

# Higher gauge theory and a non-Abelian generalization of 2-form electrodynamics

Hendryk Pfeiffer\*

*Perimeter Institute for Theoretical Physics, 35 King Street N, Waterloo, Ont., Canada N2J 2W9*

*Emmanuel College, St Andrew's Street, Cambridge CB2 3AP, UK*

*DAMTP, Wilberforce Road, Cambridge CB3 0WA, UK*

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## Abstract

In conventional gauge theory, a charged point particle is described by a representation of the gauge group. If we propagate the particle along some path, the parallel transport of the gauge connection acts on this representation. The Lagrangian density of the gauge field depends on the curvature of the connection which can be calculated from the holonomy around (infinitesimal) loops. For Abelian symmetry groups, say  $G = U(1)$ , there exists a generalization, known as  $p$ -form electrodynamics, in which  $(p - 1)$ -dimensional charged objects can be propagated along  $p$ -surfaces and in which the Lagrangian depends on a generalized curvature associated with (infinitesimal) closed  $p$ -surfaces. In this article, we use Lie 2-groups and ideas from higher category theory in order to formulate a discrete gauge theory which generalizes these models at the level  $p = 2$  to possibly non-Abelian symmetry groups. An important feature of our model is that it involves both parallel transports along paths and generalized transports along surfaces with a non-trivial interplay of these two types of variables. Our main result is the geometric picture, namely the assignment of non-Abelian quantities to geometrical objects in a coordinate free way. We construct the precise assignment of variables to the curves and surfaces, the generalized local symmetries and gauge invariant actions and we clarify which structures can be non-Abelian and which others are always Abelian. A discrete version of connections on non-Abelian gerbes is a special case of our construction. Even though the motivation sketched so far suggests applications mainly in string theory, the model presented here is also related to spin foam models of quantum gravity and may in addition provide some insight into the role of centre monopoles and vortices in lattice QCD.

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\* Fax: +15195697611.

E-mail address: [hpfeiffer@perimeterinstitute.ca](mailto:hpfeiffer@perimeterinstitute.ca).

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## 1. Introduction

In the present article, we are concerned with gauge theories in a discretized formulation on some sort of lattice, triangulation or cellular decomposition. This includes the standard formulation of lattice gauge theory, see, for example [1,2], but we can also think of a smooth manifold with a large collection of embedded curves and surfaces on which we place the variables of the theory. The main point is that we keep group elements for the parallel transports along the curves, but that we do not pass to the differential picture and do not replace the group by its Lie algebra. The discrete structure is represented by an abstract simplicial complex, i.e., a collection of vertices, edges, triangles, etc. which we call *lattice*. We do not discuss here how these simplices are mapped to or embedded in some given manifold.

The theory can be classical, in this case we have to define some action and the variables in the Lagrangian picture, or it can be a quantum theory in the path integral formulation whose path integral is the sum or integral over all classical configurations.

Let us consider a gauge theory whose gauge group is any Lie group  $G$ . We concentrate on pure gauge fields. There are no dynamical matter fields. The fundamental variables are taken to be the parallel transports of the gauge connection along the edges (links) of the lattice, i.e., one associates a group element  $g_e \in G$  with each edge  $e$ . We call the source and target vertices of the edge  $s(e)$  and  $t(e)$ , respectively.

The gauge connection encodes what happens to charged point particles if we propagate them on the lattice. A charged particle is a vector  $w \in W$  in some representation  $\rho$  of  $G$ . If we propagate such a particle from  $s(e)$  along the edge  $e$  to  $t(e)$ , then the parallel transport  $g_e$  acts on the vector,  $w \mapsto \rho(g_e)w$ . Observe that composition and orientation reversal of the edges correspond precisely to the group structure of  $G$ . Since the choice of internal reference frame, essentially the choice of basis of  $W$ , is arbitrary, all physically meaningful quantities are required to be invariant under *local gauge transformations*

$$g_e \mapsto h_{s(e)}^{-1} \cdot g_e \cdot h_{t(e)} \quad (1.1)$$

for each edge  $e$ . The *generating function*  $h$  assigns a group element  $h_v \in G$  to each vertex  $v$  and thus parameterizes the local changes of basis. The action of the theory is a physical quantity and therefore gauge invariant. It turns out that the easiest way to obtain a gauge invariant expression is to calculate the product of group elements around a closed loop and then to evaluate a character of  $G$ . The action originally proposed by Wilson makes use of the loop around an elementary square (plaquette) and calculates the real part of the character of the fundamental representation of  $G$ .

Now let us try to generalize the setting. The charged particles are replaced by line-like objects which are to be propagated along surfaces. We therefore need additional variables associated with the faces (plaquettes) of the lattice. As the faces are two-dimensional objects, we have to deal with various ways of composing individual faces

to larger surfaces, for example putting them below each other (*vertical composition*) or next to each other (*horizontal composition*). The algebraic structure we use has to reflect these geometrical conditions.

