Vagueness, Truth and Logic
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VAGUENESS, TRUTH AND LOGIC[1]

This paper began with the question 'What is the correct logic of vagueness?' This led to the further question 'What are the correct truth-conditions for a vague language?', which led, in its turn, to a more general consideration of meaning and existence. The first half of the paper contains the basic material. Section 1 expounds and criticizes one approach to the problem of truth-conditions. It is based upon an extension of the standard truth-tables and falls foul of something called penumbral connection. Section 2 introduces an alternative framework, within which penumbral connection can be accommodated. The key idea is to consider not only the truth-values that sentences actually receive but also the truth-values that they might receive under different ways of making them more precise. Section 3 describes and defends the favoured account within this framework. Very roughly, it says that a vague sentence is true if and only if it is true for all ways of making it completely precise. The second half of the paper deals with consequences, complications and comparisons. Section 4 considers the consequences that the rival approaches have for logic. The favoured account leads to a classical logic for vague sentences; and objections to this unpopular position are met. Section 5 studies the phenomenon of higher order vagueness: first, in its bearing upon the truth-conditions for a language that contains a definitely-operator or a hierarchy of truth-predicates; and second, in its relation to some puzzles concerning priority and eliminability.

Some of the topics tie in with technical material. I have tried to keep this at a minimum. The reader must excuse me if the technical undercurrent produces an occasional unintelligible ripple upon the surface.

Let us say, in a preliminary way, what vagueness is. I take it to be a semantic notion. Very roughly, vagueness is deficiency of meaning. As such, it is to be distinguished from generality, undecidability, and ambiguity. These latter are, if you like, lack of content, possible knowledge, and univocal meaning, respectively.

These contrasts can be made very clear with the help of some artificial

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examples. Suppose that the meaning of the natural number predicates, nice_1, nice_2, and nice_3, is given by the following clauses:

(1) (a) n is nice_1 if $n > 15$
(b) n is not nice_1 if $n < 13$
(2) (a) n is nice_2 if and only if $n > 15$
(b) n is nice_2 if and only if $n > 14$
(3) n is nice_3 if and only if $n > 15$

Clause (1) is reminiscent of Carnap's (1952) meaning postulates. Clauses (2) (a)–(b) are not intended to be equivalent to a single contradictory clause; somehow the separate clauses should be insulated from one another. Then nice_1 is vague, its meaning is under-determined; nice_2 is ambiguous, its meaning is over-determined; and nice_3 is highly general or un-specific. The sentence 'there are infinitely many nice_3 twin primes' possibly undecidable but certainly not vague or ambiguous.

Any type of expression that is capable of meaning is also capable of being vague; names, name-operators, predicates, quantifiers, and even sentence-operators. The clearest, perhaps paradigm, case is the vague predicate. A further characterization of vagueness will not, I think, be theory-free; for it will rest upon an account of meaning. In particular, if meaning can have an extensional and intensional sense, then so can vagueness. Extensional vagueness is deficiency of extension, intensional vageness deficiency of intension. Moreover, if intension is the possibility of extension, then intensional vagueness is the possibility of extensional vagueness. Turn to the clear case of the predicate. A predicate F is extensionally vague if it has borderline cases, intensionally vague if it could have borderline cases. Thus 'bald' is extensionally vague, I presume, and remains intensionally vague in a world of hairy or hairless men. The distinction is roughly Waismann's (1945) vagueness/open-texture one, but without the epistemological overtones.

Extensional vagueness is closely allied to the existence of truth-value gaps. Any (extensionally) vague sentence is neither true nor false; for any vague predicate F, there is a uniquely referring name a for which the sentence Fa is neither true nor false: and for any vague name a there is a uniquely referring name b for which the identity-sentence $a = b$ is neither true nor false. Some have thought that a vague sentence is both true and false and that a vague predicate is both true and false of some object.

However, this is a part of the general confusion of under- and over-determinacy. A vague sentence can be made more precise; and this operation should preserve truth-value. But a vague sentence can be made to be either true or false, and therefore the original sentence can be neither.

This battle of gluts and gaps may be innocuous, purely verbal. For truth on the gap view is simply truth-and-non-falsehood on the glut view and, similarly, falsehood is simply falsehood-and-non-truth. However, it is the gap-inducing notion that is important for philosophy. It is the one that directly ties in with the usual notions of assertion, verification and consequence. The glut-inducing notion has a split sense; for it allows truth to rest upon either correspondence with fact or absence of meaning.

Despite the connection, extensional vagueness should not be defined in terms of truth-value gaps. This is because gaps can have other sources, such as failure of reference or presupposition. What distinguishes gaps of deficiency is that they can be closed by an appropriate linguistic decision, viz. an extension, not change, in the meaning of the relevant expression.

1. The truth-value approach

It is this possibility of truth-value gaps that raises a problem for truth-conditions. For the classical conditions presuppose Bivalence, the principle that every sentence be either true or false, and so they are not directly applicable to vague sentences. In this, as in other, cases of truth-value gap, it is tempting to treat Neither-true-nor-false, or Indefinite, as a third truth-value and to model truth-value assessment along the lines of the classical truth-conditions.

The details of this and subsequent suggestions will first be geared to a first-order language. Only later do we consider the complications that arise from extending the language. Let us fix, then, upon an intuitively understood, but possibly vague, first-order language L. There are three sources of vagueness in L: the predicates, the names, and the quantifiers. To simplify the exposition, we shall suppose that only predicates are vague. Indeed, it could be argued that all vagueness is reducible to predicate vagueness. For possibly one can replace, without any change in truth-value, each vague name by a corresponding vague predicate and each quantifier over a vague domain by an appropriately relativised quantifier over a more inclusive but precise domain. We shall also suppose, though only to avoid talking of satisfaction, that each object in the domain has a name.

We now let a partial specification be an assignment of a truth-value – True (T), False (F) or Indefinite (I) – to the atomic sentences of L; and we call a specification *appropriate* if the assignment is in accordance with the intuitively understood meanings of the predicates. Thus an appropriate specification would assign True to 'Yul Brynner is bald', False to 'Mick Jagger is bald' and Indefinite to 'Herbert is bald', should Herbert be a borderline case of a bald man. Then the present suggestion is that the truth-value of each sentence in L be evaluated on the basis of the appropriate specification. The valuation is to be truth-functional in the sense that the truth-value of each type of compound sentence be a uniform function of the truth-values of its immediate sub-sentences.

The possible truth-conditions can be subject to two natural constraints. The first is that the conditions be faithful to the classical truth-conditions whenever these are applicable. Call a specification *complete* if it assigns only the definite truth-values, True and False. Then the Fidelity Condition F states that a sentence is true (or false) for a complete specification if and only if it is classically true (or false); evaluations over complete specifications are classical.

The second constraint is that definite truth-values be stable for improvements in specification. Say that one specification u *extends* another t if u assigns to an atomic sentence any definite truth-value assigned by t. Then the Stability Condition S states that if a sentence has a definite truth-value under a specification t it enjoys the same definite truth-value under any specification u that extends t; definite truth-values are preserved under extension.

The two constraints work together: definite truth-values for a partial specification must be retained upon the classical evaluation of any of its complete extensions. Indeed, if quantifiers are dropped, the two constraints are equivalent to the classical necessary truth and falsehood conditions:

(i) $\models -B \rightarrow \;=\!\!|\, B$
$=\!\!|\, -B \rightarrow \;\models B$

(ii) $\models B \,\&\, C \rightarrow \;\models B$ and $\models C$
$=\!\!|\, B \,\&\, C \rightarrow \;=\!\!|\, B$ or $=\!\!|\, C$.

Similarly for the other truth-functional connectives. (I use '$\models A$' for 'A is true', '$=\!\!| A$' for 'A is false', and '$\rightarrow$' for informal material implication.)

However, the conditions still allow some latitude in the formulation of truth-conditions. One can move in the direction of minimizing or of maximizing the degree to which sentences receive definite truth-values under a given specification [2]. At the one extreme, the indefinite truth-value dominates: any sentence with an indefinite subsentence is also indefinite. Sentences are only definite under a classical guarantee. At the other extreme, the indefinite truth-value dithers: a sentence is definite if its truth-value is unchanged for any way of making definite its immediate indefinite subsentences [3]. In effect, the arrows in the clauses (i) and (ii) above are reversed so that the only divergence from the classical conditions lies in the rejection of Bivalence. To illustrate, a conjunction with indefinite and false conjuncts is indefinite on the first account, but false on the second. There are intermediate possibilities, but they are not very interesting. Indeed, clause (i) uniquely determines the conditions for negation, for the weak and strong senses are excluded, and the above alternatives are the only ones for commutative conjunction.

Is any account along truth-value lines acceptable? Any account that satisfies the conditions F and S would always appear to make correct allocations of definite truth-value. However, even the maximizing policy fails to make many correct allocations of definite truth-value. For suppose that a certain blob is on the border of pink and red and let P be the sentence 'the blob is pink' and R the sentence 'the blob is red'. Then the conjunction $P \,\&\, R$ is false since the predicates 'is pink' and 'is red' are contraries. But on the maximizing account the conjunction $P \,\&\, R$ is indefinite since both of the conjuncts P and R are indefinite.

