace{\cancel{\ln 1}}_{0}) =$$

div.

$$= \lim_{b \to +\infty} \ln b = +\infty$$



rostoucí, neomez.

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \qquad f(x) = \frac{1}{x^2}$$

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^2} = \lim_{b \to +\infty} -\frac{1}{x}\Big|_1^b =$$

$$= \lim_{b \to +\infty} \left(\underbrace{\left(-\frac{1}{b}\right)}_{\to 0} + 1\right) \Rightarrow \text{konverguje (int. kr.)}$$

$$\boxed{\sum_{n=2}^{+\infty} \frac{1}{n(n-1)}} \quad - \, ?$$

$$\frac{1}{x(x-1)} = f(x)$$

$$\int_2^{+\infty} \frac{dx}{x(x-1)} = \int_2^{+\infty} \frac{dx}{x^2-x}$$

$$\underbrace{\frac{1}{n(n-1)}}_{a_n} < \overbrace{\boxed{\frac{1}{(n-1)^2}}}^{b_n} \qquad n(n-1) > (n-1)(n-1) = (n-1)^2$$

$n \geq 2$

Srovn. kr. $0 < a_n \leq b_n$ $(n \geq n_0)$

$\sum b_n$ konv. $\Rightarrow$ $\sum a_n$ konv.

$\sum a_n$ div. $\Rightarrow$ $\sum b_n$ div.

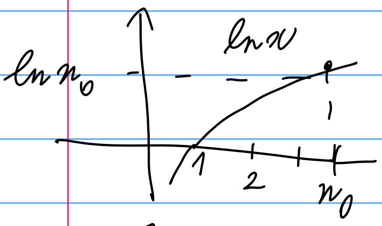
$$\sum b_n = \sum \frac{1}{(n-1)^2} \qquad \sum \frac{1}{n^2}$$

$$\sum_{n=1}^{+\infty} \frac{\ln n}{\sqrt[5]{n}}$$

$a_n = \frac{\ln n}{\sqrt[5]{n}} > 0$ pro $n > 1$

$\frac{1}{\sqrt[5]{n}} = \frac{1}{n^{1/5}}$ $\qquad \frac{1}{5} < 1$

pomaleji než $\frac{1}{n}$



$n > n_0$: $\ln n > \ln n_0$

$\ln x \nearrow +\infty$

$\frac{\ln n}{\sqrt[5]{n}} > \frac{\ln n_0}{\sqrt[5]{n}}$ pro $n > n_0$

$0 < a_n \leq b_n$

$$\sum_{n=n_0+1}^{+\infty} \frac{\ln n_0}{\sqrt[5]{n}} = \ln n_0 \cdot \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{1/5}}$$

↓ diverguje podle integr. kr.

$\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^{1/5}}$ ...

$$\sum_{n=1}^{+\infty} \frac{1}{n^x}$$

$\sum_{n=1}^{+\infty} f_n(x)$ ř. funkční

$x = x_0 \to$ konv. / div. / oscil.

— konv. bodově na E

$\forall x \in E: \sum f_n(x)$ konv.

— konv. stejnom. na E

$[\![s_n(x) \rightrightarrows s(x)$ na E

$$f_n(x) = a_n (x-x_0)^n$$

$$\sum_{n=0}^{+\infty} a_n (x-x_0)^n \quad \text{ř. mocninná}$$

R $\qquad \frac{1}{R} = \limsup_{n\to+\infty} \sqrt[n]{|a_n|}$

$|x-x_0| < R$ absolutně

$|x-x_0| > R \to$ diverguje

$\sum b_n$

Pokud $\sum |b_n| < +\infty$

$$\sum_{n=1}^{+\infty} \frac{(n+1)!\,(x+3)^n}{2n} \qquad a_n = \frac{(n+1)!}{2n} \qquad x_0 = -3$$

$$R = \lim_{n\to+\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

(existuje-li limita)

