NON-STANDARD PROBLEMS AS RESOURCE TO VERIFY MULTIPLICATION UNDERSTANDING IN PRIMARY SCHOOL

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The paper presents the results of a research focused on understanding of multiplication in primary school. An experiment was carried out simultaneously in Czech Republic and in Italy in two classes with pupils aged 8-9 years. Pupils solved two arithmetic tasks based on rectangular model of multiplication, its role was studied as a resource to promote multiplication understanding. Pupils' solutions were analyzed in detail during interviews and submitted worksheets. The results of the research show that suitable tasks can promote the transition from the additive to the multiplicative field.

INTRODUCTION

It is very known that multiplication is significantly more difficult than addition and subtraction. From the conceptual point of view, these operations are very different, even if in the usual activities they are reduced to calculations on numbers. In school, different models are used to introduce multiplication of whole numbers. It is important to be aware of the different features and potentiality of these models, and to take in account the possible problems connected with to transition from a model to another.

In the Czech Republic (CR), elementary mathematics traditionally bases the methodology of multiplication on the manipulation of concrete objects arranged in rows and columns. This arrangement is described as ' a rows of b elements' or ' a groups of b objects', later simplified to ' a by b ', and finally expressed as ' a times b '. Pupils can also solve multiplication examples using a square grid or by cutting them out of square paper. It is emphasized that this initial phase focuses solely on the pupil's understanding of the essence of the multiplication operation, with the goal of performing calculations by heart reserved for the second phase (Divíšek et al., 1989; Nováková & Blažková, 2022).

Usually in Italy (IT) the first approach to the multiplication of whole numbers is based on making a groups of b objects, or on arrays of objects and the teacher poses the problem of counting them. Pupils observe the presence of equipotent rows and columns, and they use repeated addition to count the totally of objects. Subsequently, other representations are introduced and utilized, but very soon the models are neglected and multiplication become only an activity on numbers.

THEORETICAL FRAMEWORK

The transition from the additive to the multiplicative field is complex because their structures are very different. As documented by many researchers (Mariotti & Maffia, 2018), presenting multiplication as repeated addition can hinder the understanding of multiplicative structure.

In mathematical terms, when we write $a \cdot b$ the symbols a and b represent numbers, while when we say "a repeated b times" the symbol a denotes a number, while b represents a numeral adjective: the first is an element of the 'language', the second of the metalanguage (Marchini, 2001/2002, p. 13).

A research (Briand, 1993) shows that pupils 7-8 years old, in front of an arrangement of objects in rows and columns, utilize multiplication for counting them, but if the arrangement is incomplete and it becomes necessary to uncover or reconstruct its structure (see the following 'Task 1' and 'Task 2'), the calculation procedures undergo a complete transformation. One possible explication is that the procedures learnt in class to enumerate a row-column arrangement are not interiorized; they become destabilized when the conditions of the arrangement's conditions change. Consequently, some researchers suggest to work on row-column arrangements starting from kindergarten (Rozek & Urbanska, 1998).

In his 'theory of semiotic representation,' Duval (2006) emphasizes the role of the transition from one representation to another, distinguishing between two types of transformation, 'treatment' and 'conversion', which correspond to different cognitive processes. In our study, treatment occurs when we perform calculations as $a \cdot b = c$ remaining within the arithmetic register, while conversion involves, for instance, transforming a visual representation of a rectangle into a linguistic expression, such as "it is a rectangle $a \cdot b$" and subsequently conducting the relative calculation. According to Duval's theory, in the latter second case, transitioning from the geometrical to the arithmetical register could enhance understanding of multiplication.

Thus, it can be useful to work with various approaches on multiplicative structures, such as hopping along the number line, creating grids, generating areas, and more. These models have distinct features and should be utilized in complementary ways.

