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Wittgenstein and Turing

Abstract: A Just-So story, intended as plausible philosophical reconstruction, of the mutual impact of Wittgenstein and Turing upon one another. Recognizably Wittgensteinian features of Turing's diagonal argumentation and machine-model of human computation in "On Computable Numbers, with an Application to the Entscheidungsproblem" (OCN) and his argumentation in "Computing Machinery and Intelligence" (Turing 1950) are drawn out, emphasizing the anti-psychologistic, ordinary language and social aspects of Turing's conception. These were indebted, according to this story, to exposure to Wittgenstein's lectures and dictations. Next Wittgenstein's manuscripts on the foundations of mathematics 1934–1942 are interpreted in light of the impact of Turing's analysis of logic upon them. Themes will include the emergence of rule-following issues, the notion of *Lebensform*, a suggestion about a strand in the private language remarks, and anti-psychologism. The payoff is a novel and more adequate characterization, both of Turing's philosophy of logic and of Wittgenstein's.

1 Introduction

Three assumptions about Wittgenstein and Turing should be surrendered, and it is the argument of this essay that they should be rejected as a whole. First, it is usually assumed that Wittgenstein and Turing were mutually "alien" to one another, standing on opposite sides of a dichotomy between methods of ordinary language and methods of formal logic.[1] Second, it is assumed that in his later philosophy Wittgenstein was concerned to reject Turing's machine model as an analysis of logic: witness the criticisms of talk of processes, states, and experiences in Wittgenstein's famed discussion in *Philosophical Investigations* of "the machine symbolizing its own modes of operation".[2] Third, it is assumed that Turing himself was a computational reductionist, that is, a mechanistic functionalist about the mind. Although Kripke 1982 does not argue for the last two points explicitly – in fact in a long footnote Kripke says he would like to return to this point (Kripke 1982: 35 – 37, n. 24) – his arguments assume that the "dispositionalist" model of the mind

1 Monk 1990.

2 *Philosophical Investigations*, PI §§193ff.

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is under attack by Wittgenstein in the famed remarks (PI §§193 – 194), a view promulgated, but then later rejected, by Putnam.[3]

My main claims are these:

– Wittgenstein and Turing shared a matrix of foundational ideas about the nature of logic.
– They also discussed the nature, limits, and foundations of logic over several years.
– They drew from one another, as they both recognized, developing a confluence of ideas forged over many years, not a conflict.

Given current scholarly understandings, I have to make the case in two directions, Wittgenstein → Turing, and Turing → Wittgenstein. The latter is more difficult, and I will merely aim to briefly sketch my story here, relying on previously published papers for details of the arguments.[4]

My story will be justified by appealing to background features of the Cambridge context of, and argumentation in, Turing's great paper "On Computable Numbers, with an Application to the "*Entscheidungsproblem*" (OCN) and Turing's subsequent writings 1937–1954, as well as considerations based on Wittgenstein's construction of the rule-following passages and the emergence of his later style of writing.

The latter came into view beginning in the fall of 1936, with Wittgenstein's failed revision of *The Brown Book* (EPB). Wittgenstein would have learned of Turing's result before leaving Cambridge for Norway in summer 1936. The impact of Turing reached through Wittgenstein's subsequent development, culminating in an explicit remark from 1947, as we shall see in Section 4.2 below.

In the spring of 1937 there was, I shall argue, an especially important series of reactions Wittgenstein had to Turing's paper, as indicated by the fact that the themes of *Regelmässigkeit*, rule-following, technique (*Technik*), and especially forms of life (*Lebensformen*) appear for the first time at this point. They are embedded in Wittgenstein's signature interlocutory style, emerging also at just this time.

In turn, as we shall see, Turing's paper was indebted to the Cambridge milieu in which Wittgenstein's *The Blue and Brown Books* (BB) were handed around and discussed among the mathematics students.

3 Putnam's early functionalist theories (Putnam 1960, Putnam 1967) rejected logical behaviorism and endorsed computationalism, but his subsequent doubts were voiced in Putnam 1988b and Putnam 2009. See Floyd 2017a: 108 and Shagrir 2005 for discussions of the evolution of Putnam's own views.
4 Floyd 2012b, Floyd 2013, Floyd 2016, Floyd 2017c, Floyd 2018b.

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As a package, these issues show us much about Wittgenstein's later conception of philosophy, and the stimulus we may see him having received from reacting to Turing's work. "Forms of life" emerge as fundamental and ubiquitous, but only *after* Wittgenstein read Turing's "On Computable Numbers" (OCN) in the spring of 1937. This chronology mirrors a kind of conceptual regression to what is most fundamental, what is "given" in logic (and philosophy).

The story I shall tell is forwarded as a plausible analytical and philosophical account. It requires us to regard Wittgenstein differently, but also Turing. Encapsulated, the proposed Wittgensteinian re-reading of Turing is this.

1. Turing's philosophical attitude has been distorted by controversies in recent philosophy of mind (Putnam): computationalist and behaviorist reductionisms, functionalism and the idea of an era in which machines will inevitably become the primary drivers of cultural change and creativity. (Of course this is not to deny that Turing pioneered philosophical discussion of computational explanation and modeling in such far-flung fields as cognitive science, artificial intelligence, neurological connectionism.)
2. Turing was neither a behaviorist nor a reductive mental mechanist. Philosophy of logic, not philosophy of mind, was central for his work on foundations. A Cartesian/behavioristic reading of the "Turing Test" (1950) for over 50 years focused on the individual mind at the expense of the social, despite the fact that for Turing it was the delicate, meaning-saturated human-to-human relations in the presence of machines that was fundamental to the test, not human-machine interface per se. Turing himself regarded intelligence as an "emotional" concept, one that is irreducible, response- and context-dependent, socially embedded and driven by human communicative evolution on a global scale.[5]
3. Turing learned from Wittgenstein that the evolution of our symbolic powers, individual and collective, lies within the forms of life and contingencies of contexts in which words are repeatedly embedded in life, types and categories evolving under the pressure of speech and action. To this end, in all his work Turing focussed on taking what we *say* and *do* with words seriously, and on the *limits* of formal methods, not only their power.
4. Everyday language, including our "typings" of objects as they occur naturally in science and everyday life, are an evolving framework or technology. Influenced in part by Wittgenstein, Turing stressed human conversation,

5 See Turing 1969, Proudfoot 2017, Floyd 2017c.

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"phraseology", and "common sense", as foundational. In this sense he was a Cambridge philosopher of his time, as well as a pragmatist (Misak 2016).

The structure of the paper that follows is this.

First, in Section 2 we reconstruct the evolution of Wittgenstein's thought, focussing on the key transitions that were made in 1937–1939, as part of his response to Turing's OCN. We use the notion of *simplicity* as a thread through this story.

Next, in Section 3 we explain the importance of Wittgensteinian aspects of Turing's analysis that have been widely appreciated. We draw out first the history of Turing's engagement with Wittgenstein (Section 3.1) and the distinctive nature of Turing's analysis of what a formal system (in the relevant Hilbertian sense) is, emphasizing its philosophical aspects (Section 3.2).

Finally, in Section 4 we consider a distinctive form of diagonal argumentation that both Turing, and then later Wittgenstein – responding to Turing – emphasize. We treat first Turing's own version (Section 4.1) and then Wittgenstein's rendition of the proof (Section 4.2). The latter draws Turing's argument into the orbit of Wittgenstein's mature philosophy quite explicitly.

2 Wittgenstein

2.1 Wittgenstein on Simplicity

To achieve a synoptic overview let us first consider Wittgenstein's development to have taken place in four stages, driven forward by a signal concept for him (and for Turing): the notion of *simplicity*. This notion took on a variety of forms in Viennese philosophy and philosophy of science in the wake of Mach's emphasis on the importance of "economy" in mathematics and logic.[6] Simplicity is not a simple notion.[7] However, roughly but not too controversially, we may regard Wittgenstein's thinking about the role of simplicity in logic as having unfolded in four roughly distinct phases:

- Simplicity as an absolute ideal (1914-1921)
- Simplicity as relative to *Satzsystem* (1929-1932)
- Simplicity given in language-games (1933-1936)

6 Stadler 2018 gives a nice overview of this principle's influence on much subsequent philosophy of science.

7 Floyd 2017b.

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– Simplicity as fluid and ubiquitous (1937-1951)

What I shall argue is that the final step, earmarked by what we may think of as Wittgenstein's *mature* conception of simplicity, was secured by his reading of Turing's OCN.[8]

In the **first** stage (1914–1927), contained in the *Tractatus* (TLP), simplicity in logical analysis is an *absolute* ideal. All propositions are truth-functions of elementary propositions. Objects are simple and indefinable. They show forth in our picturing of possible situations. There is a "calculus of indefinables".[9] The totality of what can be said may be presented (schematically) *via* a "form series" variable, expressing the form of a well-founded ordering of propositions according to a rule, collected by a form-series (step-by-step symbolically specified) rule utilizing truth functions:

$$[\overline{p}, \overline{\xi}, N(\overline{\xi})]$$ [10]

In the **second** stage (1929–1933), the "Middle Wittgenstein" reacts against this absolutist ideal, surrendering the general form of proposition and becoming a relativist about analysis. On his new view, a kind of compromise between the *Tractatus* and what would come later on, simplicity is *relative* to a grammatical "*Satzsystem*". Thus it is no longer essentially truth-functional. For there are many different *Satzsysteme*, or "calculi", each with their own simples (indefinables). These are relative to our forms of representation. The perspective remains a hybrid with the earlier *Tractatus* view, however, for within each *Satzsystem* simplicity is still absolute.

The idea of "aspects" of grammar enters as a newly-centered focus in this relativized conception of simplicity: logical "features" are not merely *Züge* in the sense of formal truth-functional operations on propositions, as in the earlier view, but grammatical features of uses of language. They are drawn out in "perspicuous representations" of grammar.

Influenced by Ramsey and pragmatism about logic, Wittgenstein construes beliefs as purpose-relative hypotheses, open generalizations, tools for organizing expectations. Grappling with Hilbert, Brouwer, Weyl, and Waismann, Wittgenstein develops the idea that generality in mathematics uses templates, schemata, step-by-step "logic-free" definitions. Proofs offer decision procedures, determining the "meaning" of mathematical propositions in particular "spaces" of grammar.