A more general argument applies to any three-valued approach, regardless of whether it satisfies the conditions F or S. For $P \,\&\, P$ is indefinite since it is equivalent to plain P, which is indefinite, whereas $P \,\&\, R$ is false. Thus a conjunction with indefinite conjuncts is sometimes indefinite and sometimes false and so '&' is not truth-functional with respect to the three truth-values, True, False and Indefinite.

A similar argument also applies to the other logical connectives. For example, the disjunction $P \vee P$ is indefinite since it is equivalent to plain P, which is indefinite; whereas the disjunction $P \vee R$ is true since the predicates 'is pink' and 'is red' are complementary over the given colour range. Again, the conditional $P \supset -P$ is presumably not true, whereas $P \supset -R$ is true. It is more difficult to find examples for the quantifiers. But for the universal quantifier, say, we may consider the sentence 'All pretenders to the throne are the rightful monarch', where the domain of quantification consists of several pretenders who are all borderline cases of the predicate 'is a rightful monarch'. The whole sentence is false, yet its immediate subsentences are indefinite.

Nor is there any safety in numbers. The argument can be extended to cover any finite-valued approach or any multi-valued approach that requires a conjunction with indefinite conjuncts to be indefinite. Such approaches are common and include those that are based upon degrees of truth [4] and those that satisfy a fidelity and stability condition with respect to a trichotomy of True, False, and Indefinite truth-values.

The specific examples chosen should not blind us to the general point that they illustrate. It is that logical relations may hold among predicates with borderline cases or, more generally, among indefinite sentences. Given the predicate 'is red', one can understand the predicate 'is non-red' to be its contradictory: the boundary of the one shifts, as it were, with the boundary of the other. Indeed, it is not even clear that convincing examples require special predicates. Surely $P \& -P$ is false even though P is indefinite.

Let us refer to the possibility that logical relations hold among indefinite sentences as *penumbral connection*; and let us call the truths that arise, wholly or in part, from penumbral connection, *truths on a penumbra* or *penumbral truths*. Then our argument is that no natural truth-value approach respects penumbral truths. In particular, such an approach cannot distinguish between 'red' and 'pink' as independent and as exclusive upon their common penumbra.

Placing the Indefinite on a par with the other truth-values is analogous to basing modal logics on the three values Necessary, Impossible and Contingent, or to basing deontic logic on the values Obligatory, Forbidden and Indifferent. For here, too, truth-functionality may be lost: a conjunction of contingent sentences is sometimes contingent, sometimes impossible; a conjunction of indifferent sentences is sometimes indifferent, sometimes forbidden. In all of these cases there appears to be a dogmatic adherence to a framework of finitely many truth-values. Perhaps our understanding of sentential operators is, in some sense, finite.

but this is not to say that it is based upon a finite substructure of truth-values.

2. An alternative framework

How can we account for penumbral connection? Consider again the blob that is on the border of pink and red and suppose that it is also a borderline case of the predicate 'small'. Why do we say that the conjunction 'The blob is pink and red' is false but that the conjunction 'The blob is pink and small' is indefinite? Surely the answer must rest on the fact that in making the respective predicates more precise the blob cannot be made a clear case of both the predicates 'pink' and 'red' but can be made a clear case of both the predicates 'pink' and 'small'. In other words, the difference in truth-value reflects a difference in how the predicates can be made more precise.

Such a suggestion can be made precise within the following framework. A (specification) *space* consists of a non-empty set of elements, the specification-points, and a partial ordering $\geqslant$ (also read: extends) on the set, i.e. a reflexive, transitive and antisymmetric relation. A space is *appropriate* if each point corresponds to a precisification, one point for each precisification. We regard the ways of precisifying in a generous light and, in particular, do not tie them to the expressions of any given language. The nature of the correspondence is this: each point is assigned a specification that is appropriate to the precisification to which it corresponds; points extend one another just in case they correspond to precisifications that extend one another in the natural sense. Thus, at the very simplest, the specifications could be regarded as the precisifications themselves and the partial ordering as the natural extension-relation on precisifications. Then the suggestion is that truth-valuation be based, not upon the appropriate specification, but upon an appropriate specification space, i.e. upon the specification-points that correspond to the different ways of making the language more precise. The truth-valuation is to be uniform in the sense that it only makes use of the specification-points at which the given subsentences are true or false. There may, of course, be several appropriate spaces, but then their differences should make no difference to the truth-valuation.

This framework could be generalized in various ways. For example, with each space could be associated a subset of points. The new space would be appropriate only if the subset determined a space that was appropriate in the old sense. This would allow the truth-definition to call upon specifications that did not correspond to precisifications. However, such generalizations appear to have little intuitive foundation and will not be considered further.

The account of appropriacy uses the intensional notion of precisification. A strictly extensional account could avoid this in various ways. Perhaps the simplest is to identify the specification-points with the specifications themselves. Thus a specification space is, in effect, a collection of specifications partially ordered by the natural extension-relation. A space is appropriate if the specifications are the *admissible* ones. Unofficially, a specification is admissible if it is appropriate for some precisification; officially, the notion of admissibility is primitive.

There are various conditions one can impose upon a specification space. One is that it has a base-point, the appropriate specification-point. This corresponds to the precisification of which all other precisifications are extensions. Another is Completeability. It states that any point can be extended to a complete point within the same space, i.e.

$$\text{C.} (\forall t)\, (\exists u \geqslant t)\, (u \text{ complete})$$

where a point is complete if its specification is complete.

There are also conditions one can impose upon the truth-definition. The main ones are the appropriate modifications of the earlier fidelity and stability conditions. Fidelity will state that the truth-values at a complete point are classical, i.e.

$$\text{F.}\, t \vDash A \leftrightarrow t \vDash A \text{ (classically) for } t \text{ complete.}$$

Stability will state that truth-values are preserved under extensions of points within a given space, i.e.

$$\text{S.}\, t \vDash A \text{ and } t \leqslant u \rightarrow u \vDash A$$
$$t \,=\!\!\!| A \text{ and } t \leqslant u \rightarrow u \,=\!\!\!| A.$$

As with the truth-value approach, there is the problem of how to tag truth values to the different specifications. One can tend to minimize or to maximize the amount of truth and falsehood tagging. Minimizing gives nothing new. However, maximizing gives something altogether different: a sentence is true (or false) at a partial specification point if and only if it is true (or false) at all complete extensions. A sentence is true simpliciter if and only if it is true at the appropriate specification-point, i.e. at all complete and admissible specifications. Truth is super-truth, truth from above[5].

In contrast to the truth-value approach, there are now many interesting intermediate truth-definitions. The most notable is the bastard intuitionistic account, which follows the intuitionistic conditions for $-$, &, $\vee$, $\supset$ and $\exists$, and the classical definition of $-\exists-$ for $\forall$[6]. Given that the domain of quantification is constant, the clauses run like this:

I (i) $t \models -B \leftrightarrow (\forall u \geqslant t)$ (not-$u \models B$)

(ii) $t \models B \mathbin{\&} C \leftrightarrow t \models B$ and $t \models C$

(iii) $t \models B \vee C \leftrightarrow t \models B$ or $t \models C$

(iv) $t \models B \supset C \leftrightarrow (\forall u \geqslant t)(u \models B \rightarrow u \models C)$

(v) $t \models (\exists x)\, B(x) \leftrightarrow t \models B(a)$ for some name a

(vi) $t \models (\forall x)\, B(x) \leftrightarrow (\forall u \geqslant t)(\exists v \geqslant u)(v \models B(a))$ for each name a.

There are two common factors in the rival approaches to truth-conditions. One is the insistence that the procedures for truth-valuation be uniform. The other is the insistence that the appropriate form of stability be satisfied.

These factors can be made explicit within an abstract theory of extensions. The standard Fregean theory has a principle of Functionality:

(1) The extension of a compound $\phi(A_1, \ldots, A_k)$ is a function f_ϕ of the extensions of its parts $A_1, \ldots, A_k$.

This corresponds to the appropriate form of Uniformity. We now suppose that the extensions are partially ordered by a relation of extending and add a principle of Monotonicity:

(2) If extensions $x_1', \ldots, x_k'$ extend extensions $x_1, \ldots, x_k$, respectively, then f_ϕ applied to $x_1', \ldots, x_k'$ extends f_ϕ applied to $x_1, \ldots, x_k$.

This corresponds to the appropriate form of stability.

The most important common factor is Monotonicity. This was not argued for in the previous section, but it can be given an intensional foundation. First we must graft a theory of intensions onto the earlier theory. We shall not be too concerned with the nature of intensions. A specific model can be obtained by indexing extensions with possible worlds. (I presume that the specification-points remain constant from world to world.) However, such a model would suffer from familiar difficulties. For example, if the meaning of A is relevant to the meaning of $A \vee -A$, then the vagueness of A should be relevant to the vagueness of $A \vee -A$. Thus $A \vee -A$ should be vague for vague A, though on the super-truth account, say, it is equally and completely precise for all A.