Another possible model is the Laisant's [1] table, sometimes employed by teachers as a tool for introducing multiplication. This table, also referred to as the "decanomial" in Montessori's (1934/2016) activities, provides a new semiotic

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[1] Charles-Ange Laisant, French mathematician who invented this table (Laisant, 1915).

representation for multiplication. In Laisant's table, both columns and rows increase by one, moving respectively from left to right and from top to bottom. As Maffia & Mariotti (2020, p. 28) write

> Laisant's table incorporates the rectangular model, presenting any rectangle as an ordered multiplication. Such possibility constitutes the core of the semiotic potential of this artifact.
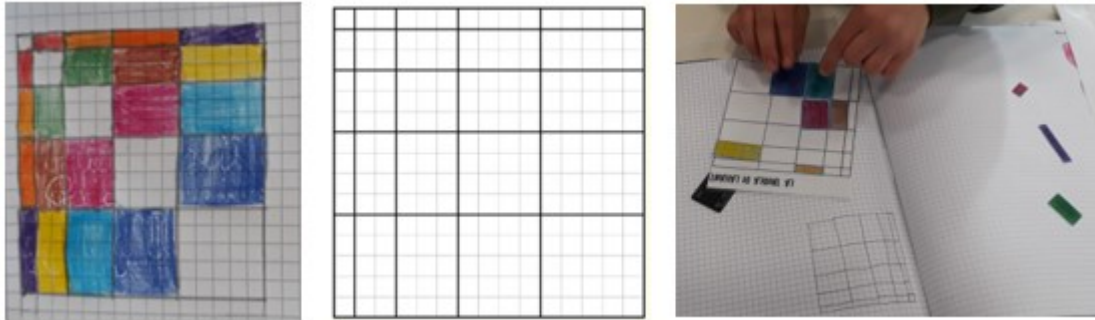

Fig. 1 Laisant's table

Initially the construction of the table can be a drawing activity, a task assigned by the teacher based on the respect of some rules. In fact, the table allows a geometrical introduction of the multiplication, enhancing a visual perception of quantities. Essentially, as the table is constructed, it immediately reveals rectangles (or squares), that appear during the construction of the table itself. Consequently, it feels natural, for example, to observe the pink rectangle and describe it as "a rectangle three times four" using everyday language, pre-empting the linguistic expressions usually employed whit multiplication. Subsequently the teacher can move pupil's attention on the small squares comprising the pink rectangle and he can ask to count them (twelve in our example). The next step involves connecting the two initial numbers with the third: $3 \cdot 4 = 12$. It's important to underscore a significant difference between working on multiplication with arrays and Laisant's table. When we work on arrays, a and b, and $a \cdot b$ represent numbers. However, with Laisant's table, the scenario changes entirely: a and b represent linear measures, the lengths of the rectangle sides, while $a \cdot b$ represents the number of squares that forming this rectangle. Thus, there is a transition from the additive field to the multiplicative field, from linear measures to area measures. It can prepare the work in geometry with linear or two-dimensional geometrical figures. Another positive aspect of this table is the possibility to cut rectangles and to superimpose appropriately them onto the rectangles drawn in the table, using manipulation (Fig. 1). The table maintains the structure while if we manipulate objects in an array their disposition changes.

In the present research, we use two problems, named 'Task 1' and 'Task 2', with the aim to investigate on the three main questions.

Research questions

In the present research, we employ two problems, designated as 'Task 1' and 'Task 2', with the aim of investigating the following questions:

1. Are 'Task 1' and 'Task 2' a resource for diagnosing pupil's preconceptions and/or internalization of multiplication?
2. Can 'Tasks 1' and 'Task 2' serve as valuable resources for exploring the transition from additive structure to multiplicative structure?
3. Is the Laisant's table a resource for constructing multiplication structure?

Task 1 and it's a-priori analysis

*How many tiles will be on the floor when it will be finished?*
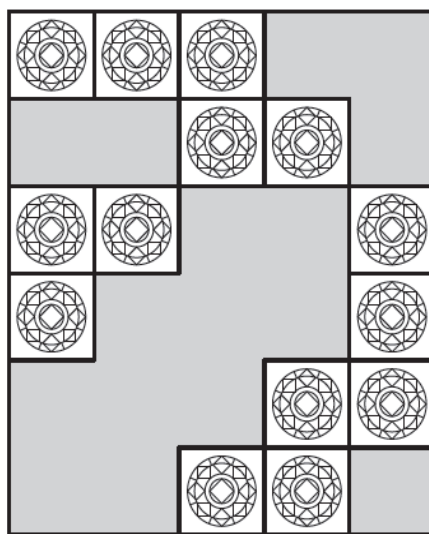
'Task 1' originates from an assessment question presented by INVALSI (Italian National Institute for the Evaluation of Instruction and Formation Educational System), the authority responsible for conducting periodic and systematic tests on pupils' knowledge and abilities. These tests are administered in all Italian schools, in the same day at the end of the school year.