It is known that one possible solution is to label the faces with elements of an Abelian group  $G$  and to consider these as the fundamental variables. The edges are not labeled. For  $G = U(1)$ , such a model is called *2-form electrodynamics*, or more generally  $p$ -form electrodynamics if the fundamental variables are associated with the  $p$ -cells of the lattice (vertices, edges, faces, etc.). The continuum formulation of the theory for  $p = 2$  involves so-called Kalb–Ramond fields [3]. The higher level models were originally introduced in the language of lattice models in statistical mechanics [4,5] where they correspond to the  $xy$ -model ( $p = 0$ ),  $U(1)$ -lattice gauge theory ( $p = 1$ ), a theory for an antisymmetric rank-2 tensor field ( $p = 2$ ), and so on. Their continuum counterparts have been studied in [6].

Consider the case  $p = 2$ . Can we do any better than just using an Abelian symmetry group? As long as we insist on colouring only faces, this is not possible as a classic argument from algebraic topology shows. But a generalization is possible if we colour both edges and faces with suitable algebraic structures. The main result of the present article is the construction of such a generalized 2-form gauge theory with its local gauge transformations and gauge invariant expressions.

We emphasize the geometrical properties of our model, namely that we have an assignment of non-Abelian quantities to geometrical objects in a coordinate free way. There exists a considerable literature on the interplay of Lie algebra valued 1- and 2-forms and their extended ‘gauge’ symmetries, but there is usually [7] no geometrical interpretation for the non-Abelian 2-form comparable to the parallel transport along curves which can be constructed from a connection 1-form. In this article, we pursue a complementary approach. We require a consistent geometrical picture and then deduce how much freedom we have in choosing the structure of our model.

At the technical level, the only thing we do is to combine two recent ideas. The first [8] is the construction of *Lie 2-groups* which serve as the algebraic structure in order to label edges and faces in a consistent way with non-Abelian quantities. The second [9] is the use of ideas from category theory in order to rephrase lattice gauge theory in a way that does admit the desired generalization.

A related construction was presented in [10] where the discrete framework is used in order to derive the corresponding continuum expressions. One can say that the construction presented here is somewhat more general than a discrete version of a theory of the 1- and 2-connections of non-Abelian gerbes. For mathematical background on gerbes, see [11]. Although we do not use the most general setting and restrict ourselves to strict Lie 2-groups, we still have plenty of examples and our construction includes a discrete version of non-Abelian gerbes as a special case.

The motivation sketched so far, namely to replace a theory for charged point particles by a theory for both charged point and charged line-like particles, seems to be mainly related to string theory. In fact, the simplest case, Abelian 2-form electrodynamics or Kalb–Ramond fields [3] play an important role in string theory so that it is interesting to understand possible generalizations.

In the context of quantum field theory, one might naively claim that in four dimensions any theory with a local non-Abelian symmetry is a gauge theory. Of course, this is not the full truth, and it is of general interest to understand in conceptual terms which other theories can have local symmetries.

Beyond these general ideas, we sketch how the model constructed here might provide further insight into the refined state sum models of quantum gravity and into the role of centre monopoles and vortices in lattice QCD. We also comment on the hierarchy of models that generalize  $p$ -form electrodynamics to non-Abelian symmetries.

We try to keep the present article self-contained and to make it readable by physicists who are not yet familiar with category theory. Therefore, we carefully review the required background material. The article is organized as follows. In Section 2, we rephrase conventional lattice gauge theory in the language of category theory in order to prepare for the generalization. In Section 3, we recall the argument which forces the symmetry group to be Abelian if we colour only the faces, and we present the relevant algebraic structures in order to circumvent it. The construction of our non-Abelian 2-form lattice gauge theory is presented in Section 4. Section 5 contains some examples and comments on their physical relevance. Finally, in Section 6, we discuss open questions and the relationship with other approaches.

## 2. Conventional lattice gauge theory

In this section, we review the basic structures of lattice gauge theory for pure gauge fields and rephrase everything in the language of category theory. This might seem to be much too complicated at first sight, but it turns out that the category theoretic language is the key to the generalization to higher level, including dynamical variables both at the edges and at the faces of the lattice.

The idea to rephrase lattice gauge theory in the language of category theory was, to our knowledge, first proposed by Baez [12] and then generalized by Grosse and Schlesinger [9]. Here we review this construction in a language adapted to the examples we are going to present. For a mathematical introduction to category theory, see, for example [13].

Informally speaking, a category is a collection of points (*objects*) and arrows between these points (*morphisms*) with enough structure so that we can compose arrows and that we have *identities*, i.e., arrows that behave like neutral elements under composition. The precise definition is as follows. We restrict ourselves to *small* categories, i.e., the collections of objects and morphisms form proper sets.