8 For further detail on this framework of analysis, see Floyd 2016, Floyd 2018b.

9 Wittgenstein MS 111: 31; cf. Engelmann 2013: 128.

10 TLP 5.2522, TLP 6. For detailed reconstructions see Leblanc 1972, Ricketts 2014, Weiss 2017.

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In the **third** stage (1933–1936), Wittgenstein reaches the view expressed in *The Blue and Brown Books* (BB). In these texts it is language-games that are the seat of simplicity: stepwise-embedded, linearly ordered, and anthropologically cast. To imagine a language is to imagine a "culture" (*Kultur*). Simplicity in analysis is comparative, analogical, and evolutionary. Rules are given by tables followed step-by-step by humans. Humans may amalgamate, share, and hand off procedures. There are no longer any "indefinables".

A Spenglerian flavor haunts this stage of Wittgenstein's thought: an additive, linear structure is used to present differing language-games in a quasi-evolutionary way. There is no sharp or general distinction between "automatic" and "non-automatic" behavior: all is cast anthropologically. And Wittgenstein's remarks about the question, "Can a machine think?" – likely read by the undergraduate Turing[11] – treat it as a grammatical or conceptually analogical issue. There remains a contextually important emphasis on the distinction between a "calculation" and an "experiment": the contrast between necessary, internal relations and those that are empirical.

Most importantly, in §41 of *The Brown Book* Wittgenstein broaches the idea of what he calls "general training": the idea that we could teach a person to follow *any* rule couched in terms of symbols and stepwise directional movements. The idea of a rule as a table; the problem of how determinate this "general training" might be, what the scope of this image of logic is – all these things are very close to what Turing would clarify in OCN, as we shall canvas it below.

But there are clear problems, both with the vagueness in Wittgenstein's remarks here, and in his way of presenting language-games. In his mature period (1936–1946) the text of the *Philosophical Investigations* (PI) emerges, beginning in the autumn of 1936.

What are the hallmarks of this mature, **fourth**-stage view?

- Wittgenstein's ideal of simplicity is "domesticated" and the notions of "culture" [*Kultur*] and "common sense" are eliminated in favor of rule-following and simplicity as embedded in environments and "forms of life" [*Lebensformen*]. The term "*Kultur*" is deleted from the manuscript of PI in the fall of 1936, and never returns to any further version of the manuscript.
- Simplicity is now fluid and ubiquitous, achieved, then contested, still comparative, but dynamic and complex. There are analyses, but they are conducted in "investigations", partial searches testing "harmonies" among us. These are then embedded in further searches, moved, separated, amalgamated, etc.

11 See Floyd 2017c for the arguments.

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– Wittgenstein now conceives applications of the notion of "simplicity" to be "home-spun" [*hausgebachen*] (MS 152: 96): they are woven out of the embeddings of words in forms of life, revealed in what is taken to be meaningful in everyday or ordinary life.

This gives us a way to think about the Turing → Wittgenstein direction of influence, bearing in mind Turing's philosophical achievement in his OCN. For in Wittgenstein's mature philosophy there remains a unity and robustness in the logical, responding to the generality and mathematical robustness of Turing's analysis of what it is to take a "step" in a formal system of logic (see section 3 below). For this is conceived by Wittgenstein in terms of step-by-step, partially-defined, rule-governed, symbolically articulated procedures and their backdrop in interlocutory exchanges and forms of life. This recovered, *realistic* unity, a kind of norm of elucidation for philosophy – the embedding of language-games in forms of life – is what prevents Wittgenstein's mature idea of logic from hardening into a dogmatically asserted totality of propositions, a static, divided archipelago of conventional schemes, or an artificially ordered series of games.

Wittgenstein's conception of the logical after 1937 exhibits certain particularly striking features. We can explain how he got to his mature philosophy by noticing several things connected, I believe, with his response to Turing's OCN.

1. It was first in the spring of 1937 that Wittgenstein revisited themes of the *Tractatus* and of philosophical method.[12]
2. At this time, for the first time, he turned concertedly toward a detailed investigation of the idea of rule-following and *Regelmässigkeit*.
3. Wittgenstein's famed remarks about the machine that "symbolizes its own modes of operation" (PI §§193ff) are first written down in the fall of 1937.
4. For the first time Wittgenstein investigated the shading off of "calculation" and "experiment" in everyday life.
5. Wittgenstein drew in, for the first time, the notion of a form of life (*Lebensform*).[13]
6. Perhaps surprisingly, the term "*Technik*" first occurs in Wittgenstein's writings only in 1937.[14] It is explored thereafter in his writings as a notion and as an object of reflection.

12 Cf. Engelmann 2013 for a discussion.

13 See Engelmann 2013, Floyd 2016, and Floyd/Mühlhölzer forth.

14 See MS 118: 874.

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When it first enters into his writing, the notion of "technique" is marked by a reference to "Watson".[15] This is an allusion to Wittgenstein's summer 1937 discussions with Alister Watson and Turing, an important fact that connects the notion to his deepening reflections on the idea of *following a rule* and the notion of *Regelmässigkeit*, and regularity.[16] (We shall discuss this below in Section 2.) After this point this term fills the pages of his writings and lectures, becoming a signature notion of his mature philosophy (it occurs in his *Cambridge Lectures on the Foundations of Mathematics* (LFM) 117 times).

As I see it, Turing's analysis of a logical "step" in OCN got Wittgenstein to see a "dynamic" perspective as a way to conceive the nature and limits of the logical, and the notion of a "technique", devised to mark the moment in which a routine is embedded in ordinary life, reflects this.

This chronology is made sense of by the analysis I shall give, and the chronology makes sense of how my analysis works. Let us review some of the key moments in this unfolding of thought.

2.2 The *Urfassung* of PI: 1936–1937

At the end of summer 1936, his Cambridge fellowship over, Wittgenstein went to Norway and attempted to turn the dictated *Brown Book* into a book manuscript (MS 115, EPB). The first appearance of "*Lebensform*" in Wittgenstein's corpus occurs in the fall of 1936 (EPB: 108). It occurs in a discussion of a language or "culture" where there is an environment, and words for color, that are very different from our own. He struggles a bit with the idea of what it is to "think of a use of language or a language" fixing "gaps" in grammar, and after several variants ("life form"/"form of life") he settles on his mature language, the language that remains in PI: To imagine a language is to imagine, not a *Kultur* (as it was in BB), but rather a *Lebensform* (cf. PI §19).

By p. 118 of EPB, Wittgenstein drew a line through the page, writing "This whole 'attempted revision' is *worthless*". After some difficult days, he began a new manuscript (MS 142, see BEE). This would become the so-called "Original Version" of *Philosophical Investigations*, the so-called *Urfassung* (UF in KgE). Seventy-six pages of the *Urfassung* were done by Christmas 1936. Several features are especially important:

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15 See the so-called "early version" of PI, the so-called "*Frühfassung*" FF §322 (KgE: 396 = RFM I §133).

16 On these discussions, see Floyd 2001, Floyd 2017c and Section 2 below.

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- Wittgenstein's remarks on Plato's *Theaetetus* and simples are added in more or less their final position (cf. PI §§46ff).
- "Forms of life" enter the manuscript concertedly (PI §§19, 23 – 25), though this key term, being primordial and normative, only occurs five times altogether in PI.[17]
- "Culture" [*Kultur*] and "common sense" are dropped from the manuscript, never entering again.
- Wittgenstein's remarks about Ramsey, logic as a calculus, and logic as a "normative science" (cf. PI §§81ff) are written down.
- The rule-following remarks are broached, but the notion of "technique" is altogether absent.
- The manuscript stops with the question, which remains as yet unanswered, as if a task Wittgenstein leaves himself for the spring: "In what sense is logic something sublime?" (UF §86 = KgE: 130)

Turing sent Wittgenstein an offprint of "On Computable Numbers, with an Application to the *Entscheidungsproblem*" before 11 February 1937. Throughout the spring, in his notebooks Wittgenstein struggles with the notion of simplicity, which he says must be domesticated: "The simple as a sublime term and the simple as an important *form of representation* [*Form der Darstellung*] but with homespun [*hausbackener*] application" (MS 152: 96). Our argument is that it is Turing who showed that analysis in the sense of formal logic, the very idea of "simplicity" of formal steps, their transparency and gap-free character, *must* have a "homespun" use. The terms "simple" and "simplest", explicitly thematized and relied upon, occur 10 times in Turing's OCN.

2.3 From the *Urfassung* of PI to the *Frühfassung*: 1937–38

Wittgenstein completed the *Urfassung* before leaving Norway on May 1st, 1937. During the spring there is substantial development of his mature philosophy of logic: the ideal of the "sublimity" of logic is reworked. Now its "sublimity" lies precisely in our everyday applications of it, what at first seem like "rags and dust" but which allow logic the friction and sensitivities of use it requires (cf. PI §§52, 107). The themes of rule-following and *Regelmässigkeit* are worked through and developed for the first time. And Wittgenstein begins to reconsider the very idea of a "foundation" of mathematics.

17 For discussion see Floyd 2016, Floyd forth.

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Completed in spring 1937, the *Urfassung* is the manuscript source of PI §§1–189. It is surely significant that the manuscript ends with what Wittgenstein will set himself to clarify over his summer break: "But are the steps then *not* determined by the algebraical formula?" — The question contains a mistake." (UF §189 = KgE: 204)

And indeed this question *does* contain a mistake, if we think of Turing's way of analyzing the idea of "determining the steps" in something other than a miraculous or "purely formal" way. This we shall discuss below in Section 3: the *Entscheidungsproblem* shows that the demand for a free-standing answer, Yes or No, cannot be made unequivocally.

Back in Cambridge in the summer of 1937, Wittgenstein had a typescript made of the *Urfassung* (TS 220). He showed it to Moore, who noticed the introduction of the new remarks about simples alluding to Plato. According to Rhees, Wittgenstein told Moore that in *The Brown Book* he had used a "false method" (*falsche Methode*), but that now he had found the "right" or "correct" method (*die richtige Methode*). Moore told Rhees that he did not understand what this meant.[18] But I think we can, with the power of hindsight.[19]

It was the *Urfassung's* closing question about steps being or not being determined by an algebraical formula that may have inclined Wittgenstein to join Alister Watson and Turing in a summer discussion group at Cambridge that was devoted to discussing the philosophical significance for foundations of mathematics of the recent undecidability results of the 1930s, including Turing's OCN.[20] Wittgenstein had known both of these Kingsmen since their undergraduate days: so it was not the first time they had talked. But the context was new, and they were each thinking about how to characterize it. After all, these undecidability results show that a naïve conception of "determining the steps" algorithmically has its provable limitations.