The pure theory of intensions should contain the analogues of Functionality and Intensionality.

(3) The intension of a compound $\phi(A_1, \ldots, A_k)$ is a function F_ϕ of the intensions of its parts $A_1, \ldots, A_k$;

(4) If intensions $X'_1, \ldots, X'_k$ extend intensions $X'_1, \ldots, X'_k$, respectively, then F_ϕ applied to $X'_1, \ldots, X'_k$ extends F_ϕ applied to $X_1, \ldots, X_k$.

The combined theory should link intensions to extensions. Each intension X determined an extension x; and each extension is so determined. This makes for two bridge principles:

(5) The intension $F_\phi(X_1, \ldots, X_k)$ determines the extension $f_\phi(x_1, \ldots, x_k)$;

and

(6) X extends Y if and only if x extends y.

(5) states that a calculated intension determines the correspondingly calculated extension; and (6) states that one intension extends another if and only if the extension determined by the one extends the extension determined by the other.[7]

We can now derive (2), i.e. Monotonicity, from (4), (5) and (6). For simplicity, take the case of $k = 1$. Now suppose x extends y. Then X extends Y by (6); $F_\phi(X)$ extends $F_\phi(Y)$ by (4); the extension determined by $F_\phi(X)$ extends the extension determined by $F_\phi(Y)$, by (6) again; and so $f_\phi(x)$ extends $f_\phi(y)$ by (5).

The main assumptions behind this argument are (4) and (6). (4), with (3), is tantamount to the assumption that an expression is made more precise through making its simple terms more precise. This assumption is correct for the language L. For the logical constants are transparent, as it were, to vagueness; any precisification of a constituent shines through into the compound. Indeed, the converse of the assumption also holds; an expression is made more precise *only* through making its simple terms more precise. For the logical constants are already perfectly precise. Since the logical constants are also the grammatical particles, all vagueness can be blamed onto constituents as opposed to constructions.

The second assumption says, roughly, that extension does not decrease with an increase in intension. In particular, a sentence does not become indefinite upon being made more precise. This is, perhaps, partly definitional of 'making more precise'. For what distinguishes this operation from a mere change in meaning is that it preserves truth-value. To precisify *is* to rule out the possibility of certain truth-value gaps. In any case, it would be odd if definite truth-value could disappear upon precisification. Truth could then hold by default, in virtue of a lack of meaning. It could be a product of linguistic laziness and not be consequent upon a positive concordance of meaning and fact.

What is the rationale for these two assumptions? It lies, I think, in our desire for an enduring use of language. Under the pressure of their own use, the meanings of terms will need to change. The terms, in their old sense, will not be adequate to express the new truths, pose the next questions, make the right distinctions. Now clearly it is convenient that the changes in meaning be conservative, that the true records before the change remain true after the change. We may wish, for example, to settle a new case within a classificatory scheme without upsetting the principles of classification. But it is the two assumptions which guarantee that truth-value be preserved upon precisification of terms, that allow for the stability of recorded truth within the required instability of meaning.

These two assumptions tie in well with a dynamic conception of language. For language need not retain its identity upon arbitrary changes in meaning; or rather, any such identity is a matter of degree and dependent upon how much change there is. On the other hand, language can retain its identity upon precisification or conservative meaning change; for the two assumptions result in a natural constraint upon change. The identity of language is visible, as it were, in the permanence of recorded truth.

If language is like a tree, then penumbral connection is the seed from which the tree grows. For it provides an initial repository of truths that are to be retained throughout all growth. Some of the connections are internal. They concern the different borderline cases of a given predicate: if Herbert is to be bald, then so is the man with fewer hairs on his head. But many other of the connections are external. They concern the common borderline cases of different predicates: if the blob is to be red, it is not to be pink; if ceremonies are to be games, then so are rituals; if sociology is to be a science, then so is psychology. Thus penumbral connection results in a web that stretches across the whole of language. The language itself must grow like a balloon, with the expansion of each part pulling the other parts into shape.

The two approaches to truth-conditions agree on requirements, but differ on how the requirements are to be met. The agreement consists in their satisfying the principles of an abstract theory of extension; the disagreement consists in how they satisfy these principles. On the truth-value approach, the extension of a sentence is a truth-value – True, False or Indefinite. Each truth-value extends itself; True and False extend Indefinite; and that is all. The extensions and extending-relation for other parts of speech can then be determined in a natural manner. It is then easy to verify that the different accounts on the truth-value approach will satisfy the principles and that Monotonicity is equivalent to the appropriate form of stability.

On the specification space approach, there is a slight difficulty in interpretation. For the extension of B will extend that of A if B corresponds to A at a later stage of precisification. There are various ways of securing this. One is to regard each expression as an ordered pair (A, t), where A is an ordinary expression and t is a specification-point that indicates the stage of precisification. Another is to imagine that, in precisifying, an expression is not endowed with a new sense but is succeeded by an expression with that sense. Thus the language expands in an orderly manner throughout the specification space. t can then be identified from the expression as the first point at which it is introduced. Let us opt for the first solution. Then the extension of a sentence (A, t) is an ordered pair (U, V), where $u = \{U: t \leqslant u\}$ and $V = \{v \in U: v \models A\}$. One extension (U', V') extends another (U, V) if $U' \subseteq U$ and V' is $V \cap U'$. It is again then easy to verify that the principles are satisfied and that Monotonicity is the appropriate form of stability.

Note that on this approach, the extension is no longer a non-linguistic entity. For to each point is assigned a specification, which is language-based. Moreover, if there are external connections, this linguistic dependency cannot be avoided. For example, it will be part of the extension of 'red' that its completion never overlaps with the completion of the extension of 'pink'. Only for the language as a whole will extension be non-linguistic.

On the super-truth account, the definitions can be simplified. The extension of a sentence (A, t) will be an ordered pair (U, V), where $U = \{u : t \leqslant u \text{ and } u \text{ is complete}\}$ and $V = \{v \in U : v \models A\}$. The relation of extension has then a similar definition. The partial specification-points can be recovered from the complete ones so long as two further conditions are satisfied. The first is that two points are identical if the complete specifications assigned to their successors are the same. The second is that for any non-empty set of complete points there is a point extended by exactly those points in the set. For then each partial point can be identified with a non-empty set of admissible specifications. The first condition is harmless and the second can be justified. For the only constraint on admission into the appropriate space is that a point can verify all the original penumbral truths. But if they are true at all of the complete points, they are true at any point extended by a certain subset of the complete points.

The interpretation for the second approach has an important distinguishing feature. On both approaches, the extension-relation is used to formulate a constraint on extensions. But only on the second approach does this relation enter into the extensions themselves. Thus how an extension can be extended is already part of the extension. The extension of an expression at a given point uniquely determines its extension at a subsequent point.

The intensional analogue of this is that how an expression can be made more precise is already part of its meaning. Let the *actual* meaning of a simple predicate, say, be what helps determine its instances and counter-instances. Let its *potential* meaning consist of the possibilities for making it more precise. Then the point is that the meaning of an expression is a product of both its actual and potential meaning. In understanding a language one has thereby understood how it can be made more precise; one has understood, in terms of the earlier dynamic model, the possibilities for its growth.

This difference in extension (or intension) implies a corresponding difference in the notion of making more precise. On the first approach, to extend is to resolve new cases *or* to make new connections. For to exclude subsequent specifications is also to extend. For example, suppose that there are no penumbral connections between 'red' and 'scarlet'. Then to require that all scarlet objects be red is to extend on the second approach, but not on the first.

3. The super-truth theory

In this section we shall argue for the super-truth theory, that a vague sentence is true if it is true for all admissible and complete specifications. An intensional version of the theory is that a sentence is true if it is true for all ways of making it completely precise (or, more generally, that an expression has a given Fregean reference if it has that reference for all ways of making it completely precise). As such, it is a sort of principle of non-pedantry: truth is secured if it does not turn upon what one means. Absence of meaning makes for absence of truth-value only if presence of meaning could make for diversity of truth-value.

The theory is a partial vindication of the classical position. For the truth-conditions are, if not classical, then classical at a remove. There is but one rule linking truth to classical truth, viz. that truth is truth in each of a set of interpretations. This rule is of general application and not dependent upon the nature of language or interpretation. The actual work is done by the clauses for truth in a single interpretation, and these are classical.

The super-truth view is better than the others for at least two reasons. The first is that it covers all cases of penumbral connection. For example, $P \,\&\, R$ is false and $P \vee R$ is true since one of P and R is true and the other false in any complete and admissible specification. For the bastard intuitionistic account, on the other hand, $P \,\&\, R$ is false but $P \vee R$ is indefinite.

