$a_1+a_2+a_3+\dots = \sum_{n=1}^{+\infty} a_n$ $\quad \{a_n : n\geq 1\}$ posloupnost čísel.

nekonečná

$\ominus$ $1+2+3+4+\dots = \sum_{n=1}^{+\infty} n = +\infty$

$\ominus$ $1+1+1+\dots = \sum_{n=1}^{+\infty} 1 = +\infty$

$\oplus$ div. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\dots = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$

$\oplus$ konv. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$

$$\sum_{n=1}^{+\infty} a_n \overset{\text{def}}{=} \lim_{n\to+\infty} s_n \qquad s_n = a_1+a_2+\dots+a_n$$

(existuje-li limita)

1. limita existuje a je vlastní (konverguje)
2. existuje lim. nevlastní (tj. $\pm\infty$) (diverguje)
3. limita neexistuje (osciluje)

$$1-1+1-1+1-\dots = \sum_{n=1}^{\infty}(-1)^{n+1}$$

$n=2k$: $s_{2k}=0 \to 0$ $\qquad$ limita $\nexists$.

$s_{2k+1}=1 \to 1$ $\qquad$ (ř. osciluje).

Nutná podmínka konvergence č. ř.

$$\sum_{n=1}^{+\infty} a_n \text{ konverguje} \Rightarrow \lim_{n\to+\infty} a_n = 0$$

$$s_n = a_1+a_2+\dots+a_n = s_{n-1}+a_n$$

$$s_{n-1} = a_1+a_2+\dots+a_{n-1}$$

$$0 \leftarrow \overset{s}{s_n} - \overset{s}{s_{n-1}} = a_n \to 0$$

Předp. že $\sum_{n=1}^{\infty} a_n = s$

$$\sum_{n=0}^{+\infty} \frac{1}{2^n} = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}}$$

$\sum q^n$ $|q|<1$

geometr.

Podílové kritérium:

$\sum_{n=1}^{+\infty} a_n$ $a_n > 0$ (ř. s kladnými členy) || $n \geq n_0$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = q < 1 \Rightarrow \text{ř. konverguje}$$

(d'Alembertovo) $q > 1 \Rightarrow$ ř. diverguje.

($q = 1$: není odpověď)

$\sum \left(\frac{1}{2}\right)^n$ $a_n = \frac{1}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{1}{2^{n+1}} \bigg/ \frac{1}{2^n} = \frac{1}{2^{n+1}} \cdot \frac{2^n}{1} = \frac{1}{2} < 1$$

$\rightarrow$ konverguje.

Odmocninové kritérium (Cauchyho kr.)

$\sum_{n=1}^{\infty} a_n$, $a_n > 0$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = q$$

$q < 1 \Rightarrow$ konverg.

$q > 1 \Rightarrow$ div.

$a_{n+1} = \frac{1}{(n+1)!}$

$\sum_{n=1}^{\infty} \frac{1}{n!} - a_n$ $\cancel{\frac{1}{\sqrt[n]{n!}}}$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{n!}{(n+1)!} = \frac{\cancel{n!}}{(n+1)\cdot \cancel{n!}} = \frac{1}{n+1} \rightarrow 0 = q < 1$$

ř. kon.

$$\sum_{n=1}^{+\infty} \frac{3^n \cdot n!}{n^n} \qquad a_n = \frac{3^n n!}{n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n \cdot n!} = \frac{3(n+1)\cancel{n!}\; n^n}{(n+1)^{n+1}\;\cancel{n!}} =$$

$$= 3\,\frac{\cancel{(n+1)}\,n^n}{(n+1)^n\cdot\cancel{(n+1)}} = 3\,\frac{n^n}{(n+1)^n} = 3\left(\frac{n}{n+1}\right)^n \ominus$$

$$\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1-\frac{1}{n+1}$$

$$\ominus\ 3\left(1-\frac{1}{n+1}\right)^n =$$

$$= 3\,\underbrace{\boxed{\left(1+\frac{-1}{n+1}\right)^{n+1}}}_{\to e^{-1}}\cdot\overset{\to 1}{\frac{n}{n+1}} \xrightarrow[n\to+\infty]{} 3e^{-1}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to+\infty}\left(1+\frac{\alpha}{n}\right)^n = \lim_{n\to+\infty}\Bigg(\underbrace{\left(1+\frac{\alpha}{n}\right)^{\frac{n}{\alpha}}}_{\to e}\Bigg)^{\alpha} = e^{\alpha}$$