Alister Watson's *Mind* paper (Watson 1938) was one result of these discussions. Watson explicitly thanked Turing and Wittgenstein, particularly for discussions of how best to represent the philosophical significance of Gödel's incompleteness theorems.[21] He closed with the thought that we are not much further along in the foundations of mathematics from the ancient Greeks, with their puzzles about the continuum.

Another immediate result of the 1937 discussions was Wittgenstein's turn toward writing numerous remarks on the foundations of mathematics. This,

18 See Rhees's introduction to EPB: 12 – 13; the editors of PI disagree with Rhees's claim that Wittgenstein brought both TS 220 and TS 221 with him to Cambridge in the summer of 1937.

19 For more detailed discussion of my claims and the dates see Floyd 2016: 21, Floyd 2018b: 72ff.

20 See Floyd 2001, Floyd 2017c.

21 Watson 1938: 445.

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long planned as part of his envisioned book, began in earnest in the autumn of 1937 with remarks echoing those in Watson's "Mathematics and Its Foundations" (Watson 1938). Wittgenstein discussed, not only rule-following and Gödel, but the whole idea of a machine that "symbolizes its own modes of operation". Wittgenstein's focus on the foundations of mathematics lasted through 1944. Floyd/Mühlhölzer forth. discusses the non-extensionalist conception of the real numbers that Wittgenstein developed, focussing on Wittgenstein's responses to Hardy's *A Course of Pure Mathematics*. It is significant that already in the spring of 1937, in light of issues about the unique representability of real numbers, Wittgenstein was turning toward ideas about the differing ways we have of thinking about irrationality, infinity and the continuum.[22] These are themes with which Turing is struggling in OCN.

In the autumn of 1937–1938, right after the discussion with Watson and Turing, Wittgenstein's *Urfassung* of PI was immediately extended to become the so-called "Early Version", the *Frühfassung* (FF) of PI.[23] Here the mature perspective developed in the *Urfassung* is applied to logic the foundations of mathematics. This extension is the basis for what was later excised from PI, and published as RFM I.

In this manuscript we see the first occurrences of Wittgenstein's remarks about our conception of "the machine as symbolizing its own ways of operating" (PI §§193ff). It is we humans who are living creatures who self-conceive *as* machines: we know what it is to "reckon without thinking" according to a rule. The significance of this will become clearer below in Section 3, when we discuss Turing's analysis of a "step" in a formal system.

Drawing out the importance of contrasting varieties of "technique", is what, on Wittgenstein's mature view, *allows* a variety of aspects of numbers to be seen. Aspects are discovered. Techniques are by contrast invented.[24] This is a form of realism, understood in the sense of Diamond's realistic spirit: the *fitting* of concepts

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22 As I discuss in Floyd 2016: 21ff., Wittgenstein's Notebook 152, written in the spring of 1937, not only concerns the themes of simplicity and sublimity, but also begins with warmup exercises in the theory of continued fractions, in which the real numbers receive unique decimal representations (unlike our decimal sequence representations). We know Turing was concerned about the implications of this for his analysis of "computable" real numbers; on this see Floyd 2017c: 125, n. 64.

23 Published in KgE: 205 – 446.

24 In Floyd/Mühlhölzer forth., chapter 8, I explain that "techniques" are invented, whereas "aspects" are, for Wittgenstein, discovered. Textual evidence may be found at BT §134; RFM II §38, RFM III §§46ff; MS 122: 15, 88, 90; PI §§119, 124 – 129, 133, 222, 262, 387, and 536; xi: 196; PPF xi, §130. Floyd 2018a analyzes this distinction, while Kanamori 2018 applies it to the real numbers.

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to reality in forms of life.[25] The idea of a "technique" is designed to register the activity of our designing the "fitting" that goes on.

Wittgenstein lectured at Cambridge in early 1938 on Gödel in an exploratory vein, focussing on the role of negation and the concept of "provability" in Gödel's proof.[26] Oddly it seems he was anticipating questions about the range of proofs about provability only later rigorized.[27] His investigations focussed on the borders of incompleteness, looking at what would be required to establish that they must exist.

Finally, Wittgenstein submitted the *Frühversion* of PI to the Cambridge Press in September 1938[28] with a Preface emphasizing that the method is *not* "gap free" [*lückenlose*], it doesn't run along one "track" (cf. PI, Preface). This apt metaphor, explored in his manuscripts in the period 1937–1939, squares with Turing's analysis of logic as well as Wittgenstein's mature view of formal logic. For Turing shows that it is the partially, and not the totally defined function that must be taken as the basic notion in analyzing the idea of a logical "step". Given this, the embedding of routines in *Lebensformen* – where there may be drift, misunderstanding, and contingencies of application – is inevitable. We shall clarify this point in what follows.

3 Turing

3.1 Turing and Wittgenstein

As is well known, Turing attended Wittgenstein's 1939 lectures at Cambridge on the foundations of mathematics (LFM). Their discussions of contradictions are often regarded as expressing fundamental philosophical or ideological disagreements.[29]

25 See Diamond 1991.

26 See WCL: 50 – 57.

27 Henkin 1952: 160 asked a question not too far from some of the questions Wittgenstein raised:

$$\text{If } \phi \text{ is } Bew(\ulcorner\phi\urcorner)\text{, Does } \Sigma \vdash \phi?$$

Löb 1955 then showed:

$$\text{If } \Sigma \vdash Bew(\ulcorner\phi\urcorner) \rightarrow \phi\text{, then } \Sigma \vdash \phi.$$

28 See Monk 1990: 413.

29 See Monk 1990.

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But what is less emphasized is that Turing's attendance, taking place during just the time he was beginning at Bletchley, was a continuation of earlier discussions. They reflect Turing's even earlier engagement with Wittgenstein as an undergraduate, engagement I shall argue left its imprint, not only on Turing's general philosophical views about logic, but on the precise argumentation he gives in his OCN.[30] Turing's implementation of diagonal argumentation, later revisited by Wittgenstein, will be interpreted below in Section 4.1. It has a Wittgensteinian flavor, one related importantly to the later 1939 discussions between Wittgenstein and Turing.

For now the important point is to note that there was a general Cambridge context, associated Wittgenstein, Whitehead, Russell, Ramsey, Nicod and others, in which foundational issues about logic in general, and types and recursion in particular, were avidly discussed.[31] Turing was an undergraduate 1931–34, and a King's Postgraduate Fellow 1934–36; Alister Watson was "Kingsman" as well, an undergraduate 1926–1933 and then a Postgraduate Fellow 1933–1939.

In the spring of 1932 in his Cambridge course of lectures "Philosophy" Wittgenstein came up with an original analysis of equational recursive specifications in which the need for a uniqueness rule was made explicit.[32] In the autumn of 1932 he began teaching a second, separate course called "Philosophy for Mathematicians" to hash the ideas out further. He argued there that

> What counts in mathematics is what is written down: if a mathematician exhibits a piece of reasoning one does not inquire about a psychological process.[33]

In the autumn of 1933 this course was taught again, and over forty students showed up to the first few lectures.[34] Seeking dialogue and discussion, Wittgenstein dismissed the class, stating that instead of offering lectures he would dictate ideas and distribute the transcriptions to the class. This was the context in which *The*

30 See Floyd 2017c for details. Hodges 1983 reports Turing engaged with Alister Watson in discussion of methods of diagonal argumentation in 1935, perhaps before Turing's idea of a "machine" had occurred to him. As I explain in Floyd 2018b: 73, n. 19, the presence of a 0-1 array to present Cantor's method of argument in a recursive, constructive vein was already present in Wittgenstein's MS 157a, written by hand in either 1934 or 1937, and possibly in 1935. This is a precursor to Wittgenstein's presentation of the diagonal argument in RPP 1 §§106ff (MS 135: 118, TS 229, §1764), discussed below in section 4.2.

31 See Floyd 2017c for a detailed argument.

32 See Marion/Okada 2018 for details.

33 AWL: 225.

34 Notes of these lectures have been published in AWL. Recently other transcriptions of these and related discussions taken down by Francis Skinner have been found, including an alternative, longer version of *The Brown Book* and lectures on the nature of logic; these will be edited and published: see Gibson 2010.

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Blue Book (1933–1934) and *The Brown Book* (1934–1935) were dictated: mathematics students were the desired audience.

There is good reason to find it plausible that Turing was exposed to these dictations, either by attending the 1933 autumn lectures or reading the dictated notes of them. It is also possible that he attended Wittgenstein's 1932–33 version of the course. For by March of 1933 we know that Turing had avidly read Russell's *Introduction to Mathematical Philosophy* (Russell 1920), in which Wittgenstein's view that logic is tautologous was discussed. And in December 1933 Turing gave a talk to the Moral Sciences Club, arguing that

> ... the purely logistic view of mathematics is inadequate; mathematics has a variety of interpretations, not just one ...[35]

We have here a view orbiting in the circle of Wittgenstein's ideas, quite different from the conception of logic being promulgated at that time by Carnap, in his logical syntax phase. Turing regards this conception as "inadequate".[36]

Whatever the case before 1939, in 1939 Wittgenstein and Turing were continuing conversations in the classroom during that spring in a cooperative, rather than an antagonistic vein. Wittgenstein knew about Turing's famous paper, and they were continuing to discuss the implications of Wittgenstein's new-found focus on rule-following. Each learns from the other, as is evident from the very first lecture where Wittgenstein makes an inside joke with Turing about the distinction between signs and symbols.