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)!}{\cancel{2}n} \cdot \frac{\cancel{2}(n+1)}{(n+2)!} = \underbrace{\boxed{\frac{n+1}{n}}}_{\to 1} \cdot \underbrace{\boxed{\frac{1}{n+2}}}_{\to 0} \longrightarrow 0, \quad n \to +\infty$$

$\Rightarrow R = 0 \qquad x: \ |x+3| < R$

$$x = -3: \quad \frac{(n+1)!}{2n} \cdot 0^n = 0$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x+2)^n}{\sqrt{n}} \qquad a_n = \frac{(-2)^n}{\sqrt{n}} \qquad x_0 = -2$$

$$R = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| \; ; \quad \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(-2)^n}{\sqrt{n}} \cdot \frac{\sqrt{n+1}}{(-2)^{n+1}}\right| =$$

$$= \sqrt{\frac{n+1}{n}} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}\sqrt{\frac{n+1}{n}} = \frac{1}{2}\sqrt{1+\underset{\downarrow 0}{\boxed{\frac{1}{n}}}} \to \frac{1}{2}$$

$n \to +\infty$

$R = \frac{1}{2}$ $\qquad |x-x_0| < R$ znamená:

$$|x+2| < \frac{1}{2} \;, \quad -\frac{1}{2} < x+2 < \frac{1}{2}$$

$$\boxed{-\frac{5}{2}} < x < -\frac{3}{2}$$

Ř. konverguje (abs.) pro ↗

$|x+2| > \frac{1}{2}$ → diverg.

$x = -\frac{5}{2}$: $\qquad x+2 = -\frac{5}{2} + \frac{4}{2} = -\frac{1}{2}$

$$\sum \frac{(-2)^n \left(-\frac{1}{2}\right)^n}{\sqrt{n}} = \sum \frac{(-1)^n \cancel{2^n}\, \cancel{2^{-n}} (-1)^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$$

diverguje (integr. kr.)

$\int_1^{+\infty} \frac{dx}{\sqrt{x}}$ diverguje

$x = -\frac{3}{2}$ $\qquad \sum \frac{(-2)^n \left(-\frac{3}{2}+2\right)^n}{\sqrt{n}} = \sum \frac{(-2)^n 2^{-n}}{\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}} =$

$= -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \dots$ $\qquad$ Leibn. typu — alternující

$\sum \frac{(-1)^n}{\sqrt{n}} = \sum (-1)^n b_n$

$b_n = \frac{1}{\sqrt{n}}$

↓ konverguje.

Leibn. krit.:

$\sum (-1)^n b_n$, kde $b_n \searrow 0$

↳ konverguje

Obor konv.: $\left(-\frac{5}{2} < x \le -\frac{3}{2}\right)$

$\sum \frac{(-1)^n}{\sqrt{n}}$ konverguje neabsolutně (= relativně)

tj. $\sum a_n$ konv., ale $\sum |a_n|$ nikoliv.

$\sum \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum \frac{1}{\sqrt{n}}$ diverguje.

$$\sum_{n=1}^{+\infty} \frac{1}{\underbrace{\sqrt[3]{n^5}+5\ln n}} \quad - ?$$

$\sum \frac{1}{n^2}$

$\sum \frac{1}{n^\alpha}$ $\alpha>1$ konv. (zobec. harm.)

$$\frac{1}{\sqrt[3]{n^5}+5\ln n} = \frac{1}{n^{5/3}+5\ln n}$$

$\alpha = \frac{5}{3} > 1$

$$0 < \frac{1}{n^{5/3}\boxed{+5\ln n}} < \frac{1}{n^{5/3}}$$

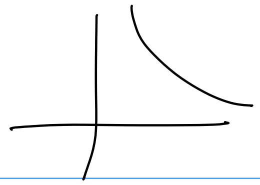
srovn. krit. ← $\sum \frac{1}{n^{5/3}}$ konverg.

$0 < a_n$, $0 < b_n$ $\lim\limits_{n\to+\infty} \frac{a_n}{b_n} = c \neq 0 \Rightarrow$

$\Rightarrow \sum a_n$ konv/div. $\Leftrightarrow \sum b_n$ konv/div.