**Definition 2.1.** A *small category*  $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \text{id}, \circ)$  consists of a set  $\mathcal{C}_0$  of *objects*, a set  $\mathcal{C}_1$  of *morphisms* and maps  $s : \mathcal{C}_1 \rightarrow \mathcal{C}_0$  (*source*),  $t : \mathcal{C}_1 \rightarrow \mathcal{C}_0$  (*target*),  $\text{id} : \mathcal{C}_0 \rightarrow \mathcal{C}_1$  (*identity*), and  $\circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$  (*composition*) such that the following axioms hold:

$$s(\text{id}_g) = g = t(\text{id}_g), \quad (2.1)$$

$$s(g \circ g') = s(g), \quad t(g \circ g') = t(g'), \quad (2.2)$$

$$\text{id}_{s(f)} \circ f = f = f \circ \text{id}_{t(f)}, \quad (2.3)$$

$$(f_1 \circ f_2) \circ f_3 = f_1 \circ (f_2 \circ f_3) \quad (2.4)$$

for all objects  $g, g' \in \mathcal{C}_0$  and morphisms  $f \in \mathcal{C}_1$  and  $f_1, f_2, f_3 \in \mathcal{C}_1$  where composable. We have denoted by

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 := \{(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_1 : s(f_2) = t(f_1)\} \quad (2.5)$$

the set of all pairs of composable morphisms. We write  $f : g_1 \rightarrow g_2$  for a morphism  $f \in \mathcal{C}_1$  from the source  $g_1 = s(f)$  to the target  $g_2 = t(f)$ . Notice that we read the composition ( $\circ$ ) from left to right. A morphism  $f : g_1 \rightarrow g_2$  is called *isomorphism* if it has a two-sided inverse, i.e., if there exists a morphism  $f^{-1} : g_2 \rightarrow g_1$  such that

$$f \circ f^{-1} = \text{id}_{g_1}, \quad f^{-1} \circ f = \text{id}_{g_2}. \quad (2.6)$$

A category in which every morphism is an isomorphism, is called *groupoid*.

It is instructive to visualize all this using diagrams. For a morphism  $f : g_1 \rightarrow g_2$  we draw an arrow

$$g_1 \xrightarrow{f} g_2 \quad (2.7)$$

Composition of morphisms and the identity morphisms are shown in the following diagram:

$$\begin{array}{ccc} \text{id}_{g_1} \circlearrowleft g_1 & \xrightarrow{f_1} & g_2 \\ & \searrow f_1 \circ f_2 & \downarrow f_2 \\ & & g_3 \end{array} \quad (2.8)$$

We will make use of categories for two purposes. First, the gauge group gives rise to a category and second, the lattice forms a category as well. This is stated in the following two examples.

**Example 2.2.** Let  $G$  be a Lie group. Then there is a groupoid  $\mathcal{G}^G$  associated with  $G$  which has only one object,  $\mathcal{G}_0^G = \{*\}$ , and whose morphisms are the group elements,  $\mathcal{G}_1^G = G$ . Composition is the multiplication in  $G$ . Obviously, the source and target maps  $s, t : G \rightarrow \{*\}$  are trivial and the identity map associated with the object  $*$  is the unit in  $G$ ,  $\text{id}_* = 1 \in G$ .

**Example 2.3.** Let  $(V, E)$  be a directed graph, given by a set  $V$  of *vertices* and a set  $E$  of *edges* together with maps  $s : E \rightarrow V$  and  $t : E \rightarrow V$  indicating the source and target of each edge. Then there is a category  $\mathcal{C}^{V,E}$  whose objects  $\mathcal{C}_0^{V,E} = V$  are the vertices. The set of morphisms  $\mathcal{C}_1^{V,E}$  comprises

- all edges  $e \in E$ ,
- for each edge  $e \in E$  its orientation reversed counterpart  $e^* \in E$  (such that  $s(e^*) = t(e)$  and  $t(e^*) = s(e)$ ),
- for each vertex  $v \in V$  one morphism  $\text{id}_v$  such that  $s(\text{id}_v) = v = t(\text{id}_v)$  and

- all formal compositions of these edges, subject to the relations

$$e \circ e^* = \text{id}_{s(e)}, \quad (2.9)$$

$$e^* \circ e = \text{id}_{t(e)}, \quad (2.10)$$

$$\text{id}_{s(e)} \circ e = e = e \circ \text{id}_{t(e)} \quad (2.11)$$

for all  $e \in E$ .

The maps  $s$  and  $t$  are given by the directed graph and extended to compositions such that (2.2) holds.

This construction looks complicated at first sight, but the only thing we have done is to take vertices as objects and edges as morphisms and make everything else fit the picture.