The Authors of the current paper utilized the figure of the task 'D9', which was originally presented to 7-8-year-old pupils in the year 2019.[2] However, they modified the question in alignment with their research inquiry. Specifically, the original test question focused on determining the number of omitted tiles. While the aim of the present research is to observe whether pupils utilized multiplication, such as 6·5 or 5·6, when facing Task 1 or not. We can suppose that pupils had to mentally visualize the omitted tiles and count them with 'mental eyes'. Alternatively, they could draw them, but in this case the drawings must be accurate. The solution could also be reached by counting 'in horizontal' or 'in vertical' or 'in groups of tiles'.

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[2] More detailed information see: https://invalsi-areaprove.cineca.it

Task 2 and it's a-priori analysis

How many tiles will be on the floor when it will be finished?
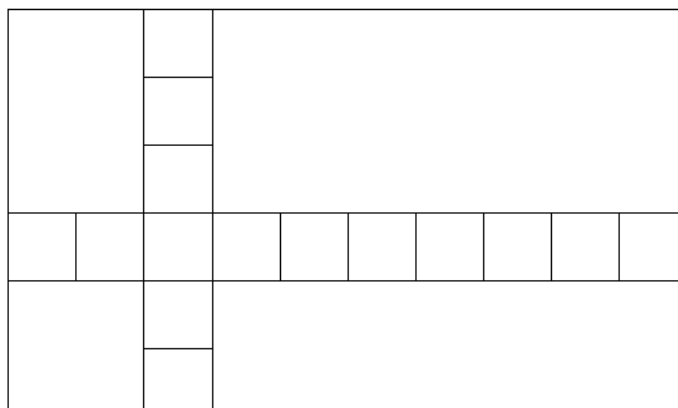
'Task 2' presents the same question of 'Task 1', the 'floor' once again is a rectangle, which includes an interior 'cross' created by two intersecting square lines and four empty white rectangles [3]. We hypothesize that Task 2 can serve as an 'educational resource' to stimulate the necessity of multiplication and promote its understanding. When presented with an array of objects, pupils tend to utilize multiplication. However, in Task 2 the conditions of enumeration are different, allowing pupils to organize their calculations in diverse ways.

Several strategies can be employed, including:

- Calculation by multiplication: 6·10 or 10·6
- Addition: 10 +10 +10 +10 +10+10 or 6+6+6+6+6+6+6+6+6+6
- Addition of blocks of tiles and the drawn tiles: 6 + 4 + 21 + 14 + 15.

METHODOLOGY

The research was conducted in the second grade of primary school with pupils aged 8-9 years old[4].

In classroom, we presented 'Task1' by a worksheet. Immediately after, individual interviews were conducted by the researcher, prompting each pupil to explain their answer and their reasoning behind it. Thus, it was possible to use artefacts of a dual nature for further research: written problem-solving responses and subsequent interview records, which were documented in writing. Both sets of research data were then analyzed systematically.

The interview commenced with the following question: "What object was suggested from the drawing presented in the worksheet?". This question was designed to put the pupils at ease, as they provided various responses such as

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[3] The drawing comes from Briand (1993).
[4] We acknowledge teachers Lenka Sýkorová (Bosonožská school, Brno) and Patrizia Coppola (P. Maupas school, Vicofertile Parma) for the collaboration during the experiment.

"floor", "wall paintings", "tablecloth", and so on. Following this, the pupils explained their solutions, providing insights into their thought processes and reasoning.

The 'Task 2' was introduced in the classroom using a multimedia interactive blackboard. Pupils were asked to observe the projected figure and provide a written answer to a question identical to that presented in 'Task 1', but in this case without having the drawing of the floor in a paper. This choice is motivated by the intention to discourage drawing and instead encourage observation of the figure and reasoning skills. In this way, we want observe if the recourse on multiplication appears. We believe that this choice may have increased the task's difficulty, which may have more incorrect solutions than 'Task 1'.