It is also clear that Turing continued working on philosophical aspects of logic afterwards, while at Bletchley. He explicitly states that his unpublished paper "The Reform of Mathematical Notation and Phraseology" (Turing 2001b, 1942–44) was influenced by Wittgenstein's lectures, in particular (as he says) the idea of handling types with ordinary ways of speaking.[37] He argues here that what needs to be taken seriously is the end-user, the ordinary "phraseology" of mathematics, rather than the "anti-democratic" ideal of a single, overarching formalism, which would serve as a kind of "straightjacket" to thought.[38]

35 See Hodges 1999: 6, discussed in Floyd 2017c: 126.

36 See Floyd 2012a and Floyd 2017c for arguments to this effect.

37 see Floyd 2012b.

38 I discuss Turing 2001b in my Floyd 2013.

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Moreover, in notebooks from the early 1940s Turing continued taking what he called “Notes on Notations". He made analyses and investigations of the specific symbolic devices worked with by Leibniz, Boole, Peano, and others.[39]

These facts serve to correct the portrait of Wittgenstein and Turing as “alien” to one another, or engaged in ideological discussion for and against the use of mathematical logic in philosophy. Instead, they are thinking through foundational issues about logic *with* one another.

But what about the well-known dispute between Turing and Wittgenstein over contradictions in Wittgenstein’s 1939 *Cambridge Lectures on the Foundations of Mathematics*?

As is well-known, Wittgenstein insists in LFM on a non-extensional view of contradiction in conversation with Turing. The presence of a formal contradiction allows, by the rules of classical logic, the problem of cascading or explosion: anything becomes derivable in the system. A non-extensional view allows that when formal contradictions are found, one can put them to the side and move elsewhere in the system, giving one or another practical, purposeful reason for so doing.

Wittgenstein is concerned to emphasize with Turing that it is the *uses* of the system that matter to foundations, not only and primarily the ultimate classical logical properties of the sentences of the language with their formal deductive consequences treated ideally, apart from this. This is the idea of the “homespun” character of formal logic discussed in section 2 above. It is instanced today in our hand calculators: punching in a large enough number will cause the addition program to fail. But we still regard the calculator as “adding”.

Although we should see Wittgenstein working up a philosophical view that is largely congenial to Turing’s OCN, Turing of course pushes back in LFM. Classical logic has its uses, especially in complex empirical situations: there may be situations where these dropping-to-the-side of formalisms would be dangerous, if we are embedding software in powerful and complicated technological projects (such as building bombs, bridges or airplanes). But what will guide us, in addition to issues of consistency and explosion, are approximations, decisions as to scope and probabilities of failure, values about what matters for the purpose at hand.

Wittgenstein’s response, then, is that formal issues of consistency in the sense of classical deductive logic are not necessarily the primary, sole foundation of what matters to the objectivity of applications of arithmetic in everyday life. This tells us something important about foundations. The “homespun” idea is that indeed,

39 These notebooks, from the estate of Robin Gandy, were sold at Bonham’s in 2016 in New York into private hands; see Hodges/Hatton 2015 and Floyd 2017c: 140 n. 100.

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for certain purposes and in certain situations, a contradiction is something we may wish to eliminate. But not because it violates an eternal law of logic that is irreversible or somehow set in abstract stone; instead as a matter of technique, a matter of adapting our formalism to actual cases and situations. – The point, actually fully consistent with Turing's argumentation in his OCN, is to reconstrue what debates over the "reality" of the law of excluded middle come to.

Wittgenstein's notebooks from 1939 include much exploration of the method of diagonalization as a technique that reveals new aspects of concepts. He is interested in exploring the differing guises under which we represent, both diagonalization itself as a method and the real numbers. As we shall see in Section 4.1, this reflects an engagement with Turing over the method of diagonal argumentation that Turing himself used in his OCN. Wittgenstein explicitly takes his own philosophical perspective to be reflected in this (see Section 4.2 below). Indeed, Turing's proof has a distinctly Wittgensteinian flavor, as we shall now argue. In particular, Turing sidesteps debates over the general applicability of the law of excluded middle when he frames his argument resolving the *Entescheidungsproblem*.

3.2 Articulations of the *Entscheidungsproblem*

Wittgenstein was perhaps the earliest person to frame the general decision problem for logic.[40] For he wrote to Russell in 1913:

> The big question now is, How must a system of signs be constituted in order to make every tautology recognizable as such *IN ONE AND THE SAME WAY*? This is the fundamental problem of logic! [*Grundproblem der Logik*]![41]

In terms of an overarching conception of logic, Wittgenstein had already begun to forward the following ideas, characteristic of his philosophy throughout his life:

– The propositions of logic are tautologies (or contradictions), "senseless" (*sinnlos*) but not "nonsense" (*unsinnig*), evincing the limits of true-false talk, i.e., sentences with sense (*Sinn*).
– There are no fundamental axioms ("laws") of logic in the sense that axiomatization does not in and of itself reveal to us what is fundamental to logic itself.

40 See Dreben/Floyd 1991 for a discussion.
41 Wittgenstein to Russell November or December 1913, see letter 30 in WC: 56ff.

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- Logic is to be understood symbolically, in terms of step-by-step procedures that can be written down and recognized by us.
- Philosophy, a part of logic, reflects on the character and limits of this perspective.

The *Entscheidungsproblem* asks whether there exists a definite method that can determine, for every statement of mathematics expressed formally in an axiomatic system (using first-order logic), whether or not that statement can be deduced from the axioms. Hilbert believed in 1930 that the answer would be positive, that there would be no such thing as an "unsolvable" problem.

In 1935 Turing took Newman's course covering the open problems of metamathematics, including the *Entscheidungsproblem*.[42] We know that he was reported discussing diagonal arguments with Alister Watson and Braithwaite at this time. By May 1936 he had resolved the *Entscheidungsproblem* in the negative. It has been an outstanding question how it was that Turing so quickly resolved the question analyzing the notion of a formal system in terms of his "machines". Emphasizing the backdrop to his work in the Cambridge philosophical tradition of discussing the *nature* of logic helps us make clearer sense of this.

The heart of the *Entscheidungsproblem* involved answering the question, What *is* a "definite method"? To satisfactorily resolve it in the negative, one would ultimately have to analyze what is meant *in general* by a "formal system" and a "step" in a formal system in the relevant Hilbertian sense. (Had the problem been answered positively, one would simply have exhibited a Decision Procedure for first-order logical validity). It is crucial that the required general analysis could not be accomplished by simply writing down just another formal system. Nor could it be done by setting out in the metalanguage various kinds of different axiomatic systems. This is why the ("logic-free" versions of) λ-definability and the Herbrand-Gödel-Kleene equational systems were used in the earliest work attempting to clarify what is meant by an "effective" calculation. It is also why Turing devised his machines with command-tables, in a "logic-free", i.e., non-formalized-system-of-logic way. His point was to avoid entanglement with the vagaries of this or that formalization of logic, in order to get to the essence of what a "step" in a formal system *is*.

As is well-known, in 1935 Church, Kleene and Rosser showed that the class of functions calculable in the Herbrand-Gödel-Kleene equational calculus is coextensive with the class of λ-definable functions.[43] In his "Note on the Entscheidungsproblem" (Church 1936) Church, building on Gödel (Gödel 1931), demon-

42 Hodges 1983.
43 See Kleene 1981a, Gandy 1988, Sieg 2009.

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strated that there is no "effectively calculable" function which decides whether two λ-definable expressions are equivalent. This resolved the *Entscheidungsproblem* in the negative. Next, Turing showed, independently of Church, that no "machine" of the type set out in his OCN can "compute" the desired general procedure as an "application" of his wholly novel analysis, also resolving the *Entscheidungsproblem* in the negative. Faced with having been scooped by Church on the result, Turing nevertheless was able to publish his paper because of its conceptual novelty. (In an Appendix he showed that the functions his "machines" can "compute" are just those that are λ-definable.)

It was the clarification of what a formal system or an algorithm or computation *is* that was new in what Turing achieved. As Kleene later put it,

> Turing's computability is intrinsically persuasive, but λ-definability is not intrinsically persuasive and general recursiveness scarcely so either (its author Gödel being [in 1934] not at all persuaded [that it analyzed the idea of "effective calculability" or "calculation in a logic"]).[44]

As Turing's student Gandy wrote of Turing's way of thinking,

> The approach is novel, the style refreshing in its directness and simplicity. The bare-hands, do-it-yourself approach does lead to clumsiness and error. But the way in which he uses concrete objects such as exercise books and printer's ink to illustrate and control the argument is typical of his insight and originality. Let us praise the uncluttered mind . . .
>
> What Turing did, by his analysis of the processes and limitations of calculations of human beings, was to clear away, with a single stroke of his broom, this dependence on contemporary experience, and produce a characterization which – within clearly perceived limits that will stand for all time.[45]

The point is that Turing's particular way of resolving the *Entscheidungsproblem* was *not* the application of a preexisting blueprint of ideas and methods in the metamathematics literature. When he first handed it to Newman, Newman thought it too elementary and nearly discarded it.[46] Instead, Turing offered – in contrast to Gödel, Kleene and Rosser – a philosophically informed, analytic exercise. What he achieved was an intuitively satisfying simplification of . . . *simplicity*! (Here of course we mean "simplicity" in the logician's sense of a transparent, unproblematic simplest *step* in a formal system.) He did so by picturesquely drawing in the idea of a human being operating with a table of rules according to a certain routine.

44 Kleene 1981b: 49; compare the discussion in Kennedy 2017.
45 Gandy 1988: 78, 93.
46 See Hodges 1983: 112.

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This last point has been widely acknowledged.[47] So is the fact that E.L. Post's analysis of logic in terms of "workers" (drawing in the human element as well) is more or less equivalent, an independent achievement.[48] What I am arguing is that Turing's deployment of his central argument also bears the stamp of Wittgenstein's way of thinking about logic "anthropologically", rather than "metamathematically": the idea of simplicity as something "homespun", rather than sublime.

Turing analyzed what a step in a formal system *is* by thinking through what it is *for*, i.e., what is *done* with it. The comprehensiveness of his treatment – its lack of "morals" – lies here. Turing made the very idea of a formal system *plain*, unvarnishing it. It is this, I believe, that Wittgenstein responded to beginning in the spring of 1937. Turing took up a "form of life" or "language-game" stance, not an ideological or metaphysical perspective: he *de*-psychologized the notion of "logic". Unlike Post 1936 and Gödel 1972, Turing did not take his analysis to rest on or even necessarily apply to limits of the human mind *per se*. This was part of his Wittgensteinian inheritance.