So far we have introduced two categories. One of them,  $\mathcal{C}^{V,E}$ , encodes the information about the lattice while the other one,  $\mathcal{G}^G$ , is the gauge group. We will see that configurations of lattice gauge theory are maps from the former to the latter. Structure preserving maps between categories are known as *functors* and are defined as follows.

**Definition 2.4.** Let  $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \text{id}, \circ)$  and  $\mathcal{C}' = (\mathcal{C}'_0, \mathcal{C}'_1, s', t', \text{id}', \circ')$  be small categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{C}'$  is a pair  $(F_0, F_1)$  of maps,  $F_0 : \mathcal{C}_0 \rightarrow \mathcal{C}'_0$  and  $F_1 : \mathcal{C}_1 \rightarrow \mathcal{C}'_1$ , sending objects to objects and morphisms to morphisms such that

$$F_0 s(f) = s'(F_1 f), \quad F_0 t(f) = t'(F_1 f), \quad (2.12)$$

$$F_1 \text{id}_g = \text{id}'_{F_0 g}, \quad (2.13)$$

$$F_1(f_1 \circ f_2) = F_1 f_1 \circ' F_1 f_2 \quad (2.14)$$

for all objects  $g \in \mathcal{C}_0$  and morphisms  $f \in \mathcal{C}_1$  and  $(f_1, f_2) \in \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ . We are going to omit the indices 0, 1 of  $F_0, F_1$  if it is obvious from the context which map we refer to.

**Example 2.5.** Let  $(V, E)$  be a directed graph and  $G$  be a Lie group. A functor  $F : \mathcal{C}^{V,E} \rightarrow \mathcal{G}^G$  is a pair of maps,  $F_0 : V \rightarrow \{*\}$  and  $F_1 : E \rightarrow G$ . The edges are therefore labeled by group elements whereas the vertices are not labeled at all so that a functor  $\mathcal{C}^{V,E} \rightarrow \mathcal{G}^G$  is just a configuration of lattice gauge theory.

Observe that the identity edges  $\text{id}_v$  at each vertex  $v \in V$  (see Example 2.3) are labeled by the group unit  $1 \in G$ . This implies that orientation reversed edges are assigned the inverse group element,  $F_1(e^*) = (F_1 e)^{-1}$ .

So far we have said in the new category theoretic language what a configuration of lattice gauge theory is, namely a functor  $F : \mathcal{C}^{V,E} \rightarrow \mathcal{G}^G$ . From category theory, we know how to compare two functors, and this concept naturally leads to the familiar local gauge symmetry.

**Definition 2.6.** Let  $\mathcal{C}, \mathcal{C}'$  be small categories and  $F, \tilde{F} : \mathcal{C} \rightarrow \mathcal{C}'$  be functors. A *natural transformation*  $\eta : F \Rightarrow \tilde{F}$  is a map  $\eta : \mathcal{C}_0 \rightarrow \mathcal{C}'_1$  that associates with each object  $g \in \mathcal{C}_0$  a morphism  $\eta_g : Fg \rightarrow \tilde{F}g$  in  $\mathcal{C}'$  such that

$$Ff \circ \eta_{g_2} = \eta_{g_1} \circ \tilde{F}f \quad (2.15)$$

holds for all morphisms  $f : g_1 \rightarrow g_2$  in  $\mathcal{C}$ . This means that the following diagram commutes:

$$\begin{array}{ccc} Fg_1 & \xrightarrow{Ff} & Fg_2 \\ \eta_{g_1} \downarrow & & \downarrow \eta_{g_2} \\ \tilde{F}g_1 & \xrightarrow{\tilde{F}f} & \tilde{F}g_2 \end{array} \quad (2.16)$$

A natural transformation is called *natural equivalence* if  $\eta_g$  is an isomorphism for any  $g \in \mathcal{C}_0$ .

Natural equivalences can now be used in order to compare two configurations of lattice gauge theory. Let us specialize the preceding definition to our situation.

**Example 2.7.** Let  $F, \tilde{F} : \mathcal{C}^{V,E} \rightarrow \mathcal{G}^G$  be two functors. A natural equivalence  $\eta : F \Rightarrow \tilde{F}$  is a map  $\eta : V \rightarrow G$  such that for each edge  $e : v \rightarrow w$ , the following diagram commutes:

$$\begin{array}{ccc} * & \xrightarrow{Fe} & * \\ \eta_v \downarrow & & \downarrow \eta_w \\ * & \xrightarrow{\tilde{F}e} & * \end{array} \quad (2.17)$$

In the category  $\mathcal{G}^G$  in which composition is the group product, this means that

$$\tilde{F}e = \eta_v^{-1} \cdot Fe \cdot \eta_w, \quad (2.18)$$

so that the configuration of lattice gauge theory given by  $\tilde{F}$  is locally gauge equivalent to the configuration given by  $F$ , cf. (1.1). The map  $V \rightarrow G, v \mapsto \eta_v$  in the definition of the natural equivalence plays the role of the generating function.