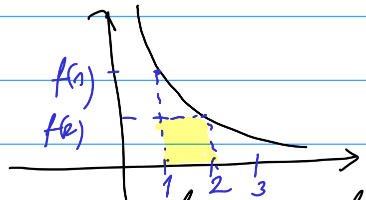
$$t = \frac{\alpha}{n}$$

$$e \approx 2{,}8 \Rightarrow \frac{3}{e} > 1$$

$$\lim \frac{a_{n+1}}{a_n} > 1 \Rightarrow \text{ř. diverguje.}$$

Integrální krit.

$$\sum_{n=1}^{+\infty} a_n \qquad a_n = f(n)$$



$f \geq 0$, $f \downarrow 0$

Pak platí:

$$\sum a_n \text{ konv./div.} \Leftrightarrow \int_1^{+\infty} f(x)\,dx \text{ konv./div.}$$

$$\sum_{n=1}^{+\infty}\frac{1}{n^2} \qquad \frac{1}{n^2} = f(n), \qquad f(x) = \frac{1}{x^2}$$

$$\int\frac{dx}{x^2} = -\frac{1}{x} \qquad \int x^{-2}\,dx = \frac{x^{-1}}{-1}$$

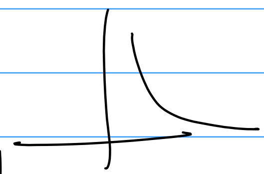
$$\int_1^{+\infty}\frac{dx}{x^2} = \lim_{b\to+\infty}\int_1^b \frac{dx}{x^2} =$$

$$= \lim_{b\to+\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to+\infty}\Big(1-\underbrace{\frac{1}{b}}_{\to 0}\Big) = 1 < +\infty$$

int. konverg. $\Rightarrow$ konverg. i ř.

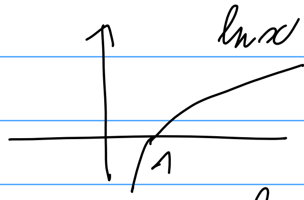
ř. harmonická

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\qquad \frac{1}{n} = f(n)$ $\qquad f(x) = \frac{1}{x}$



$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} \left(\ln b - \underbrace{\ln 1}_{0}\right) =$$

$$= \lim_{b\to+\infty} \ln b = +\infty .$$



Integrál diverguje ⇒ ř. také. $\qquad \ln x \nearrow +\infty$

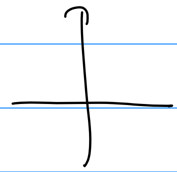
$\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt[2n]{4n-1}}}_{=a_n}$ — ?

$\sqrt[n]{n} \to 1, \ n\to+\infty$

$\sqrt{\sqrt[n]{n}}$

$\lim_{n\to+\infty} a_n = 1$, tj. $\lim a_n \neq 0$ ⇒ neplatí nutná p.
⇒ div.

$\sum_{n=2}^{+\infty} \frac{1}{n(n-1)}$ — ? $\qquad f(x) = \frac{1}{x(x-1)}$



$\int \frac{dx}{x(x-1)}$

$\frac{1}{n(n-1)} = \frac{1}{n^2-n}$

$\approx \frac{1}{n^2}$ (?)

Kriter. srovn.

$\sum a_n, \sum b_n$; $\quad 0 \leq a_n \leq b_n$, pak platí:

$\sum b_n$ konv. ⇒ $\sum a_n$ konv.

$\sum a_n$ div. ⇒ $\sum b_n$ div.

$a_n = \frac{1}{n(n-1)} < \frac{1}{(n-1)(n-1)} = \frac{1}{(n-1)^2} = b_n$; $\quad \sum_{n=2}^{+\infty} b_n = \sum_{n=2}^{+\infty} \frac{1}{(n-1)^2} =$

$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ — konverguje.

Ř. konverguje (srovn. krit.)

$\sum \frac{1}{\sqrt[3]{n^5}+6\ln n}$ — ?

$\ln n > 0$ pro $n > 1$

$\sum \frac{1}{n} = +\infty$

$\sum \frac{1}{n^2} < +\infty$

$\sum \frac{1}{n^3}, \sum \frac{1}{n^{5/2}}$ — 2 konv.

$a_n = \frac{1}{n^{5/3}+6\ln n} < \frac{1}{n^{5/3}} = b_n$

zobecn. harm. ř. $\rightarrow$ $\sum \frac{1}{n^\alpha}$ konv. pro $\alpha > 1$

$\alpha = \frac{5}{3} > 1$ $\quad \sum \frac{1}{n^{5/3}}$ – konv.