Indeed, one can argue that the super-truth view is the only one to accommodate all penumbral truths. For consider the following clauses:

A (i) $\quad$ not-$t \vDash A \rightarrow (\exists u \geqslant t)(u \Dashv A)$
not-$t \Dashv A \rightarrow (\exists u \geqslant t)(u \vDash A)$,
for A atomic

(ii) $t \models -B \leftrightarrow t \mathrel{=\!\!|} B$
$t \mathrel{=\!\!|} -B \leftrightarrow t \models B$

(iii) $t \models B \,\&\, C \leftrightarrow t \models B$ and $t \models C$
$t \mathrel{=\!\!|} B \,\&\, C \leftrightarrow (\forall u \geqslant t)\,(\exists v \geqslant u)\,(v \mathrel{=\!\!|} B$ or $v \mathrel{=\!\!|} C)$

(iv) $t \models (\forall x)\, B(x) \leftrightarrow t \models B(a)$ for any name a
$t \mathrel{=\!\!|} (\forall x)\, B(x) \leftrightarrow (\forall u \geqslant t)\,(\exists v \geqslant u)\,(v \mathrel{=\!\!|} B(a)$ for some name $a)$

Clause (i) is a Resolution Condition R for atomic sentences and states that an indefinite atomic sentence can be resolved in either way upon improvement in precision. The necessary truth- and falsehood conditions are to the effect that all truth-functional pledges are to be redeemed. For example, clause (iii) for & requires that whenever $B \,\&\, C$ is false it is possible to point to a subsequent specification-point at which either B or C is false.

All of these clauses are reasonable with the possible exception of the sufficient falsehood conditions for & and ∀. But these clauses are required to account for such penumbral falsehoods as 'The blob is pink and red' or 'All pretenders to the throne are the rightful monarch'. Similar considerations apply to the other logical constants ∨, → and ∃. Now given the ancillary conditions F, S and C, the A clauses are equivalent to the super-truth account. Thus the claims of penumbral connection force one to adopt our favoured view.

The second reason for preferring the super-truth view is that it follows an optimizing strategy: maximize one's advantage within the given constraints. The theory maximizes the extent of truth and falsehood subject to the constraints F, S and C. The argument can be put another way. The Resolution Condition R should hold for all sentences, so that any indefinite sentence can be resolved in either one of two ways. The value of indefinite sentences lies in the possibility of this bipolar resolution: they are born, as it were, to be true or false. There is no point in withholding truth from a sentence that can be made true by improving any improvement in precision. Now the super-truth account is the only one to satisfy the four conditions F, C, S and R. Thus placing the right value on indefiniteness also forces one to adopt our favoured view.

These arguments are essentially claims of the following form: such and such theory is the only one to satisfy the reasonable conditions X, Y and Z. Such claims are of great importance, for they provide a point or rationale for the theory in question: if you want the conditions then you must accept the theory. All too often, truth-conditions for different languages have been constructed with insufficient regard for rationale. Their basis has often been a scanty set of intuitions. Thus a great advantage of the present approach is its possession of a uniquely determining rationale.

One might object to the previous arguments on the grounds that they presuppose Completeability, which is unreasonable. However, there is a perfectly a priori argument for this condition. Suppose that the 'limit' of a chain of admissible specifications is also admissible. This is a slight restriction on penumbral connection: for example, it excludes the requirement that the specifications be finite (in an obvious sense) or that they verify decidable theories. Then by Zorn's Lemma, any admissible specification can be extended to a maximally admissible specification. Now suppose that each atomic sentence can always be settled in at least one of two ways, i.e. that no atomic is ever always indefinite. This is a very weak form of Resolution. Then it follows that the maximally admissible specification is complete.

Even without Completeability, our arguments will still go through. In place of the super-truth theory we use an anticipatory account that makes a sentence A true if $--A$ is true on the (bastard) intuitionistic account, i.e. if A is always going to be intuitionistically true. In effect, we mould intuitionism to the Resolution Condition: a sentence whose truth can always be anticipated is already true. This account is the maximal one to satisfy Stability and the necessary A-clauses. The latter consist of A(i), Resolution for atomic sentences, and the left-to-right parts of A(ii)–(iv), Redemption of truth-functional pledges. Moreover, for countable domains, anticipatory truth turns out to be a form of super-truth.[8] Say that a sequence of specification points is *complete* if

(a) each member of the sequence extends its predecessor, and
(b) any sentence is settled by some member of the sequence.

Then a sentence is true on the anticipatory account iff it is true in all generic specifications, i.e. in all limits of complete sequences.

Thus quantification over generic (complete) specifications can be eliminated in favour of quantification over partial specifications. The generic models figure as ideal points; they do not 'exist', but truth-values can be calculated as if they did. This reformulation lends itself to a nominalistic interpretation. The partial specifications are identified with the corresponding collections of predicates. One requires that any borderline case be under our control in the sense that it can be settled by making the predicate more precise. But one does not require that any predicate can be made perfectly precise.

The objection to Completeability may really be a question about our understanding of vague sentence. How, it may be asked, do we *grasp* all of those complete and admissible specifications, the existence of which is necessary to determine truth-value?

There are, I think, three main possibilities. The first is that we understand each of the predicates that make the given predicate perfectly precise. We then grasp the complete and admissible specifications indirectly, as those appropriate to the perfectly precise predicates.
Thus a vague sentence, say:

The blob is red

is like the scheme:

The blob is *R*

where '*R*' stands in for perfectly precise predicates that we are able to enumerate. The main objection to this account is that in understanding a vague predicate we may not understand all or, indeed, any of the predicates that make it perfectly precise.

The second possibility is that we directly grasp all of the admissible and complete specifications. Thus the vague sentence:

The blob is red

is like the open sentence:

The blob belongs to *R*,

where *R* is a variable that ranges over complete and admissible extensions of 'red'. In case of penumbral connection, there will be restrictions on several variables; and in case 'admissible' is vague, it will give way to a third-order variable, and so on. But in any case the principle is the same: one grasps the specifications as being sets of a certain sort. The trouble with this account is that 'admissible' contains a hidden quantifier over non-extensional entities. An admissible specification is one that is appropriate for some precisification. For example, an admissible and complete extension for 'red' is one that is determined by a suitable pair of sharp boundary shades; and a shade is, or corresponds to, a property as opposed to a set.

Thus the third possibility is that we grasp all of the perfect precisifications. The sentence:

The blob is red

is now like the open-sentence

The blob has *R*,

where *R* is a variable that ranges over all of the properties that perfectly precisify 'red'. The perfect properties are grasped, not individually, but as a whole – in one go. There are, perhaps, two main ways in which this can be done. First, they may be understood from below, as the limits of relevant imperfect properties; examples are provided by 'chair' and 'game'. Second, they may be understood from above, in terms of some more direct condition; an example is the sliding scale for 'red'.

Perhaps the main objection to this account is that grasping all properties of a certain kind requires that one be able, in principle, to find a predicate for one such property. But I do not see why any but a constructivist should accept this. One can quantify over a domain without being able to specify an object from it. Surely one can understand what a precise shade is without being able to specify one?

These accounts bring out well the connection and contrast with ambiguity. Vague and ambiguous sentences are subject to similar truth-conditions; a vague sentence is true if true for all complete precisifications; an ambiguous sentence is true if true for all disambiguations. Indeed, the only formal difference is that the precisifications may be infinite, even indefinite, and may be subject to penumbral connection. Vagueness is ambiguity on a grand and systematic scale.

However, how we grasp the precisifications and disambiguations, respectively, is very different. Ambiguity is understood in accordance with the first account: disambiguations are distinguished; to assert an ambiguous sentence is to assert, severally[9], each of its disambiguations. Vagueness is understood in accordance with the third account: precisifications are extended from a common basis and according to common constraints: to assert a vague sentence is to assert, generally, its precisifications. Ambiguity is like the super-imposition of several pictures, vagueness like an unfinished picture, with marginal notes for completion. One can say that a super-imposed picture is realistic if each of its disentanglements are; and one can say that an unfinished picture is realistic if each of its completions are. But even if disentanglements and completions match one for one, how we *see* the pictures will be quite different.

4. The Logic of Vagueness

This completes our discussion of the truth-conditions for the language L. We now turn to logic and consider how the preceding analyses affect the notions of validity and consequence.

On the truth-value approach, a formula is *valid* if it takes a designated value for every specification. If True is the sole designated value, then no formulas are valid on any account that conforms to the stability and fidelity conditions. For they require that any sentence is indefinite if all of its atomic subsentences are. If, somewhat unaccountably, True and Indefinite are the designated values, then validity is classical on any account that conforms to the conditions. For if a sentence is false for a specification, it is false for any of its complete specifications and so is not classically valid. Thus the truth-value approach leads either to classical logic or to the trivial logic, in which there are no valid formulas at all.