Which expressions are gauge invariant and can therefore correspond to physically meaningful quantities? We know that the simplest such expressions are Wilson loops, group characters evaluated at the holonomy around loops.

Consider Fig. 1. It shows a triangle  $(1, 2, 3)$  labeled in two ways. The inner triangle is labeled by a functor  $F : \mathcal{C}^{V,E} \rightarrow \mathcal{G}^G$ , the outer one by  $\tilde{F}$ . We have set  $g_{ij} := Fe_{ij}$  where  $e_{ij}$  are the edges such that  $s(e_{ij}) = i$  and  $t(e_{ij}) = j$  and similarly  $\tilde{g}_{ij} := \tilde{F}e_{ij}$ . The figure contains three commutative squares of the form (2.17). Commutativity implies that

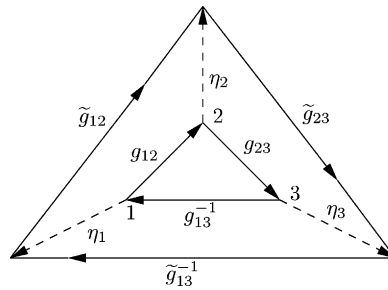


Fig. 1. The holonomy  $g_{12}g_{23}g_{13}^{-1}$  around some triangle  $(1, 2, 3)$ . The inner triangle is labeled by a functor  $F : \mathcal{C}^{V,E} \rightarrow \mathcal{G}^G$ , the outer triangle by  $\tilde{F}$ . The functors  $F$  and  $\tilde{F}$  are related by a natural equivalence  $\eta$ .

$$\tilde{g}_{12}\tilde{g}_{23}\tilde{g}_{13}^{-1} = \eta_1^{-1}g_{12}g_{23}g_{13}^{-1}\eta_1, \quad (2.19)$$

so that any group character of the holonomy,  $\chi(g_{12}g_{23}g_{13}^{-1})$ , is a gauge invariant quantity.

Of course, in standard lattice gauge theory, the properties mentioned so far are much easier to verify in direct computations. The category theoretic language presented here, however, provides a structural framework which would have allowed us to derive all these properties in a systematic fashion even if we had not known them in advance. We will exploit this conceptual advantage when we generalize lattice gauge theory to the next level, colouring both edges and faces with dynamical variables. In Section 4, we will encounter the higher level analogues of all the diagrams used here. Without any help from category theory it would hardly be possible to guess the appropriate assignment of variables and the relevant symmetries.

Our plan for the following section is to review suitable generalizations for the notions of category, functor and natural transformation in order to pass on to the next level, including interesting examples for which we can perform explicit computations. The higher level generalizations are known as 2-categories, 2-functors, etc.

### 3. Mathematical background

#### 3.1. The Eckmann–Hilton argument

Let us first review the argument of Eckmann and Hilton [14] which explains why we are forced to use Abelian groups as long as we colour only the faces.

We assume that the plaquettes of the lattice are labeled by the elements  $f \in G$  of some algebraic structure  $G$ . We further assume that there are two composition laws as illustrated in Fig. 2. *Horizontal composition* is denoted by a dot  $(\cdot)$ , *vertical composition* by a circle  $(\circ)$ . Finally, we assume that both compositions have two-sided units, denoted by  $1$  and  $1_\circ$ . On larger lattices, there will occur mixed compositions such as that shown in Fig. 2c. We require that this composition is well defined without parentheses, i.e., it does not depend on the ordering by which horizontal and vertical compositions are performed. This means that the *exchange law*,



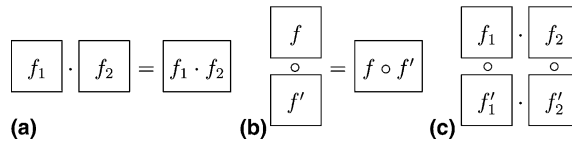


Fig. 2. (a) Horizontal composition of faces is denoted by a dot ( $\cdot$ ). (b) Vertical composition is indicated by a little circle ( $\circ$ ) which is read from left to right in our equations. (c) Parentheses are not necessary provided the exchange law (3.1) holds.

$$(f_1 \cdot f_2) \circ (f'_1 \cdot f'_2) = (f_1 \circ f'_1) \cdot (f_2 \circ f'_2), \quad (3.1)$$

is satisfied.