Differently put, Turing made the notion of a formal system (or definite method) *surveyable* (*übersichtlich*, *überschaubar*), "open to view". This in turn makes the very idea of surveyability … surveyable! And this would explain as well why it is that the very notion of "surveyability" becomes such a focus in Wittgenstein's manuscripts in 1939.[49] Wittgenstein is exploring, in the wake of his discussions with Turing, what it means to say that a proof is "surveyable", "reproducible", "communicable" and so on.

In the end, to clarify the foundations of logic one must draw in the notion of *a human calculator*. This requires, not a psychological account, but a *logical* one: the idea of a *shareable* human calculating procedure that may be offloaded to a machine or another human prover or calculator.

As Sieg puts it,

> Most importantly in the given intellectual context [the move from arithmetically motivated calculations to general symbolic processes that underlie them] has to be carried out programmatically by human beings: the *Entscheidungsproblem* had to be solved by *us* in a mechanical way; it was the normative demand of radical intersubjectivity between humans that motivated the step from axiomatic to formal systems . . . .
>
> It is for this very reason that Turing most appropriately brings in human computers in a crucial way and exploits the limitations of their processing capacities, when proceeding mechanically.[50]

47 For a discussion see Kennedy 2017.

48 See Post 1936 and Sieg/Mundici 2017.

49 For a detailed commentary and explication of RFM III, from 1939, see Mühlhölzer 2010.

50 Sieg 2006: 200, my emphasis.

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Turing's comparison, in analyzing the idea of a "simplest step" in a formalism, is that:

- (OCN) §9 I: A human computor works locally, step-by-step, and can only take in a certain number of symbols at a glance.
- (OCN) §9 I: The computor takes in "simple operations . . . so elementary that it is not easy to imagine them further divided".
- (OCN) §9 III: As Turing himself puts it, we "avoid introducing the notion of a 'state of mind' by considering a more physical and definite counterpart: it is always possible for the computor to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions (written in standard form) explaining how the work is to be continued. This note is the counterpart of the 'state of mind'.

This last point makes very clear that Turing is *not* relying on any theory of mentality, but only presupposing the human *communicability* of a "step" in calculation. The notion of a shareable routine of reckoning-according-to-a-rule is taken as basic in his model.

4 The Diagonal Argument

Why, on our story, would Wittgenstein have been so struck by Turing's 1936 paper?

It is important here to understand certain philosophical aspects of Turing's method of proof in "On Computable Numbers" (OCN). As we have just argued, what Turing offered was a remarkable analysis of our very idea of a "step" in a formal system. And he did this by embedding the idea of "calculation-in-a-logic" in a shared human world: an analogical simplification.

His analysis does *not* turn on a theory of mental states, mathematics, or logic, but instead on the idea that logic is *written down*, just as Wittgenstein had argued it should be in his Cambridge lectures 1932–1935.[51] Turing takes the everyday human ideas of a "command" and a "calculation" as basic elements of logic and works out a (mathematically robust) "comparison" between the activities of a human and that of a machine. In other words, like Wittgenstein Turing takes the *human*

51 In addition to BB there is the so-called "Yellow Book" and transcriptions of the 1932–33 "Philosophy for Mathematicians" (cf. AWL: 43 – 73, 205 – 225).

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notion of calculation as basic or simple, and builds his analogy with machines from there.[52]

In effect, Turing used the method of what Wittgenstein called *Vergleichsobjekte* (cf. PI §130), objects of comparison.[53] He states explicitly that we may *compare* the activities of a human computor[54] and a machine (OCN, §1). This was a distinctive move, one that probably would not have been made by a mathematician such as Gödel, Church, Rosser or Kleene: it is remarkably simple, down-to-earth, everyday.

This is why, revisiting Turing's paper in a remark written in 1947, subsequently published in *Remarks on the Philosophy of Psychology*, Vol. I (RPP 1 §1096), Wittgenstein says: "These machines [Turing's 'Machines'] are *humans* who calculate."

4.1 Turing's Diagonal Argument

Let us next turn to the actual diagonal proof used in Turing's OCN to resolve the *Entscheidungsproblem* in the negative. I have made a careful reconstruction of the proof elsewhere (Floyd 2012b) and will simply give an overview of the salient philosophical points here.

It is philosophically crucial that OCN does *not* rely fundamentally on the now readily applied "Halting Argument" in order to show that there is no decision procedure for pure logic. Instead, Turing constructs an idiosyncratic machine, utilizing a kind of *positive* argument that does not turn on the production of a contradiction, or the construction of a machine capable of negating the behavior of another machine, as the Halting Argument does.[55]

Instead, Turing's argument turns on the fact that his machine turns up something analogous to the following command, as I have argued elsewhere:[56]

Do What You Do

This expresses a rule that *cannot* be followed. This makes its point deeply philosophical, not only logico-mathematical. For the fact that we can see that this command is, without further supplementation, unuseable demonstrates that the

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52 Floyd 2012b.

53 This reading is laid out in Floyd 2012b, Floyd 2017c, Floyd 2018b.

54 Until the late 1940s "computer" referred to a person, often a woman, who carried out calculations and computations in the setting of an office or research facility. Nowadays "computor" is used to make the human user explicit.

55 Floyd 2012b reconstructs the argument carefully; cf. Floyd 2016, Floyd 2018b.

56 Floyd 2012b, Floyd 2017c.

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human interface, the human context of a shareable command, is fundamental to the nature of logic.

For "Do What You Do" tells you nothing without a specific context of application. It is like a pair of fingers pointing straight at one another. Of course, in an ongoing stream of life, embedded in a conversation or activity with a purpose (e.g., I am showing you how to ride a bike or type the return key on a keyboard repeatedly) "Do What You Do" makes perfect sense, indicating perhaps that you should continue on, doing the same as what you are doing now. Without being embedded in a form of life, however, "Do What You Do" does not issue a command that can be followed (imagine drawing a card in a game with this printed on it). This is what Turing's proof ultimately reveals. The machine he constructs is not contradictory, and does not generate an infinite regress. Rather, we must *see* that such a machine, imagined put into service of a Decision Method for determining first-order logical validity, must stop in the face of its own tautology-like self-inscription. This shows the fundamental need for a context, that is, a form of life in which words and symbols are being embedded.[57]

Right at the beginning of OCN, §9, anticipating his application of the diagonal "process" (as he calls it), Turing notes that he *could* have run his argument differently, by way of contradiction in the manner of the Halting Problem:

> The simplest and most direct proof … is by showing that, if this general process [of determining whether a machine is "circle free"] exists, then there is another ["contradictory"] machine β. This proof, although perfectly sound, has the disadvantage that it may leave the reader with a feeling that "there must be something wrong".

What might be "wrong" is a concern that Turing has assumed, against the intuitionist, that the law of excluded middle applies univocally to all specifications of all Turing Machines. So Turing says,

> The proof which I shall give has not this disadvantage, and gives a certain insight into the significance of the idea "circle-free". It depends not on constructing β [the "Contrary" machine familiar from the Halting Argument, in which machines that halt are changed to those that do, and vice versa, along the diagonal], but on constructing β', whose nth figure is $\phi_n(n)$.

Turing's β' machine is constructed so as to follow its own commands perfectly, without any difficulty, through a series of stages. The difficulty comes when it reaches the particular stage that embodies the machine that it itself is. At this

57 See Floyd 2012b, Floyd 2016, Floyd 2018b for further discussion of the "Do What You Do" argument of Turing.

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point, it comes to the command to do what it itself does: and then it cannot do anything.[58]

An analogy would be with the "positive" Russell Paradox, that is, the issue of the set of all sets that *are* members of themselves. This is the exact complement, so to speak, of the usual Russell set of all sets that are *not* members of themselves. Think of it as the *positive* Russell set. In a certain sense, S "comes before" Russell's set, is more primordial, for there is no use of negation within its definition. And it is not contradictory.

Define

$$S = \{x \mid x \in x\}.$$

Now ask

$$\text{Is } S \in S?$$

And the answer is:

$$\text{If Yes, then } S \in S.$$
$$\text{If No, then } S \notin S.$$

So we have that:

$$S \in S \iff S \in S.$$

There is no inconsistency or paradox here. But there is a problem. For all that we can deduce here is that:

$$S \in S \iff S \in S, \text{ and also } S \notin S \iff S \notin S.$$

We are caught in a kind of circular thought of the form, "it is whatever it is". This is surely not incoherent or inconsistent. The trouble is deeper: the thought cannot be *implemented* or applied.

We have here what might be regarded, following Turing and Wittgenstein, as a kind of performative or empty rule. You are told to do something depending upon what the rule tells you to do, but you cannot do anything, because you get into a loop or tautological circle. This set membership question cannot be a question

58 I explain the argument in detail in Floyd 2012b.

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that can be applied, because one cannot apply the set's defining condition at every point.

An analogous line of reasoning may be applied to, e.g., "autological" in the Grelling paradox if we ask, "Is autological autological?". Without using negation, one does not get a contradiction. But one may generate a question with the concept that may be sensibly answered with either Yes or No. And in this sense it is an unanswerable question. The trouble is, one cannot get to a decision point here. One cannot *play* the game of Yes and No. "Falls under the concept" and "$\in$" cannot be *used* if they are directly equated.

In the above argument an apparently unproblematic way of thinking is applied, but two different ways of thinking about *S* are involved. For there is the thinking of *S* as an object or element that is a member of other sets, and the thinking of *S* in terms of a concept, or defining condition. Similarly, in Turing's OCN proof, there is the unproblematic characterization of a particular machine, and then there is the difficulty that it must, at one precise point or another, get stuck in a loop, confronted with the command to do what it does.

What is important here is that Turing crafts his argument in OCN carefully, in several respects:

- Even an intuitionistic logician who rejects the law of the excluded middle in infinite contexts can accept his analysis of the idea of a "step" in a formal system: "Do What You Do" is not a contradiction so that the proof is not an indirect one.
- Turing does not build into his notion of a "machine" that it must utilize negation, or change halting to non-halting behavior, in its specification.
- Turing's proof demonstrates clearly that is not part of our notion of "following a rule step-by-step" that we do or do not obey the law of excluded middle.
- More generally, Turing's analysis of a "step" in a formal system is altogether independent of *which* formal system we are speaking of, or which particular "states of mind" are actually used, so that the particular choice of formalism or formalized language is not at issue.
- The internal consistency or precise strength of a command structure is not at issue, nor is the internal coherence or strength of a metastance at issue.