Formula B is a *consequence of* formula A if, for any specification, B takes a designated value whenever A does. If True is the sole designated value, then B is a consequence of A on the minimal account iff B is a classical consequence of A and any predicate (or sentence) letter in B is also in A. The maximal account leads to a different consequence-relation with A to $B \vee -B$ being the characteristic non-consequence. If True and Indefinite are the designated values then B is a consequence of A on either account iff $-A$ is a consequence of $-B$ with True as sole designated value.

On the specification space approach, A is valid if it is true in all specification spaces and B is a consequence of A if, for any specification space, B is true whenever A is. This approach gives rise to numerous logics. For example, the bastard intuitionistic truth-conditions lead to a slight extension of intuitionistic logic. On the other hand, the super-truth and anticipatory accounts lead to classical logic. For if a formula is classically valid, i.e. true in all classical models, it is true for all specification spaces, since it is true for each complete specification within the space; and conversely, if a formula is true for all specification spaces, it is classically valid, since each classical model is a degenerate case of a specification space. A similar argument establishes that the consequence-relation is classical for the language at hand. Thus the supertruth theory makes a difference to truth, but not to logic.

Can we maintain that there is no special logic of vagueness? Let us consider two objections against this, one against classical validity and the other against classical consequence.

The first objection is that the Law of the Excluded Middle may fall for vague sentence. For suppose that Herbert is a borderline case of a bald man but that the disjunction 'Herbert is bald ∨ – (Herbert is bald)' is true. Then one of the disjuncts is true. But if the second disjunct is true the first is false. So the sentence 'Herbert is bald' is either true or false, contrary to the supposition that Herbert is a borderline case of a bald man.

The argument here rests on two assumptions. The first is that the classical necessary truth-conditions for 'or' and 'not' are correct. From this it follows that the Law of the Excluded Middle implies the Principle of Bivalence. The second assumption is that borderline cases give rise to sentences without truth-values, i.e. to breakdowns of Bivalence. So from both assumptions it follows that LEM fails for such sentences.

It would be perverse to deny the force of this argument; both of its assumptions are very reasonable. However, I think that one can make out that the argument is a fallacy of equivocation. If truth is super-truth, i.e. relative to a space, then the necessary truth-conditions for 'or' and 'not' fail, though truth-value gaps can exist. If on the other hand, truth is relative to a complete specification then the truth-conditions hold but gaps cannot exist.

An analogy with ambiguity may make the equivocation more palatable. An ambiguous sentence is true if each of its disambiguations is true. Now let J be the ambiguous sentence 'John went to the bank'; let J_1 and J_2 be its disambiguations, viz. 'John went to the money bank' and 'John went to the river bank'; and suppose that John is after fish rather than money. Then the disjunction $J \vee -J$ is true, for its disambiguations, $J_1 \vee -J_1$ and $J_1 \vee -J_2$ are true. However, neither disjunct is true, for each disjunct has a false disambiguation. Thus a truth-value gap exists for assertible or unequivocable truth, whereas the classical truth-conditions hold for truth as relative to a given disambiguation.

Mere ambiguity does not impugn LEM. So why should vagueness? There is, however, a good *ontological* reason for disputing LEM. Suppose I press my hand against my eyes and 'see stars'. Then LEM should hold for the sentence $S =$ 'I see many stars', if it is taken as a vague description of a precise experience. However, LEM should fail for S if it is taken as a precise description of an intrinsically vague experience. Again if the universal set V is taken to be vague, then the sentence '$V \in V \vee - V \in V$' is, I imagine, not true. More generally, a set is vague if it is not the case of every object that it either belongs or does not belong to the set. One cannot but agree with Frege (1952, p. 159) that "the law of the excluded middle is really just another form of the requirement that the concept should have a sharp boundary".[10]

The second objection against the classical solution is that it gives rise to the sorites-type of paradox. Consider the following instance, which is said to go back to Eubulides:

> A man with no hairs on his head is bald
> If a man with n hairs on his head is bald then a man with $(n + 1)$ hairs on his head is bald.
> ∴ A man with a million hairs on his head is bald.

The conclusion follows from the premisses with the help of a million applications of modus ponens and universal instantiation.

The objection now runs like this. The first premiss is true. The second premiss is true: for if not, it is false; but then there is an n such than a man with n hairs on his head is bald and a man with $(n + 1)$ hairs on his head is not bald; and so the predicate 'bald' is precise after all. The conclusion is false. Therefore the reasoning, which is classical, is at fault.

This argument contains two non-sequiturs. The first is that the non-truth of the second premiss implies its falsity; Bivalence may fail for vague sentences. The second is that the existence of the hair-splitting n implies that the predicate 'bald' is precise. One need no more accept this than accept that Herbert is bald or not bald implies that Herbert is a clear case of a bald man.

In fact, on the super-truth view, the second premiss is false. This is because a hair splitting n exists for any complete and admissable specification of 'is bald'. I suspect that the temptation to say that the second premiss is true may have two causes. The first is that the value of a falsifying n appears to be arbitrary. This arbitrariness has nothing to do with vagueness as such. A similar case, but not involving vagueness, is: if n straws do not break a camel's back, then nor do $(n+1)$ straws. The second cause is what one might call truth-value shift. This also lies behind LEM. Thus $A \vee -A$ holds in virtue of a truth that shifts from disjunct to disjunct for different complete specifications, just as the sentence 'for some n, a man with n hairs is bald but a man with $(n+1)$ hairs is not' is true for an n that shifts for different complete specifications.

It is, perhaps, worth pointing out that no special paradoxes of vagueness can arise on the super-truth view, at least for a classical language. For suppose that intuitively false B is a classical consequence of intuitively true A. Then for some complete and admissible specification, A is true and B is false, and this is a classical paradox within a second-order language. This paradox can be brought to the level of the original language if there are predicates to correspond to the complete specification.

Thus the two objections against classical logic for vague sentences cannot be sustained. I do not wish to deny that LEM is counter-intuitive. It is just that external considerations mitigate against it. In particular, an adequate account of penumbral connection appears to require that the logic be classical.

One could, of course, still attempt to construct a logic that was more faithful to unreformed intuition. However, such an attempt would soon run into internal difficulties. One is that our unreformed intuitions on validity do not enable us to decide between the various ways of avoiding LEM. For example, if LEM goes, then so does $A \supset A$ or the standard definition of $\supset$ in terms of $\vee$ and $-$. But which? Again, if LEM goes, then one of $-(A \,\&\, -A)$, de Morgan's Laws, or the substitutability of A for $--A$ must go. Or again, if modus ponens holds but the logic is not classical then either the $(\rightarrow, \vee)$ or $(\rightarrow, -)$ fragment is non-classical.

Another difficulty is that it is hard to motivate a departure from classical logic. Perhaps the best that can be done is this. One interprets 'A or B' as 'clearly $A \vee$ clearly B', 'if A then B' as 'clearly $A \supset B$', 'A and B' as '$A \,\&\, B$', and 'not A' as 'clearly $-A$'. The standard natural deduction rules for disjunction, implication and conjunction will then hold. For example, one still has the Deduction Theorem: if B is a consequence of A

then 'if A then B' is valid. Only negation bears the burden of non-classicality. Also, this account discriminates in a fairly plausible way between conjunction and disjunction. The conjunctions 'P and not P' and 'P and R' are false, while the disjunctions 'P or not P' and 'P or R' are not true. Shifts on conjuncts are allowed, shifts on disjuncts are not.

However, such an alternative does not, in any way, create a challenge for classical logic. For the connectives have merely been re-interpreted within an extension of classical logic. The underlying logic remains classical. There are, then, at least three reasons for adopting a classical solution. The first is that it is a consequence of a truth-definition for which there is strong independent evidence. The second is that it can account for wayward intuitions in an illuminating manner. And the last is that it is simple and non-arbitrary.

5. Higher-order vagueness

One distinctive feature of vagueness is penumbral connection. Another is the possibility of higher-order vagueness. The vague may itself be vague, or vaguely vague, and so on. For suppose that James has a few fewer hairs on his head then his friend Herbert. Then he may well be a borderline case of a borderline case or a borderline case of a borderline case of a borderline case of a bald man.

This feature of vagueness can be expressed with the help of the operator 'D' for 'it is definitely the case that'. Let us define the operator 'I' for 'it is indefinite that' by:

$$IA =_{\text{df}} -DA \,\&\, -D-A\,.$$

This is in analogy to the definition of the contingency operator in modal logic. But note that 'D', unlike the adjective 'definite' or the truth-value designator 'T', is biased towards the truth. Then $I^nFa = \overbrace{II\ldots I}^{n}Fa$ expresses that what a denotes is an n-th order borderline case of F. For example, the first of the two possibilities for James is expressed by: II(James is bald).

The same possibility can be put in terms of the truth-predicate. One says: the sentence 'James is bald is neither true nor false' is neither true nor false. Thus higher-order vagueness can be expressed in the material mode, with the help of the definitely-operator, or in the formal mode, with the help of the truth-predicate.