This implies first that the units actually agree because

$$1_\circ = 1_\circ \circ 1_\circ = (1_\circ \cdot 1_\circ) \circ (1_\circ \cdot 1_\circ) = (1_\circ \circ 1_\circ) \cdot (1_\circ \circ 1_\circ) = 1_\circ \cdot 1_\circ = 1. \quad (3.2)$$

Therefore, we can write  $1 = 1_\circ$ . The exchange law further implies that both compositions agree because for any  $f, g \in G$ ,

$$f \cdot g = (f \circ 1) \cdot (1 \circ g) = (f \cdot 1) \circ (1 \cdot g) = f \circ g, \quad (3.3)$$

and finally that this composition is Abelian,

$$f \cdot g = (1 \circ f) \cdot (g \circ 1) = (1 \cdot g) \circ (f \cdot 1) = g \circ f = g \cdot f. \quad (3.4)$$

Drawing the diagrams corresponding to (3.4), it becomes obvious that the two-dimensionality of the situation allows  $f$  and  $g$  to move around in the plane and thereby to change places.

If we wish to escape this Abelianness, we have to change some of the initial assumptions. There are a priori various conceivable approaches. Since we aim for a setting similar to lattice gauge theory, we have to require that Fig. 2c is well-defined without parentheses so that the exchange law (3.1) is not in question. It turns out that it is a viable strategy to colour both edges and faces with different algebraic structures in such a way that there is a non-trivial interplay and that the identities no longer agree.

An observation related to the Eckmann–Hilton argument was made independently by Teitelboim [7] in a physical context. The idea is as follows. In conventional gauge theory, there is the notion of a ‘path ordered product’ by which one defines the parallel transport along some curve and which is independent of the parameterization and thus a geometrical quantity. At higher level, however, one is forced to use Abelian labels because it seems to be impossible to define a ‘surface ordered product’ for generic non-Abelian quantities in a way that is independent of the choice of coordinates.

### 3.2. Lie 2-groups

In order to sidestep the Eckmann–Hilton argument, we follow ideas from higher category theory as explained in [8]. The picture is as follows. The edges, going from one vertex to another, are labeled with elements  $g_1, g_2$  from one algebraic structure. Composition of edges has to be reflected in this algebraic structure

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \quad \mapsto \quad \bullet \xrightarrow{g_1 \cdot g_2} \bullet \quad (3.5)$$

We therefore require an associative product  $g_1 \cdot g_2$ . In addition to this, there are faces, going from one edge to another, which are labeled with elements  $f$  from another algebraic structure. Here we use *bi-gons* as the fundamental faces, i.e., their boundary consists of only two edges that have the same source and target vertex

$$\bullet \begin{array}{c} \xrightarrow{g} \\ \Downarrow f \\ \xrightarrow{g'} \end{array} \bullet \quad (3.6)$$

These bi-gons can then be composed horizontally,

$$\bullet \begin{array}{c} \xrightarrow{g_1} \\ \Downarrow f_1 \\ \xrightarrow{g'_1} \end{array} \bullet \begin{array}{c} \xrightarrow{g_2} \\ \Downarrow f_2 \\ \xrightarrow{g'_2} \end{array} \bullet \quad \mapsto \quad \bullet \begin{array}{c} \xrightarrow{g_1 \cdot g_2} \\ \Downarrow f_1 \cdot f_2 \\ \xrightarrow{g'_1 \cdot g'_2} \end{array} \bullet \quad (3.7)$$

and vertically,

$$\bullet \begin{array}{c} \xrightarrow{g} \\ \Downarrow f \\ \xrightarrow{g'} \end{array} \bullet \begin{array}{c} \xrightarrow{g'} \\ \Downarrow f' \\ \xrightarrow{g''} \end{array} \bullet \quad \mapsto \quad \bullet \begin{array}{c} \xrightarrow{g} \\ \Downarrow f \circ f' \\ \xrightarrow{g''} \end{array} \bullet \quad (3.8)$$

The mixed composition,

$$\bullet \begin{array}{c} \xrightarrow{g_1} \\ \Downarrow f_1 \\ \xrightarrow{g'_1} \end{array} \bullet \begin{array}{c} \xrightarrow{g_2} \\ \Downarrow f_2 \\ \xrightarrow{g'_2} \end{array} \bullet \quad (3.9)$$

is required to be independent of the ordering of the horizontal and vertical compositions which is precisely stated by the exchange law

$$(f_1 \circ f'_1) \cdot (f_2 \circ f'_2) = (f_1 \cdot f_2) \circ (f'_1 \cdot f'_2). \quad (3.10)$$

Lie 2-groups as introduced by Baez [8] form a suitable structure with these properties. Note that the Eckmann–Hilton argument can be avoided here because we have labeled the edges by  $g_1, g_2$  and we perform some sort of computation  $g_1 \cdot g_2$  whenever edges are composed. The units for the vertical composition ( $\circ$ ) will in general depend on the  $g_i$  and therefore need not agree. When we list examples below, we mention in which cases the Eckmann–Hilton argument applies and in which it does not.