In general, Turing is exploiting the fact that formalization alone doesn't settle the analysis. He refuses to ascend to a "metalevel" in a general way, and instead takes on the needed analogy with human activity, working it out mathematically.

Gödel also resisted the idea that the undecidability results tell us anything general about "human reason", holding instead that they reveal something about

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"the potentialities of pure formalism in mathematics".[59] What we learn is something about what formal systems *cannot* do. But the idea of a human being and what he or she can take in as "simple", "gap-free" or "transparent" is at the heart of our very idea of a formal system, and it is this that Turing, and not Gödel, was able to draw out.

To be clear, Gödel was unstinting in his praise of Turing's analysis of the general notion of "formal system". He argued that the precise scope of his own 1931 incompleteness result was only determined by Turing's work, writing:

> The precise and unquestionably adequate definition of the general concept of formal system [made possible by Turing's work allows the incompleteness theorems to be] proved rigorously for *every* consistent formal system containing a certain amount of finitary number theory.[60]

The point here was that a kind of potential "gap" remained in our understanding of the scope of applicability of Gödel's 1931 paper until Turing clarified what we mean *in general* by a "formal system of the relevant kind".[61]

Moreover, Gödel argued, the universality of Turing's analysis made it special, freeing it of entanglement with this or that particular formalism:

> With Turing's analysis of computability one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen . . . . In all other cases treated previously, such as demonstrability or definability, one has been able only to define them relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, although it is merely a special kind of demonstrability or definability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion.[62]

I would argue that it is hardly a "miracle" that Turing's analysis dodges the issue of relativity-to-language in the way Gödel suggests. Rather, it is a by-product of his starting point. As to what Gödel means by calling Turing's analysis "absolute": unsurprisingly this remark has been much discussed, since this notion is notoriously difficult to make sense of.[63] However, if we focus on the details of Turing's OCN diagonal argument with Wittgenstein mind, I think what it comes to in this context becomes clearer.

59 Gödel 1964: 370, discussed in Webb 1990: 292ff.
60 Gödel 1964: 369.
61 Compare Kennedy 2017 for a discussion.
62 Gödel 1946: 1.
63 See however Kennedy 2017 for a recent discussion of "formalism freeness" as a wide-ranging logical phenomenon.

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First of all, note that Turing demonstrates that the partially defined, and not the totally defined function, is the basic and more general notion. He does this by framing his Universal Computing Machine *U*, arguing that one machine can do the work of all, suitably alphabetized in a series of finite coded sequences of particular Turing Machines (see OCN, §6). Given *U*, we see that if we suppose we have a total listing of all the machines that compute real decimal expansions, given those machines that are undefined on certain inputs, we cannot diagonalize out à la Cantor. In Table 4.1, the downward arrows act like holes in Swiss cheese: they prevent the diagonal method from being applied in such a way that the enumeration may be said to fail:

Table 1: Turing's Partial Functions Prevent Diagonalization à la Cantor

↓	1	1	0	↓	…
1	0	0	0	1	…
0	1	↓	0	0	…
1	1	0	↓	0	…
1	1	1	1	1	…

Turing shows that an analysis of formal logic cannot be "gap free".

4.2 Wittgenstein's Diagonal Argument

In 1947 Wittgenstein wrote down the following remark, subsequently published in RPP 1 §1096ff:

> Turing's "Machines". These machines are *humans* who calculate. And one might express what he says also in the form of games. And the interesting games would be such as brought one *via* certain rules to nonsensical instructions [*unsinnigen Anweisungen*]. I am thinking of games like the "racing game". One has received the order "Go on in the same way" when this makes no sense, say because one has got into a circle. For that order makes sense only in certain positions. (Watson.)[64]

Wittgenstein is remembering or alluding to his 1937 discussions with Watson and Turing here. And his remark makes it clear that he is fully aware of the distinctive argument that lies at the heart of Turing's negative resolution of the *Entscheidungsproblem* in §9 of his OCN.

64 RPP I §1098 (MS 135: 117, 1947).

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This is clear, because what Wittgenstein does next is to write down an "everyday", "language-game", "forms of life"-embedded version of Turing's proof in OCN. This reformulation casts Turing's argument and its result in a more general manner, one suited to Wittgenstein's mature conception of rule-following and simplicity. On Wittgenstein's view of Turing's argument the idea of a shareable command is shown to be fundamental, and with it the need for techniques and the embedding of words in forms of life. The idea of a rule that is partial, i.e., not everywhere defined, is the basic notion, and not the idea of a rule everywhere defined.

Wittgenstein considers first a list or series of rules – or, as he also say, "laws" – for the expansion of forms of decimal representations of "computable" real numbers

$$\ldots .a_{k1}a_{k2}a_{k3} \ldots.$$

He calls this list $\phi(k, \ldots)$. According to his notation, $\phi(k, n)$ is the nth decimal place determined by the kth rule in the list.

He then argues as follows:

> A variant of [C]antor's diagonal proof:
> Let $v = \phi(k, n)$ be the form of the laws for the expansion of decimal fractions. $\underline{v}$ is the nth decimal place of the $\underline{k}$th expansion. The law of the diagonal then is:
>
> $$v = \phi(n, n) =^{def.} \phi'(n).$$
>
> It is to be proven that $\phi'(n)$ cannot be one of the rules $\phi(k, n)$. Assume it is the 100th. Then we have the formation rule
>
> of $\phi'(1)$: $\phi(1, 1)$
> of $\phi'(2)$: $\phi(2, 2)$
> etc.,
>
> But the rule for the formation of the 100th place of $\phi'(n)$ becomes $\phi(100, 100)$, that is, it tells us only that the 100th place is supposed to be equal to itself, and so for $n = 100$ is *not* a rule.
>
> [I have always had the feeling that the Cantor proof did two things, while appearing to do only one].
>
> The rule of the game runs "Do the same as..." – and in the special case it becomes "Do the same as you are doing".[65]

65 MS 135: 118; the square brackets indicate a passage later deleted when the remark made its way into TS 229/§1764, published as RPP I §1097. As I explain in my Floyd 2012b, in *Zettel* §694 only

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In order to understand this proof, we need to read the law $\phi'(n)$ as an instruction or command, in the way that Turing reads his quintuples specifying his "machines" in his 'On Computable Numbers'. For $n = 1$ it says: calculate the first decimal place provided by the law $\phi(1, \ldots)$; for $n = 2$: calculate the second decimal place provided by the law $\phi(2, \ldots)$; . . . .

There will be no trouble at all until we try to say *which* rule on our list, in particular, this instruction is. Suppose (without loss of generality) that it is the 100th. Then at $n = 100$ we have the following command: calculate the 100th decimal place provided by the law $\phi(100, \ldots)$. But we just presupposed that the law $\phi(100, \ldots)$ is the *same* as $\phi'(n)$! Therefore, this instruction, namely "Calculate $\phi'(100)$ by calculating $\phi(100, 100)$", is identical with the instruction: "Calculate $\phi(100, 100)$ by calculating $\phi(100, 100)$", which is empty. It is not a rule that we can follow as we can the others on the list, and in that sense it is "*not* a rule", as Wittgenstein says.

This is what I called in the last section the "Do What You Do" argument. It is evidently drawn from Turing's argument in OCN, §9. It is free of any tie to a particular formalism or picture or diagramming method or way of representing decimal expansions or rules. And, since it doesn't use negation to formulate the appeal to the diagonal method, it depends upon no restrictions or extensions of the application of any particular logical law.

What Wittgenstein's version of Turing's diagonal argument proves is that there is a new rule (or command) that is not like the other rules on the list, in that it cannot be followed, because it is quasi-tautologous. In this sense his old view of logic holds up: as shown by Turing, the "limits of logic" lie in rules or instructions that *cannot* be applied. Differently put, the idea of a routine everywhere defined from all perspectives is in a sense incomplete.[66]

The mechanism of the argument clearly depends upon our ability to *see* that a rule cannot be followed, rather than our getting one another to agree or disagree about the status or scope of the law of the excluded middle, or a general point of view on negation or contradictions. In this sense Wittgenstein's diagonal argument draws out something fundamental also to Turing's diagonal argument: that it is fundamental to our very idea of logic – more fundamental, in fact than the idea of any particular logical law holding or not holding – that we have a hold on

this second remark concerning the proof is published, thereby separating it from the mention of Turing and Watson – one reason that the close connection with Turing's (OCN) was not noticed by scholars before me.

66 Kreisel later reported (Kreisel 1950: 281 n.) that Wittgenstein's remark about Turing offers a "neat" way of looking at incompleteness, the limitative result being reachable by a command of the form "write what you write".

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everyday ways of applying rules, rule-following, and shareable commands. Logic does not need to depend upon community-wide agreement on philosophical theses or conventions about what is to count as a correct logical "law". It is not a question of consensus, but of forms of life.

For this reason Wittgenstein's argument does not work if one considers the decimal expansions *extensionally*, that is, if one severs the results of the expansion rules from the rules themselves. Then all the expansions are pictured as simply spread out before us, and nothing seems to prevent the unaltered diagonal $\phi'(n)$, $n = 1, 2, \ldots$, of the given series from occurring somewhere in the series itself. Yet as soon as one thinks of the rules as genuine commands, i.e., instructions or procedures given that are to be followed in everyday life, the situation changes radically, as Wittgenstein's argument shows. And this draws out in a beautiful way the richness of Wittgenstein's remarks about rules and rule-following.

It is clear that Wittgenstein was not in any way aiming to *refute* the extensional, completed infinite here. There is nothing wrong with it, intrinsically. But it is not adequate on its own to reveal the foundations of logic. And we get into conceptual trouble when we try to think that it is. Instead, Wittgenstein is emphasizing that there are two different points of view that may be taken up on Cantor's diagonal argument. From the extensional point of view, Cantor is showing us something about the limited nature of a list of sequences to catch (and number) the real numbers. From the non-extensional point of view Cantor has given us a "positive recipe" for constructing more and more sequences. Both points of view are valid in their way. But the nature of the limits of each differ.