The above notations would appear to belie undue scepticism over the existence of higher-order vagueness. For if *IFa* can be true, then so surely can *IIFa*, *IIIFa*, and so on. Or again, if *a* can denote a borderline case of the predicate *F*, then surely the sentence *Fa* can be a borderline case of the predicate 'is neither true nor false'. In both instances higher-order vagueness is a species of first-order vagueness: in the first instance, the higher order consists in the correct application of *I* to a statement of indefiniteness, and in the second, the higher-order consists in the truth-predicate possessing borderline cases. This makes a sudden discontinuity in the orders appear unreasonable.

In any case, artificial examples of higher-order vague predicates can be constructed. One might stipulate which borderline cases are to be clear and which not. Indeed, most, if not all, vague predicates in natural language are higher-order vague. Though some, such as 'red', have a higher concentration of 'lateral' or first-order vagueness, whilst others, such as 'few', appear to have a higher concentration of 'vertical' or higher-order vagueness.

How can we characterize higher-order vagueness? We shall consider two equivalent forms of this question. The first is: what are the truth-conditions for a language with the definitely-operator? The second is: what are the truth-conditions for a language with a hierarchy of truth-predicates? To answer the first question, we let L' be the result of enriching the original language L with the operator D, and, to simplify the answer, we first take care of the case of mere first-order vagueness. We consider the truth-value and rival approaches in turn; though, in view of earlier criticisms, the former consideration is an act of generosity.

On the truth-value approach, D should satisfy the following clauses:

$$\models DA \leftrightarrow\ \models A$$
$$\dashv DA \leftrightarrow \text{not} \models A$$

The extended language will no longer satisfy the stability condition, for DA is false for A indefinite, but true for A true. Indeed, *all* three-valued truth-functions can be defined in terms of maximal &, $-$, D and a constant for the Indefinite, whilst all three-valued functions satisfying stability can be defined in terms of maximal $\wedge$, $-$ and constants for the Indefinite and the True.[11]

On the specification space approach, D can receive the following clause:

$w \models DA \leftrightarrow w_s \models A$, where w_s is the base-point, i.e. the admissible specification-point, of the specification space S. In this case, Stability will still hold for the enriched language. However, the proper form for stability is:

$$w \models A \text{ (for the space } S) \text{ and } w \leqslant v \rightarrow v \models A \text{ (for the space } R),$$

where R is obtained from S by taking v as the base point. Now B may be indefinite at w but true at v. So $A = DB$ will be false at w (in S) but true at v (in R). Thus internal Stability holds, but external Stability does not.

The reason for this divergence is that the clause for D ignores any improvement in specification that may have taken place. If it were not ignored, the clause would be:

$$w \models DA \leftrightarrow w \models A$$
$$w \dashv DA \leftrightarrow \text{not} - w \models A$$

The original clause is analogous to that for the operator 'Now' in tense logic (see Prior, 1968) or to the reference clause for certain rigid designators as in Kaplan (unpublished) or Kripke (1972). In all of these cases, the reference (or truth-value) of an expression at an arbitrary point is given in terms of the reference of a simpler expression at a privileged point, the appropriate specification or the present time or the actual world. Reference is, as it were, frozen at the privileged point.

The intensional aspect of the distinction is that the sentences of L' are not necessarily made more precise through making their predicates more precise. Suppose that 'bald' is only first order vague. Then the sentence:

Definitely Herbert is bald

is not made more precise through making 'bald' more precise. Indeed, the sentence is, in the relevant way, already perfectly precise. More generally, the compound sentence will suffer from n-th order vagueness only if its constituent sentence suffers from $(n + 1)$-th order vagueness.

The definitely-operator is not the only one to behave in this way. For example, the sentence:

Casanova believes that he has had many mistresses

may be a precise report of a vague belief or a vague report of a precise belief. In the latter case, the sentence can be made more precise through making 'many' more precise; but in the former case, it cannot.

A compound sheds, as it were, the n-th order vagueness of its constituent and comes under the control of its $(n+1)$-th order vagueness. The phenomenon is formally similar to that behind Frege's distinction between direct and indirect reference. Reference may depend upon indirect reference, indirect reference upon indirectly indirect reference, and so on. Similarly, zero-order vagueness may depend upon first-order vagueness, first-order vagueness upon second-order vagueness, and so on. If indirect reference is taken to be sense, and indirect sense to be itself, then the reference hierarchy has essentially only two terms. The vagueness hierarchy will, of course, have as many terms as there are orders of vagueness.

The logics for D on the different accounts are quite distinctive. Indeed, one might say that the characteristic logical feature of vagueness is not a non-classical logic but a non-classical notion. For the truth-value approach there are six logics in all, one for each independent choice of maximal or minimal and of $\{T\}$ or $\{T, I\}$ as the set of designated values. We shall not go into details. However, for all choices, the unacceptable formula $D(B \vee C) \supset (DB \vee DC)$ is valid. On the super-truth view, the set of valid formulas is given by the modal system S5. This is because a sentence is true at a complete specification-point if and only if it is true at the base specification-point, which holds if and only if the sentence is true at all of the complete points.

On all of the accounts, the Deduction Theorem does not hold for the consequence-relation. This again distinguishes the presence of D from its absence. In particular, DA is a consequence of A but $A \supset DA$ is not valid. For the truth of A guarantees the truth of DA, but the indefiniteness of A implies the falsity of $A \supset DA$. Thus in one sense A and DA are equivalent, for to assert A is to assert DA; while, in another sense, A and DA are not equivalent, for to assert $-A$ is not to assert $-DA$. With the exception of the truth-value accounts with $\{T, I\}$ as designated values, the relationship between consequence and validity is given by: B is a consequence of A if and only of $DA \supset B$ is valid. In the presence of higher-order vagueness, the relationship takes the form: B is a consequence of A if and only if the set $\{-A, B, DB, DDB, \ldots\}$ is not satisfiable.

It is worth noting that the truth of $DA \supset A$ is not completely straightforward. For it involves a sort of penumbral connection between orders of vagueness. Thus on the super-truth view, any complete specification for the predicates of A must be a member of the first-order space that helps to determine the truth-value of DA. This point is even clearer for the truth-predicate. If the sentence Fa is made a clear case of 'true', then the denotation of a must also be made a clear case of F. There is a penumbral connection between 'true' and F.

We must now consider how higher-order vagueness affects the truth-conditions for D. On the truth-value approach, we can no longer be satisfied with the trichotomy True, False and Indefinite. For example, DA will be true if A is definitely true but indefinite if A is indefinitely true. Thus D will not be truth-functional with respect to the three truth-values.

In order to determine the truth-value of DA we need to know whether A is definitely, indefinitely or definitely-not true. But DA may itself come under the scope of a D-operator. So we need to know whether these qualifications apply definitely, indefinitely or definitely-not, and so on. In general, a truth-value of order $n \geqslant 0$ is a 3-valued truth-function f of degree, i.e. number of arguments, n. Thus the ordinary truth-values – T, F and I – are the 0-order functions. That sentence A has 'truth-value' f means that for any ordinary values $x_1, \ldots, x_n$, $O_{x_n} O_{x_{n-1}} \ldots O_{x_1} A$ has value $y = f(x_1, \ldots, x_n)$. O_{x_i} is the operator corresponding to x_i, $i = 1, \ldots, n$. Thus O_T is D, O_I is I and O_F is $D-$.

A sentence can contain any finite number of nested D's. So we must also define an infinite-order value. This may be regarded as an infinite sequence $f^0 f^1 f^2 \ldots$ such that:

(a) f^i is an i-th order value

(b) $f^{i+1}(x_0, x_1, \ldots, x_{i-1}, f^i(x_0, x_1, \ldots, x_{i-1})) \neq$
$\neq F$ for any $x_0, x_1, \ldots, x_{i-1}$, $i = 0, 1, 2, \ldots$.

(b) is a compatibility condition: if f^{i-1} says that $O_{x_{i-1}} O_{x_{i-2}} \ldots O_{x_0} A$ has value x_i, then f^i must not say that $O_{x_i} O_{x_{i-1}} \ldots O_{x_0} A$ has value F.

The truth-conditions are more involved. Suppose second-order functions f and g are assigned to B and C respectively. Then what second-order function $h = f \cup g$ should be assigned to $B \,\&\, C$ upon the maximal account? Let us illustrate the construction of h by putting $x_0 = I$ and $x_1 = T$. The ordinary truth-value of $DI(B \,\&\, C)$ equals that for $D[IB \,\&(DC \vee$ $\vee IC) \vee IC \,\&(DB \vee IC)]$, which equals that for $DIB \,\&(DDC \vee DIC) \vee$ $\vee DIC \,\&(DDB \vee DIC)$. So that if $f(I, T) = g(T, T) = T$ and $f(T, T) =$ $= g(I, T) = I$, say, then $h(I, T) = I$.