In CR, the pupils were not yet familiar with the operation of multiplication. In particular, they never used the scissors and square grid. This context allows for a clearer observation of pupils thought processes and problem-solving approaches.

In IT, the teacher clarified that despite being in the third grade, the pupils' competencies were similar to those of second-grade pupils due to disruptions caused by the COVID-19 epidemic, which had slowed down the execution of usual activities. Multiplication had been recently introduced in the current school year, and the pupils had limited experience with it. However, in the previous year, pupils worked with Laisant's table. A week later, the researcher went in classroom submitting again Task 2 by blackboard, and giving a white paper to each pupil asking to write not only its answer to the question, but also her/his reasoning. The aim was to verify if pupils use multiplication or not.

On the contrary, in CR researcher presented the 'Task2' furnishing also to the pupils a paper for writing their solutions since when the task was given on the board (as well as IT) the pupils asked to redraw the picture. They were allowed to draw.

RESULTS

Analysis of pupils' solutions to Task 1

In CR 18 pupils, 9 girls and 9 boys, are involved in the experimentation. In IT, the total number was 22, 12 boys and 10 girls participated.

Two basic phenomena emerged from the analysis. The first was the need for the drawings of missing tiles, as an integral part of the solution to the problem. By sketching vertical and horizontal lines, all tiles were visible on the floor, allowing them to focus on determining the total number of tiles, both present and missing. The second phenomenon was the method of determining the number of tiles. From the pupils' solutions, their written comments and the subsequent interview, we traced five different strategies.

a. After illustrating the missing tiles, pupils proceeded to count the tiles in each row one by one, numbering them sequentially from 1 to 30. To facilitate the counting process, each tile was marked, either with a dot or a circle or a number.
b. Calculation of the number of tiles drawn on the floor and the number of missing tiles and addition 14+16 (four pupils).
c. Addition of all tiles in the rows 5+5+5+5+5 (eight pupils), or of all tiles in the columns 6+6+6+6+6 (two pupils).
d. Multiplication 6·5 or 5·6 (two pupils). One of them knew multiplication from older sibling.

Only two pupils did not take advantage of the opportunity to draw the missing tiles in their solutions. In one instance, a pupil determined the number of tiles by mentally counting them one by one, row by row. Another girl counted the current number of tiles shown; while counting the missing tiles, she pointed to the locations of each missing tile. She then added the two counts together.

One boy did not solve the problem correctly. He made a mistake when reciting the series of natural numbers, omitting the number 16. The pupil who did not solve the problem correctly used a functional strategy.

In IT, 7 pupils used multiplication (strategy d), 11 used addition (strategy a), and 3 counted only the missing tiles.

We want notice that the question 'How many …?' suggests the use of counting, influencing the chosen strategies of solution. Moreover, the possibility to draw on the worksheet, promote the counting one by one of the tiles. Some pupils separately calculated the number of drawn tiles (14), the number of omitted tiles (16) and after they made addition 14+16=30 (strategy b). Sometimes they stop after the counting of omitted tiles, a girl finished with this curious statement: "The tiles will be available after 15 days". Pupils who used multiplication without hesitation, explain in this way: 5 in horizontal, 6 in vertical, so 5·6=30. This language was employed in the previous year during the activities with Laisant's table. Some of them confused 'horizontal' with 'vertical'.

Analysis of pupils' solutions to Task 2

In CR the development of the research investigation was the same as for the first task. After solving the task independently (about 10 minutes) each child was again interviewed by the researcher. In individual interviews with the pupils, they verbally explained, justified and commented their procedures recorded in writing.