This may be seen if we imagine a first-person version of the argument. Consider

I Do What I Do

Bernhard Ritter has suggested a remarkable connection between the Do What I Do argument and the private language argument in Wittgenstein at PI §258. Ritter points to Wittgenstein's "Motor Roller",[67] a story of a steamroller Wittgenstein's father once conceived without seeing at first that turned out to be unable to work. The inner and outer sides of the roller of this "machine" have no friction, the machine, as Wittgenstein says, "admits everything" or is "always right". This is an analogy for the idea that however one behaves, what is going on "privately" "inside" one is somehow metaphysically independent of this.

Ritter's suggestion is that as in the case of the diagonal arguments we have considered, Wittgenstein's point is not to emphasize the need for stage-setting

67 Cf. Ritter forth., ch. 18; MS 131: 219 – 222 from 8–9 September 1946.

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and context in the use of language (as he does in other remarks on "privacy"), but rather to argue that the "private" diarist cannot use his sensation *itself* to say or explain *which* sensation in particular he is having. As in the positive diagonal argument we have considered, the conclusion must be seen directly, not indirectly, in the very attempt to apply itself to itself.

That a connection is to be drawn with the "vanishing" of the "I" is clear from MS 157a: 17r. This diagonal argument, written in Wittgenstein's hand (in 1934–1937), embeds the usual form of Cantor's diagonal argument, where the numbers along the diagonal are altered, directly in considerations about the vanishing of the "I". "I do", Wittgenstein remarks, has "no volume of experience" but rather "seems like a pointless point, the tip of a needle", something "detached" from phenomena of agency when regarded arbitrarily.

In the context of Turing's OCN, we have seen that there is no diagonalizing out of the class of computable numbers. In this sense the class is robust: Turing's parameter of taking a "step" in a calculation impervious to the vagaries of any particular system of representing them, just as Gödel noted. And yet this "absoluteness" is relative to something else, on the view Wittgenstein thinks Turing's analysis is driven to, in the end: our ability to take in, follow, and recognize one another *as* taking steps in calculation. It is not part of our concept of what it is to follow a rule that we do or do not always follow the law of excluded middle. It *is* part of our concept of following a rule that we can communicate and reach consensus on what *in particular* to *do* with it in a given situation.[68]

Bibliography

Church, Alonzo (1936): "A Note on the Entscheidungsproblem." In: *Journal of Symbolic Logic*, Vol. 1, No. 1, 40 – 41.

Diamond, Cora (1991): *The Realistic Spirit: Wittgenstein, Philosophy, and the Mind.* Cambridge (MA): MIT Press.

Dreben, Burton; Floyd, Juliet (1991): "Tautology: How Not to Use A Word." In: *Synthèse*, Vol. 87, No. 1, 23 – 50.

Engelmann, Mauro Luiz (2013): *Wittgenstein's Philosophical Development: Phenomenology, Grammar, Method, and the Anthropological View.* Basingstoke (UK): Palgrave Macmillan.

Floyd, Juliet (2001): "Prose versus Proof: Wittgenstein on Gödel, Tarski and Truth." In: *Philosophia Mathematica*, Vol. 3, No. 9, 280 – 307.

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Download Date | 12/1/19 2:37 PM

Floyd, Juliet (2012a): "Wittgenstein, Carnap, and Turing: Contrasting Notions of Analysis." In: Wagner, Pierre (ed.): *Carnap's Ideal of Explication and Naturalism*. Basingstoke (UK): Palgrave Macmillan, 34 – 46.

Floyd, Juliet (2012b): "Wittgenstein's Diagonal Argument: A Variation on Cantor and Turing." In: Dybjer, P. et al. (eds.): *Epistemology versus Ontology, Logic, Epistemology: Essays in Honor of Per Martin-Löf*. Dordrecht: Springer, 25 – 44.

Floyd, Juliet (2013): "Turing, Wittgenstein and Types: Philosophical Aspects of Turing's 'The Reform of Mathematical Notation' (1944 – 5)." In: Turing 2013, 250 – 253.

Floyd, Juliet (2016): "Chains of Life: Turing, *Lebensform*, and the Emergence of Wittgenstein's Later Style." In *Nordic Wittgenstein Review*, Vol. 5, No. 2, 7 – 89.

Floyd, Juliet (2017a): "Positive Pragmatic Pluralism." In: *Harvard Review of Philosophy*, Vol. 24, 107 – 115.

Floyd, Juliet (2017b): "The Fluidity of Simplicity: Philosophy, Mathematics, Art." In: Kossak, Roman; Ordling, Philip (eds.): *Simplicity: Ideals of Practice in Mathematics and the Arts*. New York: Springer, 155 – 178.

Floyd, Juliet (2017c): "Turing on 'Common Sense' Cambridge Resonances." In: Floyd/Bokulich 2017: 103 – 152.

Floyd, Juliet; Bokulich, Alisa (eds.): *Philosophical Explorations of the Legacy of Alan Turing – Turing 100*. New York: Springer.

Floyd, Juliet (2018a): "Aspects of Aspects." In: Sluga, Hans; Stern, David (eds.): *The Cambridge Companion to Wittgenstein*. New York: Cambridge University Press, 2018^2 (2009), 361 – 388.

Floyd, Juliet (2018b): "*Lebensformen*: Living Logic." In: Martin, Christian (ed.): *Language, Form(s) of Life, and Logic: Investigations after Wittgenstein*. Berlin: De Gruyter, 59 – 92.

Floyd, Juliet (forthcoming, expected 2019). "Wittgenstein on Ethics: Working through *Lebensformen*." In: *Philosophy and Social Criticism*.

Floyd, Juliet; Mühlhölzer, Felix (forthcoming, expected 2019): "Wittgenstein on the Real Numbers: Annotations to Hardy's A Course of Pure Mathematics." In *Nordic Wittgenstein Studies*.

Gandy, R. O. (1988): "The Confluence of Ideas in 1936." In: Herken, Rolf (ed.): *The Universal Turing Machine: A Half-Century Survey*. New York: Oxford University Press, 55 – 112.

Gibson, Arthur (2010): "Francis Skinner's Original Wittgenstein *Brown Book* Manuscript." In: Munz, Volker; Puhl, Klaus; Wang, J. *Language and World*, Part One: *Essays on the Philosophy of Wittgenstein*. Vienna: Ontos Verlag, 351 – 366.

Gödel, Kurt (1931): "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I." In *Monatshefte für Mathematik und Physik*, Vol. 38, 173 – 198. Reprinted and translated in Gödel 1986a: 144 – 195.

Gödel, Kurt (1946): "Remarks Before the Princeton Bicentennial Conference on Problems in Mathematics." In: Gödel 1986b: 150 – 153.

Gödel, Kurt (1964) "Postscriptum to the 1934 Princeton Lectures." In: Gödel 1986a: 369f.

Gödel, Kurt (1972): "Some Remarks on the Undecidability Results." In: Gödel 1986b: 305f.

Gödel, Kurt (1986a): *Collected Works*, Vol. I: *Publications 1929–1936*. Feferman, Solomon et al. (eds.). Oxford: Clarendon Press.

Gödel, Kurt (1986b): *Collected Works*, Vol. II: *Publications 1938–1974*. Feferman, Solomon et al. (eds.). Oxford: Clarendon Press.

Henkin, Leon (1952): "A Problem Concerning Provability." In: *The Journal of Symbolic Logic*, Vol. 17, 160.

Hodges, Andrew (1983): *Alan Turing: The Enigma of Intelligence*. New York: Touchstone.

Authenticated | jfloyd@bu.edu author's copy
Download Date | 12/1/19 2:37 PM

Hodges, Andrew (1999): *Turing*. The Great Philosophers. New York: Routledge.
Hodges, Andrew; Hatton, Cassandra (2015): "Turing Point." In *Bonham's Magazine*. Vo. 42, Spring 2015, 18 – 21.
Kanamori, Akihiro (2018): "Aspect-Perception and the History of Mathematics." In: Beaney, Michael; Harrington, Brendan; Shaw, Dominic (eds.): *Aspect Perception After Wittgenstein: Seeing-As and Novelty*. New York: Routledge, 109 – 132.
Kennedy, Juliette (2017): "Turing, Gödel and the 'Bright Abyss'." In: Floyd/Bokulich 2017: 63 – 92.
Kleene, Stephen (1981a): "Origins of Recursive Function Theory." In: *Annals of the History of Computing*, Vol. 3, No. 1, 52 – 67.
Kleene, Stephen (1981b): "The Theory of Recursive Functions, Approaching Its Centennial." In: *American Mathematical Society Bulletin; New Series*, Vol. 5, No. 1, 43 – 61.
Kreisel, Georg (1950): "Note on Arithmetic Models for Consistent Formulae of the Predicate Calculus." In: *Fundamenta Mathematicae*, Vol. 37.
Kripke, Saul (1982): *Wittgenstein on Rules and Private Language: An Elementary Exposition*. Cambridge (MA): Harvard University Press.
Leblanc, Hugues (1972): "Wittgenstein and the Truth-Functionality Thesis." In: *American Philosophical Quarterly*, Vol. 9, 271 – 274.
Löb, M. H. (1955): "Solution of a Problem of Leon Henkin." In: *Journal of Symbolic Logic*, Vol. 20, No. 2, 115 – 118.
Marion, Mathieu; Okada, Mitsuhiro (2018): "Wittgenstein, Goodstein and the Origin of the Uniqueness Rule." In: Stern, David G. (ed.):*Wittgenstein in the 1930s: Between the* Tractatus *and the* Investigations. Cambridge (UK): Cambridge University Press, 253 – 271.
Misak, Cheryl (2016): *Cambridge Pragmatism: From Peirce and James to Ramsey and Wittgenstein*. New York: Oxford University Press.
Monk, Ray (1990): *Ludwig Wittgenstein: The Duty of Genius*. New York/London: Free Press/Jonathan Cape.
Mundici, Daniel; Sieg, Wilfried (2017): "Turing, the Mathematician." In: Floyd/Bokulich 2017: 39 – 62.
Mühlhölzer, Felix (2010): *Braucht die Mathematik eine Grundlegung? Ein Kommentar des Teils III von Wittgensteins BEMERKUNGEN ÜBER DIE GRUNDLAGEN DER MATHEMATIK*. Frankfurt a. M.: Vittorio Klostermann.
Post, Emil L. "Finite Combinatory Processes, Formulation 1." In: *Journal of Symbolic Logic*, Vol. 1, 103 – 105.
Proudfoot, Diane (2017): "Turing and Free Will: A New Take on an Old Debate." In: Floyd/Bokulich 2017: 305 – 322.
Putnam, Hilary (1960): "Minds and Machines." In: Hook, Sidney (ed.): *Dimensions of Mind*. New York: New York University Press, 138 – 164. Reprinted in Putnam 1975: 206 – 214.
Putnam, Hilary (1967): "The Nature of Mental States." In: *Philosophical Papers*; Vol. 1: *Mind, Language and Reality*. Cambridge (UK): Cambridge University Press, 429 – 440. Originally titled "Psychological Predicates." In: Capitan, William H.; Merrill, Daniel D. (eds.): *Art, Mind, and Religion*. Pittsburgh: University of Pittsburgh Press, 1967, 37 – 48. Reprinted in: Putnam 1988a: 150 – 161
Putnam, Hilary (1975): *Philosophical Papers*; Vol. 2: *Mind, Matter and Method*. Cambridge (UK): Cambridge University Press.
Putnam, Hilary (1988a): *The Many Faces of Realism*. Cambridge (MA): MIT Press.
Putnam, Hilary (1988b): *Representation and Reality*. Cambridge (MA): MIT Press.