$a_n =$ přív. ř., $b_n = \frac{1}{n^{5/3}}$

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)\ln^2(n+2)}$$



$$f(x) = \frac{1}{(x+2)\ln^2(x+2)}$$

$$\int_1^{+\infty} \frac{dx}{(x+2)\ln^2(x+2)} = \int_1^{+\infty} \frac{d\left(\ln(x+2)\right)}{\ln^2(x+2)} =$$

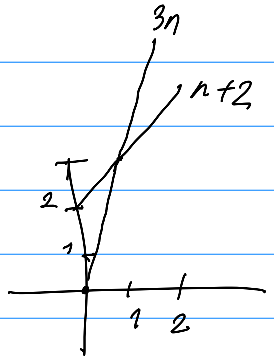
$$\boxed{d\left(\ln(x+2)\right) = \frac{1}{x+2}\,dx}$$

$$\int \frac{dt}{t^2} = -\frac{1}{t}$$

$$= \lim_{b\to+\infty} -\frac{1}{\ln(x+2)}\Big|_1^b = \lim_{b\to+\infty}\left(\underbrace{-\frac{1}{\ln(b+2)}}_{\to 0} + \frac{1}{\ln 3}\right) =$$

$= \frac{1}{\ln 3} < +\infty \Rightarrow$ konverguje.

$\ln x \nearrow +\infty$



$$\sum_{n=1}^{+\infty} \frac{1}{3n\ln^2(n+2)}$$

$n > 2 \qquad 3n > n+2$

$$3n\ln^2(n+2) > (n+2)\ln^2(n+2)$$

$$\frac{1}{3n\ln^2(n+2)} < \frac{1}{(n+2)\ln^2(n+2)}$$

↑ předch. př. + srovn. krit.

$$\sum_{n=1}^{+\infty}\boxed{\frac{7^n x^n}{(n+2)^2}} = f_n(x)$$

$a_n = \frac{7^n}{(n+2)^2}$ $\quad x_0 = 0$

$R = \lim \left|\frac{a_n}{a_{n+1}}\right|$

↳ odmocn. / podílové.

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{7^{n+1}x^{n+1}}{(n+3)^2}\cdot\frac{(n+2)^2}{7^n x^n}\right| = 7\underbrace{\left(\frac{n+2}{n+3}\right)^2}_{\downarrow 1}|x| \xrightarrow[n\to+\infty]{} 7|x|$$

Podílové kr.: $7|x|<1$ – konv.

$7|x|>1$ – div.

Konv. $|x|<\frac{1}{7}$, $\quad -\frac{1}{7}<x<\frac{1}{7}$.

$\left(R=\frac{1}{7}\right)$

$x=-\frac{1}{7}$: $\quad \sum\frac{7^n(-1/7)^n}{(n+2)^2} = \sum\frac{(-1)^n}{(n+2)^2}$

$x=\frac{1}{7}$: $\quad \sum\frac{7^n(1/7)^n}{(n+2)^2} = \sum\frac{1}{(n+2)^2}$ → konvergují podle int. kr. (zobecn. harm. ř.)

↑ konv. absolutně

Obor konv: $\left\langle -\frac{1}{7}, \frac{1}{7}\right\rangle$.

$$\sum \boxed{\frac{n\,(x-1)^{2n}}{9^n\sqrt[n]{n^5+2}}} = f_n(x)$$

mocninná

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{(n+1)(x-1)^{2n+2}}{9^{n+1}\sqrt[n+1]{(n+1)^5+2}}\cdot\frac{9^n\sqrt[n]{n^5+2}}{n\,(x-1)^{2n}}\right| =$$

$$= \underbrace{\frac{n+1}{9n}}_{\downarrow 1}\underbrace{\frac{\sqrt[n]{n^5+2}}{\sqrt[n+1]{(n+1)^5+2}}}_{\to 1}|x-1|^2 \longrightarrow \frac{(x-1)^2}{9};$$

Konv. $\quad |x-1|<3$

$R=3$