From the pupils' solutions, their written comments and the follow-up interview, we again identified different strategies. Only two girls did not develop any solution strategy. We can distinguish four strategies for solving the problem:

a. A group of four pupils chose a procedure based on counting one by one to determine the number of squares, often reaching an incorrect conclusion. Some

pupils failed to redraw the picture correctly. One boy gave an incorrect result because he made a picture of seven rows instead of six.

b. Five children first noticed the number of tiles in one row and then realized that there would be the same number of tiles in all the rows. Consequently, they counted the number of rows (i.e., the number of tiles in a column) and calculated the resulting number of tiles by repeatedly adding the number of tiles in one row (10+10+10+10+10+10).

c. Another strategy, also based on addition, was chosen by three pupils. They noticed that except for the third column with marked tiles, there were 5 tiles missing in each column. There are 9 such columns, calculating this they found that there are 45 missing tiles. Employing a method of memory addition (5+5+5+5+5+5+5+5), they added the 15 tiles that are marked in the figure to obtain the total number of tiles. This was achieved by adding the number of tiles drawn both in the row and column. They were aware of the necessity to avoid counting the same tile twice, so they added 10+5.

d. Four children utilized the same initial situation, counting 10 tiles in a row and 6 tiles in a column. However, these children approached their solution focusing on the relationship between the number of tiles in the columns and rows and they intuitively arrived at determining the result through multiplication. When expressing their solution orally, they articulated the number of tiles as the result of the $6 \cdot 10$ reasoning.

In IT only seven pupils chose strategy (a), two chose strategy (b), two chose strategy (c), six chose strategy (d). Sometimes counting occurred by imagining the omitted tiles and mentally counting them, obviously with various and approximate results such as 57, 58, 64, 52, 88.

It is interesting to observe that a new strategy appears: counting of the tiles of white rectangles $(4 + 6 + 21 + 14)$, counting of drawn squares (15) and then adding them together (4+6+21+14+15= 60). We suppose that the previous work with Laisant's table influenced their performances moving to observe the white rectangles. Some pupils mistakenly counted the square placed at the intersection of the horizontal and vertical lines twice, obtaining a total of 61 tiles. Additionally, four pupils considered only the omitted tiles obtaining 45. This indicates that they applied their multiplication knowledge on the 'small rectangles' and not on the biggest. The majority of pupils used multiplication, probably influenced from a revision made in classroom by the teacher.

Conclusions

At the end of the activities, pupils mentioned that initially the tasks seemed trivial or easy, but they found difficult to explain their reasoning.

We want to underline the role of the activities proposed in classroom.

In CR only two pupils used multiplication to solve the first problem, four pupils used multiplication in the second. One boy remarked: "Rows and columns have something in common. There are 10 squares in a row and 6 in a column. Six times ten is sixty". This observation suggests that the second task prompted the pupils to think differently.

In IT only five pupils employed multiplication in both tasks, indicating that for them this operation appears internalized. When we reintroduced 'Task 2' one week later, the percentage of Italian pupils who used multiplicative strategy passed from 32% to 58%. We believe that the difficulties and obstacles presented by the proposed tasks prompted the pupils to see the multiplication as a useful tool for organizing calculations of objects in an array.

In other words, we think that our tasks provoked the need for a link between the existing understanding of multiplication and its mental representations, promoting a deeper understanding of the concept.

With reference to 'research question 1', we can affirm that both tasks led to the identification of pupils' preconceptions in the area of multiplication. Additionally, we observed that for some children who used multiplication in both tasks, the employment of this operation came later. For the other pupils the understanding begins slowly, step by step, as the figures drawn in Tasks 1 and 2 confuse their visualization of rectangular models.

With reference to 'research question 2', we can observe that perhaps after numerous attempts children come to see multiplication as better tool to solve the problems. The transition from additive to the multiplicative field must be promoted, but it is important to emphasize that each pupil has his/her own time of understanding, which must respected.

With reference to 'research question 3', in Italian classes we observe the influence of the previous work on Laisant's table, particularly in same pupils.

The teachers of the classes involved in the experimentation initially considered the tasks difficult and hard to solve. However, by the end, they were surprise from the performances of pupils. In particular, they observe a deeper understanding of multiplication. As mentioned earlier, 'Task 2' played this role, as we can verify during the last intervention in classroom. In other words, as Barmby & all. (2009, p. 219) state

> Therefore, representations were used earlier on, but only for the purpose of illustrating multiplication and rarely for the purpose of supporting calculation.

while our experience documents the need for continuous, rather than episodic use of the rectangular model when working on multiplication.

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