This calculation can be made precise as follows. Given a function f of

$(n+1)$ arguments and a 0-order truth-value z, we let f_z be the function defined by:

$$f_z(x_1, \ldots, x_n) = f(z, x_1, \ldots, x_n).$$

We now define operations $\bar{f}, f \cup g, f \cap g$ by induction on the degree n of f and g:

$$\begin{aligned} n = 0.\quad & \bar{T} = F, \bar{F} = T, \bar{I} = I \\ & f \cup g = T \text{ if } f \text{ or } g = T \\ & \qquad\;\; = F \text{ if } f = g = F \\ & f \cap g = T \text{ if } f = g = T \\ & \qquad\;\; = F \text{ if } f \text{ or } g = F \end{aligned}$$

$$\begin{aligned} n > 0.\quad & \bar{f}_T = \bar{f_F}, \bar{f}_F = \bar{f_T}, \bar{f}_I = \bar{f_I} \\ & (f \cup g)_T = f_T \cup g_T \\ & (f \cup g)_F = f_F \cap g_F \\ & (f \cup g)_I = (f_I \cap (g_F \cup g_I)) \cup (g_I \cap (f_F \cup f_I)) \\ & (f \cap g)_T = f_T \cap g_T \\ & (f \cap g)_F = f_F \cup g_F \\ & (f \cap g)_I = (f_I \cap (g_T \cup g_I)) \cup (g_I \cap (f_T \cup f_I)) \end{aligned}$$

There are similar definitions for the minimal account.

The clauses should now go as follows. If infinite-order values $f^0 f^1 \ldots$ and $g^0 g^1 \ldots$ are assigned to B and C respectively, then $(f^0 \cap g^0)(f^1 \cap \cap g^1) \ldots$ is assigned to (B & C), $\bar{f}^0 \bar{f}^1 \ldots$ to $-B$, and $f_T^1 f_T^2 \ldots$ to DB.

It is reasonable to impose several further conditions upon what functions can be values. For example, one can require that $f(x_0, \ldots, x_{n-1}, z) \neq \neq F$ for some value z, or that if $f(x_0, \ldots, x_{n-1}, x_n) = T$ then $f(x_0, \ldots, x_{n-1}, z) = F$ for $z \neq x_n$. In case there is merely vagueness to order k, one should require of the infinite-order values that $f^1(x_0, x_1, \ldots, x_{1-1}, z) = T$ for $z = f^{l-1}(x_0, x_1, \ldots, x_{1-1})$ and $= F$ otherwise, $1 > k$.

The most natural choice for the designated value is the sequence $d^0 d^1 \ldots$, where d^i is the i-th order value such that $d^i(x_0, x_1, \ldots, x_{i-1}) = T$ if $x_0 = x_1 = \cdots = x_{i-1} = T$ and $= F$ otherwise. However, I have not worked out the logics that result from this or other choices.

It would be a bad mistake to fit the values into a discrete linear ordering. For example, one might try to work with the truth-values $T =$ true, $I^k =$ indefinite to degree k, $k > 0$, and $F = I^0 =$ false and declare that

DB had value I^k if B had value I^{k+1} and value $T(F)$ if A had $T(F)$. Such an account would ignore important distinctions. For suppose that we move our blob on the border of pink and red to the pink side of the colour spectrum. Then the sentence P might be indefinitely true but definitely not false, though the above ordering could express no such distinction. It would be an even worse mistake to treat the values as a continuous or densely ordered set, say the real closed interval between 0 and 1, as in Zadeh (1965). More distinctions would go. For example, one could no longer express the fact that Herbert was a *clear* borderline case of a bald man.

We must now consider how the rival approach fares for L' under conditions of higher-order vagueness. The general set-up is extremely complicated, so let us consider the special case of the super-truth theory.

To simplify further, we identify specification-points with specifications. Now suppose we pick upon an admissible complete specification for the language L. If the language suffers from first-order vagueness, this specification is not unique and we may pick upon an admissible set of complete specifications. If the language also suffers from second-order vagueness, this set is not unique and we may pick upon an admissible set of sets, and so on. After $(n + 1)$ such choices, we obtain what might be called an n-th-order boundary.

Let us be more precise. A zero-th order space is a complete specification and a $(n + 1)$-th order space is a set of n-th order spaces. A n-th order boundary is then a sequence $s^0 s^1 \ldots s^n$ such that s^i is an i-th order space, $i \leqslant n$, and $s^j \in s^{j+1}, j < n$; and a ω-order boundary is an infinite sequence $s^0 s^1 \ldots$ such that each $s^0 s^1 \ldots s^i$ is an i-order boundary, $i = 1, 2, \ldots$. A boundary is admissible if each of its terms are and we suppose that the members of an admissible $(n + 1)$-order space are also admissible.

We can now define the truth of L'-sentences relative to an ω-order boundary, or boundary for short. The clauses for the logical constants are standard. The clause for 'D' is:

$$b \vDash D\phi \Leftrightarrow (\forall \text{ boundaries } c)\,(bRc \Leftrightarrow c \vDash B),$$

where $b = b_0 b_1 \ldots Rc = c_0 c_1 \ldots$ if $b_i \in c_{i+1}$, $i = 0, 1, \ldots$ The justification of the clause is this: $D\phi$ is true at b if ϕ is true for all admissible ways of drawing the boundaries; but the admissible zero-order boundaries are the $c_0 \in b_1$, the admissible first-order boundaries the $c_0 c_1$ such that $c_0 \in c_1$

and $c_1 \in b_2$, and so on. Assertible or absolute truth is, in accordance with the super-truth view, truth in all admissible boundaries.

The above clause has the form of the necessity clause in the standard relational semantics for modal logic. However, the 'accessibility' relation R is not primitive but is determined from the structure of the boundary points. This structure is such that R is reflexive; and, in fact, the resulting logic is the modal system T. Further restrictions on R could be obtained by restricting the possible boundary points. For example, given any $n \geqslant 0$, one could require that each boundary $b = b_0 b_1 \ldots$ tapers after n, i.e. that $b_{i+1} = \{b_i\}$ for $i > n$. This corresponds to there being at most n-th order vagueness.

So much for the truth-conditions of L'. We must now consider the truth-conditions for a language with a hierarchy of truth-predicates. We let the meta-language M^0 of level 0 be the original language L, the meta-language M^{n+1} of level $n+1$ be the result of adding the truth-predicate for M^n to M^n (with appropriate means for referring to the sentences of M^n), and the meta-language M^ω of infinite level be the union, in an obvious sense, of the previous languages $M^0, M^1, M^2, \ldots$.

In one way, it is simpler to provide truth-conditions for M^ω than for L'. For each of the meta-languages is merely another first-order language. So any account for the original language L should, when properly generalized, lead to an account for each of the meta-languages.

However, the details for the general case are very complicated. For the truth predicate for L will be defined in terms of the following predicates, say: x is an admissible L-specification; x extends y; the atomic L-sentence A is true (false, indefinite) at x. So the truth-predicate for M^1 will be defined in terms of the corresponding primitives for the language M. But then, in particular, the third primitive must tell us whether it is true, false or indefinite at an M-specification that an atomic L-sentence is true, false or indefinite at an L-specification. The whole process must then be successively repeated for the other meta-languages. If we imagine that the truth-conditions for L are given in the form of a (labelled) tree, then those for M^1 are given by a tree whose nodes are trees that 'grow' throughout the bigger tree, and those for M^2 by a tree whose nodes are ordinary trees, and so on.

However, for particular approaches the details may be much simpler. On the truth-value approach, the truth-predicate for L is defined solely in terms of the primitives: the atomic sentence A is true (false, indefinite). Since truth-value is determined relative to a unique appropriate specification, the admissible specifications drop out of view. The truth-predicate for M^1 is then defined in terms of the primitive: the atomic M^1-sentence A is true, false, or indefinite. But the atomic M-sentences will now include the atomic L-sentences and the sentences of the form:

'A' is true (false, indefinite),

where 'A' is an atomic L-sentence. Similarly for the other meta-languages.

On the super-truth view, the truth-predicate for L is defined in terms of the primitive: 'x is a complete and admissible L-specification'. The assignments of truth-values can be regarded as internal to the specifications and so left out of view. The truth-predicate for M^1 is then defined in terms of the predicate: x is a complete and admissible M-specification. But such a specification will consist of an L-specification and an assignment of an extension to the predicate 'x is a complete and admissible L-specification'. Similarly for the other meta-languages.

Higher-order vagueness gives rise to two puzzles, to which it is difficult to give convincing answers. The first arises from the systematic correlation between the sentences of L' and M^ω. This is provided by the equivalence:

'A' is true $\leftrightarrow$ It is definitely the case that A.

For a sentence of L' can be converted into one of M^ω upon successively replacing innermost 'DA' by ''A' is true$_n$', for n an appropriate level-indicator. Accordingly, there should also be a conversion of truth-conditions. Since we have already given independent truth-definitions for L' and M^ω, this conversion should provide a check on correctness. I cannot give details, but let us observe that there will also be a conversion of conditions. For example, the conditions, given for no vagueness of order $(n+1)$ in L' will correspond to the conditions which guarantee that the truth-predicate for M^{n+1} has no borderline cases.