Strict Lie 2-groups can be obtained by a standard construction in category theory. A strict Lie 2-group is an *internal category* in the category of Lie groups. This means that we take the definition of a small category (Definition 2.1) and systematically replace the word ‘set’ by ‘Lie group’ and ‘map’ by ‘Lie group homomorphism.’

**Definition 3.1.** A *strict Lie 2-group*  $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \text{id}, \circ)$  consists of two Lie groups  $\mathcal{C}_0$  and  $\mathcal{C}_1$  with Lie group homomorphisms  $s : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ ,  $t : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ ,  $\text{id} : \mathcal{C}_0 \rightarrow \mathcal{C}_1$  and  $\circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$  such that the axioms (2.1)–(2.4) hold.

**Remark 3.2.**

1. For elements  $g_1, g_2 \in \mathcal{C}_0$  we write single arrows while elements  $f \in \mathcal{C}_1, f : g_1 \Rightarrow g_2$  are visualized by double arrows, cf. (3.6).
2. In a strict Lie 2-group, we have two composition laws for the set  $\mathcal{C}_1$ . One of them is the group product as  $\mathcal{C}_1$  is now a Lie group. We call it *horizontal composition* (see (3.7)) and write a dot  $(\cdot)$  which we sometimes even omit. Observe that the source of a horizontal composition is the product of the sources because  $s : \mathcal{C}_1 \rightarrow \mathcal{C}_0$  is a group homomorphism, etc. The other composition law is denoted by a circle  $(\circ)$ . It originates from the definition of a category and is explicitly listed in Definition 3.1. This is called *vertical composition* (see (3.8)).
3. Notice that the exchange law (3.10) holds because the composition map  $\circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$  is a Lie group homomorphism.
4. A generalization of strict 2-groups is provided by *weak* and *coherent 2-groups* [15]. We restrict ourselves to the strict case.

Concerning the structure of strict Lie 2-groups, we quote the following results from [8,16].

**Lemma 3.3.** *Let  $\mathcal{C}$  be a strict Lie 2-group. In particular we have Lie group homomorphisms  $s : \mathcal{C}_1 \rightarrow \mathcal{C}_0$  and  $\text{id} : \mathcal{C}_0 \rightarrow \mathcal{C}_1$  such that  $s(\text{id}(g)) = g$  for all  $g \in \mathcal{C}_0$ ,*

$$\mathcal{C}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\text{id}} \end{array} \mathcal{C}_0 \quad (3.11)$$

1. *Each element  $f \in \mathcal{C}_1$  has a unique decomposition of the form  $f = h \cdot \text{id}_g$  where  $g = s(f) \in \mathcal{C}_0$  and  $h \in \ker s \trianglelefteq \mathcal{C}_1$ .*<sup>1</sup>
2. *There is an isomorphism of Lie groups  $\ker s \rtimes \mathcal{C}_0 \cong \mathcal{C}_1$ , given by  $(h, g) \mapsto h \cdot \text{id}_g$ . The semi-direct product  $\ker s \rtimes \mathcal{C}_0$  is defined so that*

$$(h_1, g_1) \cdot (h_2, g_2) := (h_1 \alpha(g_1)[h_2], g_1 g_2), \quad (3.12)$$

*for  $h_1, h_2 \in \ker s$  and  $g_1, g_2 \in \mathcal{C}_0$ , where we set  $\alpha(g)[h] := \text{id}_g h \text{id}_{g^{-1}}$ .*

3. *The vertical composition of elements of  $\mathcal{C}_1$  is already fixed by the structure described so far. In the notation of the semi-direct product, it reads*

$$(h_1, g_1) \circ (h_2, g_2) = (h_2 h_1, g_1), \quad (3.13)$$

*for  $h_1, h_2 \in \ker t$  and  $g_1, g_2 \in \mathcal{C}_0$ , whenever composable.*

### 3.3. Lie crossed modules

For an introduction to internal categories in the category of groups, see [16]. One can prove [8,16] (also see [13, Chapter XII.8]) that strict Lie 2-groups are in one-to-one correspondence (up to isomorphism) with *Lie crossed modules*. The language of

<sup>1</sup> The notation  $A \trianglelefteq B$  indicates that  $A$  is a normal subgroup of  $B$ .

Lie crossed modules is most convenient in order to construct interesting examples of Lie 2-groups. They are defined as follows.

**Definition 3.4.** A Lie crossed module  $(G, H, t, \alpha)$  consists of two Lie groups  $G$  and  $H$  with Lie group homomorphisms  $t : H \rightarrow G$  and  $\alpha : G \rightarrow \text{Aut}(H)$  (i.e.,  $\alpha$  is an action of  $G$  on  $H$  that is compatible with the group structure of  $H$ ; we write it as  $\alpha(g)[h]$  for  $g \in G, h \in H$ ), such that

$$t(\alpha(g)[h]) = gt(h)g^{-1}, \quad (3.14)$$

$$\alpha(t(h))[h'] = hh'h^{-1} \quad (3.15)$$

for all  $g \in G$  and  $h, h' \in H$ .