$0 < a_n < b_n$, $\sum b_n$ konv. $\Rightarrow$ srovn. kr.

Srovn. krit. – limitní tvar

$0 < a_n \le b_n$, $\sum a_n$, $\sum b_n$

$\exists \lim\limits_{n\to+\infty} \frac{a_n}{b_n} = c \neq 0 \Rightarrow \sum a_n$ konv/div $\Leftrightarrow \sum b_n$ konv/div.

$b_n = \frac{1}{n^{5/3}}$, $\quad a_n = \frac{1}{n^{5/3}+6\ln n}$

$$\frac{a_n}{b_n} = \frac{n^{5/3}}{n^{5/3}+6\ln n} = \frac{\cancel{n^{5/3}}}{\cancel{n^{5/3}}} \cdot \frac{1}{1+6\frac{\ln n}{n^{5/3}}}$$

$n \to +\infty$

$\sum\limits_{n=1}^{+\infty} \frac{\ln n}{\sqrt[5]{n}}$ — ? diverguje

$\sum \frac{1}{\sqrt[5]{n}} = \sum \frac{1}{n^{1/5}}$ diverguje

$\left(\sum \frac{1}{n^\alpha}, \alpha = \frac{1}{5} < 1\right)$

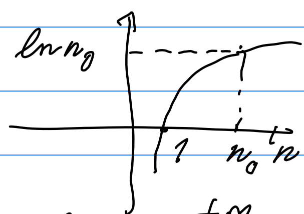
$$b_n = \frac{\ln n}{\sqrt[5]{n}} = \frac{1}{n^{1/5}} \cdot \ln n > \frac{1}{n^{1/5}} \cdot \ln n_0 = a_n$$

$0 < a_n \le b_n$

$\sum a_n$ diverg. $\quad \sum a_n = \ln n_0 \sum \frac{1}{n^{1/5}}$

$n_0 > 1$

$\ln n > \ln n_0$ $(n > n_0)$



$\ln x \nearrow +\infty$

$\lim\limits_{x\to+\infty} \ln x = +\infty$

$\sum_{n=1}^{+\infty} \frac{1}{n^x}$ $x \in \mathbb{R}$ Obor konvergence $(1, +\infty)$ $\nearrow x$

$\searrow$ $\sum_{n=1}^{+\infty} f_n(x)$ – funkční ř.

$\sum_{n=1}^{\infty} (3-x)^n$ – ? $\sum_{n=1}^{\infty} a^n$, $a = 3-x$

– geometr. $|a|<1$

$|3-x|<1$, $-1 < 3-x < 1$

$1 > x-3 > -1$, $4 > x > 2$

Konv. pro $2<x<4$.

$\sum_{n=0}^{+\infty} a_n (x-x_0)^n$ $f_n(x) = a_n(x-x_0)^n$

$\searrow$ mocninná ř.

$\exists R$: mocn. ř. konv. pro $|x-x_0|<R$
diverg. pro $|x-x_0|>R$

$R=0$, $R=+\infty$ nebo $0<R<+\infty$

$\downarrow$ jen pro $x=x_0$	$\downarrow$ konverg. na $\mathbb{R}$ absolutně	$\downarrow$ absolutně konverg. na $\{x : |x-x_0|<R\}$ (+ možná kr. body)

Platí: $\frac{1}{R} = \limsup_{n\to+\infty} \sqrt[n]{|a_n|}$

$R = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right|$ (existuje-li tato lim.)

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad \left(\frac{x^n}{n!} = f_n(x)\right) \qquad a_n = \frac{1}{n!} \qquad x_0 = 0$$

Podílové kr.: $\quad \sum |f_n(x)| \qquad |f_n(x)| \geq 0$

$$\left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{1}{n+1} \cdot |x| \xrightarrow[n \to +\infty]{} 0$$

Podíl. kr.: řada konverguje absolutně pro libov. x ($x \in \mathbb{R}$)

$R = +\infty$ (poloměr konv.).

$\sum_{n=1}^{+\infty} n 6^n x^n$ — ? $\qquad a_n (x-x_0)^n$

$a_n = n 6^n$, $\quad x_0 = 0$.