The puzzle is: should we regard 'DA' as merely elliptical for ''A' is true'? This would be to regard the definitely-operator as a device for incorporating the meta-language into the object-language. The device would, strictly speaking, be improper since it ignores use/mention and type distinctions; but it would be harmless if no quantifiable variables occurred within the scope of 'D'. On the semantic side, it is a matter of whether the extended spaces or truth-values have an independent status or whether they are merely fanciful formulations of the ordinary spaces or values, but for a richer language. An analogous question is whether necessity is best regarded as an operator on or predicate of sentences.

The ellipsis view has the general advantage of replacing a non-extensional operator with an extensional predicate. It has the general disadvantage of involving an incorrect reference to language. Suppose 'bald' has first-order vagueness and the borderline cases are just those people with 40 to 60 cranial hairs. Then 'It is indefinite that Herbert is bald' is synonymous with 'Herbert has between 40 and 60 cranial hairs', but this latter sentence is not synonymous with any claim about a sentence being true. The indefiniteness of vague sentences is as much a matter of fact as the truth or falsehood of precise ones.

Also the ellipsis view has the particular disadvantage of making for a sudden discontinuity between first- and second-order vagueness. First-order vagueness is a matter of ordinary predicates having borderline cases, but second-order vagueness is a matter of the truth-predicate having borderline cases. There is, of course, a *correlation* between the second-order vagueness of ordinary predicates and the first-order vagueness of the truth-predicate. But we feel that the latter arises from the former, and not vice versa. The truth-predicate is supervenient upon the object-language; there can be no independent grounds for its having borderline cases.

Indeed, I think that 'D' is a prior notion to 'true' and not conversely. For let 'true$_T$' be that notion of truth that satisfies the Tarski-equivalence, even for vague sentences:

$$\text{'}A\text{' is true}_T \text{ if and only if } A.$$

The vagueness of 'true$_T$' waxes and wanes, as it were, with the vagueness of the given sentence; so that if a denotes a borderline case of F then Fa is a borderline case of 'true$_T$'. Then the ordinary notion of truth is given by the definition:

$$x \text{ is true} =_{\text{df}} \text{Definitely } (x \text{ is true}_T).$$

Thus 'true$_T$' is primary; 'true' is secondary and to be defined with the help of the definitely-operator.

The second puzzle arises from the demand for a perfectly precise meta-language. So far, we have only demanded of our truth-conditions that they provide correct allocations of truth. To respect the truth-value gap, to account for penumbral connection, to yield the correct logic; these are all special cases of this more general demand. However, one may also require that the meta-language not be vague or, at least, not so vague in its proper part as the object-language. Thus it will not do to subject truth to the standard equivalences:

'A' is true if and only if A.

For then truth will be truth$_T$; the truth-conditions will be classical; and the vagueness of the truth-predicate will exactly match that of the object-language.

What we require is that the true/false/indefinite trichotomy be relatively firm. Ideally, the truth of the disjunction 'A is true, false or indefinite' should imply the truth of one of its disjuncts. It is not that the infirmity of this trichotomy in any way impugns the correctness of the previous accounts. In particular, validity is still classical on the super-truth view; for classically valid A is true in all complete and admissible specifications, regardless of whether it is clear that a particular complete specification is admissible. Rather it is that the infirmity raises another problem for truth-conditions.

This raises the puzzle: is there a perfectly precise meta-language? Certainly, each of the meta-languages M^n could be vague. One could take the whole construction into the transfinite and have, for each ordinal α, a meta-language M^α or strong definitely-operator D^α. But the same problem would arise anew. At no point does it seem natural to call a halt to the increasing orders of vagueness.

However, if a language has a semantics in terms of higher-order boundaries, then it also has a firm truth-predicate. For the boundaries will be based upon a set of admissible specifications and we can let truth (or falsehood) be truth (or falsehood) in all such specifications. Anything that smacks of being a borderline case is treated as a clear borderline case. The meta-languages become precise at some, but no pre-assigned, ordinal level. The only alternative to this is that the set of admissible specifications is itself intrinsically vague. There would then be a very intimate connection between vague language and reality: what language meant would be an intrinsically vague fact.

If higher-order vagueness terminates at some stage α then vagueness can, in a sense, be eliminated. For each sentence A can be replaced by a perfectly precise sentence $D^{\alpha}A$ that entails it. However, this method is unsatisfactory in several ways. First, one may not be able to specify the α. Second, even if one can, one may not be able to make much sense of D^{α}. Our intuitions seem to run out after the second or third orders of vagueness. Perhaps this is because our understanding of vague language is, to a large extent, confused. One sees blurred boundaries, not clear boundaries to boundaries. Finally, the method is too uniform to be discriminate. Penumbral connections may be lost: our blob, for example, is not definitely red or definitely pink. Indeed, the question of making predicates perfectly precise [12] is independent of whether higher-order vagueness terminates. The predicate 'small', as applied to numbers, may suffer from endless higher-order vagueness; yet it can still be made perfectly precise.[13]

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NOTES

[1] I should like to thank Gordon Baker for numerous stimulating conversations on the topics of this paper. My ideas would not have taken their present form without his help. I should also like to thank Michael Dummett for some valuable remarks in a discussion of the paper.

[2] Kleene's (1952, pp. 329 and 332) 'weak/strong' and 'regular' correspond to our 'minimal/maximal' and 'stable', though his motivation for introducing the terms is different from ours.

[3] Frege (1952, p. 63) and Hallden (1949) have adopted the minimal truth-value approach, though Frege would not be happy in regarding Indefinite as a third truth-value. Körner (1960, p. 166) and Åqvist (1962) have espoused the maximal approach.

[4] See the work of Zadeh (1965) and others. It is not clear that one can make much sense of degrees of truth within a closed interval for 'multi-dimensional' vagueness, as in 'chair' and 'game'. It is even less clear that any *semantical* sense can be given to the notion. Possibly there is a confusion with the higher order vagueness of Section 5.

[5] Van Fraassen (1968) has already made much of the super-truth notion, though with different applications in mind. He has also drawn out the consequences for logic and considered the possibility of minimizing and maximizing truth-value (the conservative/radical distinction of van Fraassen (1969).)

[6] The semantics for intuitionistic logic comes from Kripke (1965). The bastard account can be found in Fitting (1969).

[7] The Fregean theory and its extension have a nice algebraic formulation. The usual theory states that there is a homomorphism from the word algebra into the algebra of intensions, and from the algebra of intensions into the algebra of extensions, and hence a homomorphism from the word algebra into the algebra of extensions. The extended theory states that the extension and intension algebras both possess a monotonic partial ordering, which is respected by the homomorphism. It follows that (1) and (2) are implied by (3)–(6).

[8] The argument is Cohen's (1966).

[9] To assert, severally, sentences $P_1, \ldots, P_k$ is not to assert the conjunction or, for that matter, the disjunction of the sentences. For the conjunctive assertion is false if one of the sentences is false, whereas the multiple assertion is false only if each of the sentences is false; and the disjunctive assertion is true if one of the sentences is true, whereas the multiple assertion is true only if each of the sentences is true. These distinctions may have a useful application to the cluster theory of names. For suppose predicates $F_1, \ldots, F_k$ underly the name *a*. Then the assertion of $\phi(a)$ can be regarded as the multiple, as opposed to the conjunctive or disjunctive, assertion of ϕ(the F_1-er), …, ϕ(the F_k-er). A truth-value gap results in case some of the predicates individuate and others do not.

[10] Philosophers have been unduly dismissive over intrinsically vague entities. This attitude may derive, in part, from the view that any piece of empirical reality is isomorphic to a mathematical structure; since the structure is precise, so is the reality. Thus, the blurred outline becomes isomorphic to a set of points in Euclidean space. However, I am not even sure that all mathematical entities are precise. Perhaps one could develop an intuitive theory of vague sets. Hopefully, it would not even be interpretable within standard set theory; so that the sceptic could not then treat vague sets on the onion-model, as a 'facon de parler'.

[11] For references to other results on functional completeness, see Rescher (1969).

[12] This question is usually settled upon covertly constructivist lines. Our powers of perceptual discrimination are limited; therefore we cannot *settle* whether an object has such and such exact shade. But could not a non-constructivist take 'red', say, to mean the colour of *that*, where 'that' refers to a perfectly uniform shade? The inability to know would not affect the ability to mean.

[13] (Added in the proofs). After writing the paper, I discovered that the super-truth account of vague languages had also been espoused by the following authors: H. Kamp in 'Scalar Adjectives', to appear in the *Proceedings of the Cambridge Linguistics Conference*; D. Lewis in 'General Semantics', *Synthese* **22** (1970), 18–67; M. Przełecki in *The Logic of Empirical Theories*, Routledge and Kegan Paul, London, 1969; and P. A. Williams in Chapter 1 of his doctoral thesis. The view seems to go back to H. Mehlberg's *The Reach of Science*, Toronto University Press, Toronto, 1956.

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