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Download Date | 12/1/19 2:37 PM

Putnam, Hilary (2009): "On Computational Psychology." In *Journal of Philosophy*, Vol. 4, 55.
Ricketts, Thomas (2014): "Analysis, Independence, Simplicity and the General Sentence-Form." In: *Philosophical Topics*, Vol. 42, No. 2, 263 – 288.
Ritter, Bernhard (forthcoming). *Kant and Post-Tractarian Wittgenstein: Transcendentalism, Idealism, Illusion*. Palgrave Macmillan, London.
Russell, Bertrand (1920): *Introduction to Mathematical Philosophy*. London/New York: Allen and Unwin/Macmillan, 1920^2 (1919).
Shagrir, Oran (2005): "The Rise and Fall of Computational Functionalism." In: Ben-Menahem, Yemima (ed.): *Hilary Putnam*. Cambridge (UK): Cambridge University Press, 220 – 250.
Sieg, Wilfried (2006). "Gödel on Computability." In: *Philosophia Mathematica*, Vol. 14, No. 2, 189 – 207.
Sieg, Wilfried (2009): "On Computability." In: Irvine, Andrew (ed.): *Handbook of the Philosophy of Science: Philosophy of Mathematics*. Amsterdam: Elsevier BV, 535 – 630.
Stadler, Friedrich (2018): "The Principle of Economy, Simplicity and Parsimony From Mach to the Vienna Circle and Beyond." Paper read at the *First International Workshop INTEREPISTEME*, 13 December 2018.
Turing, Alan (1936): "On Computable Numbers, with an Application to the Entscheidungsproblem." *Proceedings of the London Mathematical Society*, Vol. 2, No. 42, 230 – 265. Corrections: "On Computable Numbers, with an Application to the Decision Problem: a Correction." In: *Proceedings of the London Mathematical Society*, Vol. 2, No. 43, 1937, 544 – 546.
Turing, Alan (1969 [1948]). "Intelligent Machinery." In: Meltzer, B.; Miche, D. (eds.): *Collected Works of A. M. Turing: Mechanical Intelligence*, 106 – 113. Edinburgh: Edinburgh University Press. Originally an unpublished report for the National Physical Laboratory; reprinted in Turing 2013: 501 – 516, and in Turing 2004: 410 – 423.
Turing, Alan (1950): "Computing Machinery and Intelligence." In: *Mind*, Vol. 59, 433 – 460.
Turing, Alan (2001a): *Collected Works of A. M. Turing: Mathematical Logic*. Gandy, R. O.; Yates, C. E. M. (eds.). Amsterdam, etc.: Elsevier.
Turing, Alan (2001b): "The Reform of Mathematical Notation and Phraseology [1944–45]." In: Turing 2001a: 211 – 222; reprinted with commentary in Turing 2013: 245 – 249; original online at http://www.turingarchive.org/search/, King's College Online Archive.
Turing, Alan (2013): *Alan Turing – His Work and Impact*. Amsterdam/Burlington (MA): Elsevier. Cooper, Barry S.; van Leeuwen, Jan (eds.); with commentary on Turing's papers by many experts.
Turing, Alan; Copeland, B. Jack (eds.) (2004): *The Essential Turing: Seminal Writings in Computing, Logic, Philosophy, Artificial Intelligence, and Artificial Life*. Oxford: Oxford Clarendon Press.
Watson, A. G. D. (1938): "Mathematics and Its Foundations." In: *Mind*, Vol. 47, No. 188, 440 – 451.
Webb, Judson C. (1990): "Remark 3; Introductory Note to *Gödel 1972a* [= "Some Remarks on the Undecidability Results"]." In Gödel 1986b: 292 – 304.
Weiss, Max (2017). "Logic in the *Tractatus*." In: *Review of Symbolic Logic*, Vol. 10, No. 1, 1 – 50.
Wittgenstein, Ludwig (TLP): *Tractatus Logico-Philosophicus*. London/New York: Routledge and Kegan Paul. Ogden, C.K. (transl.). First German edition in *Annalen der Naturphilosophie*, Ostwald, Wilhelm (ed.), Vol.14, Leipzig, 1921, 12 – 262. Reprint, second impression with a few corrections, 1931. Republished 1981. Several online free versions of the text are available with hypertext and translation comparison capability. See Kevin Klement's "Side-by-Side" edition, available at http://people.umass.edu/klement/tlp/, the University of Iowa

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Tractatus Map http://tractatus.lib.uiowa.edu/, and Luciano Bazzocchi's multiple language site http://www.bazzocchi.com/wittgenstein/.

Wittgenstein, Ludwig (PI): *Philosophische Untersuchungen / Philosophical Investigations*. Anscombe, G. E. M.; Hacker, P. M. S.; Schulte, J. (eds. and transl.); Wiley-Blackwell, Chichester (UK), 2009[4] (1953).

Wittgenstein, Ludwig (KgE): *Philosophische Untersuchungen: Kritisch-genetische Edition*. Suhrkamp Verlag, Frankfurt am Main. Schulte, J.; Nyman, H.; von Savigny, E.; von Wright, G. H. (eds.).

Wittgenstein, Ludwig (BB): *Preliminary Studies for the 'Philosophical Investigations': Generally Known as The Blue and Brown Books*. New York: Harper and Row.

Wittgenstein, Ludwig (RFM): *Remarks on the Foundations of Mathematics*. Anscombe, G. E. M. (ed. and transl.); von Wright, Georg H.; Rhees, Rush (eds.); Blackwell, Oxford, 1978[2] (1956).

Wittgenstein, Ludwig (RPP 1): *Remarks on the Philosophy of Psychology*; Vol 1. Oxford: Blackwell. von Wright, G. H.; Nyman, Heikki (eds.); Anscombe, G.E.M. (transl.).

Wittgenstein, Ludwig (Z): *Zettel*. Oxford: Blackwell, 1981.

Wittgenstein, Ludwig (EPB): *Eine Philosophische Betrachtung (Das Braune Buch)*. *Werkausgabe*; Vol. 5, Rhees, Rush (ed.), Frankfurt a. M.: Suhrkamp, 1984, 117 – 282. Wittgenstein's enlargement of *The Brown Book* (MS 115, TS 310). German translation by von Petra von Morstein. EPB is based on Wittgenstein's manuscript revision of TS 310 contained in MS 115. The revision is not complete and the published EPB concludes with a German translation of the remainder of TS 310.

Wittgenstein, Ludwig (MS / TS): *Wittgenstein's Nachlass: The Bergen Electronic Edition*. Charlottesville (VA)/Oxford: Intelex Corporation/Oxford University Press, 2000. Edited by the Wittgenstein Archive, University of Bergen, under the direction of Claus Huitfeldt. Incorporates Wittgenstein-Source Bergen Facsimile Edition, ed. Alois Pichler, with the assistance of H. W. Kruger, D. C. P. South, T. M. Bruvik, A. Lindebjerg, V. Olstad, published online 2009 (www.wittgensteinsource.org) in a web edition of about 5000 pages of facsimile, diplomatic and normalized transcriptions. CD-ROMs were originally distributed by Intelex Corporation in their Past Masters Series.

Wittgenstein, Ludwig (AWL). *Wittgenstein's Lectures, Cambridge, 1932 – 1935: From the Notes of Alice Ambrose and Margaret Macdonald*. Chicago: University of Chicago Press, 1979.

Wittgenstein, Ludwig (LFM): *Wittgenstein's Lectures on the Foundations of Mathematics: Cambridge, 1939, from the Notes of R.G. Bosanquet, Norman Malcolm, Rush Rhees and Yorick Smythies*. Hassocks, Sussex/Chicago: The Harvester Press/University of Chicago Press, 1976.

Wittgenstein, Ludwig (WC): *Wittgenstein in Cambridge: Letters and Documents, 1911–1951*. McGuinness, Brian (ed.). Malden (MA), Oxford: Blackwell, 2008[4] (1995).

Wittgenstein, Ludwig (MWL): *Wittgenstein: Lectures, Cambridge 1930–1933, From the Notes of G. E. Moore*. Stern, David G.; Rogers, Brian; Citron, Gabriel (eds.) Cambridge (UK): Cambridge University Press, 2016. Originals of the notes are located at the University Library, Cambridge University. Moore published a three part excerpted series from these notes, "Wittgenstein's Lectures in 1930–33" in *Mind*, New Series, Vol. 63, 1954, Part I: 1 – 15; Part II: 289 – 316; Vol. 64, 1955, Part III: 1 – 27, with two corrections on page 254; these were reprinted in his *Philosophical Papers*, New York, Humanities Press, 1959, 252 – 324.

Wittgenstein, Ludwig (WCL): *Wittgenstein's Whewell's Court Lectures, Cambridge, 1938–1941: From the Notes by Yorick Smythies*. Munz, Volker A.; Ritter, Bernhard (eds.); Wiley-Blackwell, 2017.

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