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

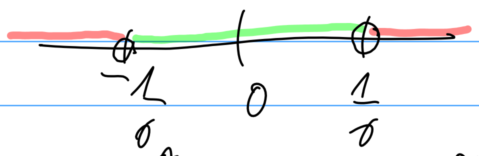
$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n 6^n}{(n+1) 6^{n+1}} = \frac{n}{n+1} \cdot \frac{1}{6} \to \frac{1}{6}, \quad n \to +\infty$$

$\frac{n}{n+1} \to 1$

$R = \frac{1}{6}$

$|x - x_0| < R$

$|x| < \frac{1}{6}$



$x = \frac{1}{6}$: $\sum_{n=1}^{\infty} n \cancel{6^n} \left(\cancel{\frac{1}{6}}\right)^n = \sum_{n=1}^{\infty} n = +\infty$ — div.

$x = -\frac{1}{6}$: $\sum_{n=1}^{+\infty} n 6^n \left(-\frac{1}{6}\right)^n = \sum_{n=1}^{+\infty} n \cancel{6^n} \frac{(-1)^n}{\cancel{6^n}} = \sum_{n=1}^{+\infty} (-1)^n n$ — osciluje

Obor. konv. je $\left(-\frac{1}{6}, \frac{1}{6}\right)$.

$$\sum_{n=1}^{+\infty} \underbrace{\frac{n\,(x-1)^{2n}}{9^n\cdot\sqrt{n^5+2}}}_{=f_n(x)} \quad - ?$$

mocninná

$\sum |f_n(x)|$; podílové kr.:

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{(n+1)(x-1)^{2(n+1)}}{9^{n+1}\sqrt{(n+1)^5+2}}\cdot\frac{9^n\cdot\sqrt{n^5+2}}{n\,(x-1)^{2n}}\right| =$$

$$= \frac{1}{9}\cdot\underbrace{\frac{n+1}{n}}_{\downarrow\, 1}\cdot\underbrace{\frac{\sqrt{n^5+2}}{\sqrt{(n+1)^5+2}}}_{\to 1}\cdot(x-1)^2 \longrightarrow \frac{1}{9}(x-1)^2, \quad n\to+\infty.$$

$n\to+\infty$

Podílové kr. pro $\sum |f_n(x)|$:

$\frac{1}{9}(x-1)^2 < 1 \quad \Rightarrow$ konverg.

$\frac{1}{9}(x-1)^2 > 1 \quad \Rightarrow$ diverg.

$(x-1)^2 < 9 \qquad -3 < x-1 < 3, \qquad \underbrace{-2 < x < 4}_{\text{konv. (abs.!)}}$

$$x=4: \sum \frac{n\,(4-1)^{2n}}{9^n\cdot\sqrt{n^5+2}} = \sum\frac{n\,\cancel{3^{2n}}}{\cancel{9^n}\sqrt{n^5+2}} = \sum\frac{n}{\sqrt{n^5+2}}$$

$\sqrt{n^5} = n^{5/2}$

srovn. kr. v lim. tvaru:

$$a_n = \frac{n}{\sqrt{n^5+2}}, \quad b_n = \frac{n}{\sqrt{n^5}} = \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$$

$$\frac{a_n}{b_n} = \frac{\cancel{n}}{\sqrt{n^5+2}}\cdot\frac{\sqrt{n^5}}{\cancel{n}} = \sqrt{\frac{n^5}{n^5+2}} = \sqrt{\frac{1}{1+\frac{2}{n^5}}} \to 1$$

$\sum \frac{n}{\sqrt{n^5+2}}$ konv. (konv. $\sum \frac{1}{n^{3/2}}$ – zobecn. harm. + srovn. kr. – lim. tvar)

$x = -2$:

$$\sum_{n=1}^{+\infty} \frac{n(-2-1)^{2n}}{9^n \cdot \sqrt{n^5+2}} = \sum \frac{n(-3)^{2n}}{9^n\sqrt{n^5+2}} = \sum \frac{n\, 9^n}{\dots}$$

$$\sum \frac{2^n x^n}{\sqrt{4n+1} \cdot 5^n} \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| =$$

$$= \lim_{n\to\infty} \frac{2^n |x|^n}{\sqrt{4n+1} \cdot 5^n} \cdot \frac{\sqrt{4(n+1)+1} \cdot 5^{n+1}}{2^{n+1}|x|^{n+1}} =$$

$$= \lim_{n\to+\infty} \frac{5}{2} \frac{\sqrt{4(n+1)+1}}{\sqrt{4n+1}}$$

konv. (abs.) $\qquad x_0 = 0$

$R = ?$