**Theorem 3.5** (see [8,16]).

1. Let  $\mathcal{C}$  be a strict Lie 2-group. Then there is a Lie crossed module  $(G, H, t, \alpha)$  defined as follows. Define  $G := \mathcal{C}_0$  and  $H := \ker s$  to be the kernel of the source homomorphism. The map  $t : H \rightarrow G$  is defined to be the restriction  $t|_H$  of the target homomorphism. Finally,  $\alpha(g)[h] := \text{id}_g h \text{id}_g^{-1}$ .
2. Let  $(G, H, t, \alpha)$  be a Lie crossed module. Then there is a strict Lie 2-group  $\mathcal{C}$  defined as follows. Set  $\mathcal{C}_0 := G$  and  $\mathcal{C}_1 := H \rtimes G$ , the semi-direct product given by (3.12) using the map  $\alpha$  provided by the Lie crossed module. The source, target, identity, and composition maps are defined as follows:

$$s : H \rtimes G \rightarrow G, \quad (h, g) \mapsto g, \quad (3.16)$$

$$t : H \rtimes G \rightarrow G, \quad (h, g) \mapsto t(h)g, \quad (3.17)$$

$$\text{id} : G \rightarrow H \rtimes G, \quad g \mapsto (1, g), \quad (3.18)$$

$$(h, g) \circ (h', g') := (h'h, g), \quad (3.19)$$

for  $g, g' \in G$  and  $h, h' \in H$ , whenever composable.

**Remark 3.6.** Let us now assume that we have constructed a strict Lie 2-group from a Lie crossed module  $(G, H, t, \alpha)$  as in the preceding theorem.

1. Horizontal composition is given by the product in  $H \rtimes G$ . In particular, horizontal composition has the identity  $(1, 1)$  and inverses

$$(h, g)^{-1} = (\alpha(g^{-1})[h^{-1}], g^{-1}). \quad (3.20)$$

2. Any pair of elements of  $H \rtimes G$  can be composed horizontally.
3. Elements  $(h_2, g_2), (h_1, g_1) \in H \rtimes G$  are vertically composable to  $(h_1, g_1) \circ (h_2, g_2)$  if and only if  $g_2 = t(h_1)g_1$ .
4. Vertical composition has the units  $\text{id}_g = (1, g)$ ,  $g \in G$ , and is always invertible

$$(h, g)^* = (h^{-1}, t(h)g), \quad (3.21)$$

where we write a star  $(*)$  for the inverse with respect to  $\circ$ .

5. The group  $H \trianglelefteq H \rtimes G$  parameterizes all morphisms that have the unit  $1 \in G$  as their source. All morphisms whose source and target agree are parameterized by  $\ker t \trianglelefteq H$ .

For proofs and more technical details, see [8,16,13]. A list of examples can be found in [8] some of which we mention here.

**Example 3.7.**

1. The *trivial 2-group*.  $G$  is any Lie group and  $H$  is trivial. This example is uninteresting except for the fact that it confirms that ordinary Lie groups form a special case of strict Lie 2-groups.
2. The *purely Abelian 2-group*.  $G$  is trivial. In this case  $H$  is Abelian by the Eckmann–Hilton argument. This example gives rise to Abelian 2-form electrodynamics.
3.  $H$  is an Abelian Lie group on which  $G$  acts by some action  $\alpha : G \rightarrow \text{Aut}H$ . The map  $t$  is trivial,<sup>2</sup>  $t : H \rightarrow G$ ,  $h \mapsto 1$ .
4. The *Euclidean 2-groups*. As a special case of (3.) we can choose  $H = V$  to be some  $\mathbb{R}$ -vector space (translations) equipped with a non-degenerate symmetric bilinear form  $\eta$ . The group  $G = SO(V, \eta)$  (rotations) acts on  $V$ . From this we obtain the *Poincaré 2-group* [8] which is employed in the refined state sum model of [17] if we choose  $V = \mathbb{R}^{3+1}$  with the scalar product of Minkowski space.
5. The *automorphism 2-group*. This is finally an example with non-trivial  $t$ . Choose any Lie group  $H$  and  $G := \text{Aut}H$  to be its group of automorphisms. The action of  $G$  on  $H$  is the action by the particular automorphism, and  $t : H \rightarrow G$  assigns the inner automorphism, conjugation by  $h$ , to each element  $h \in H$ . This example gives rise to the lattice version of a theory involving the connections on non-Abelian gerbes. This Lie 2-group is related to the bi-torsors that are usually employed in the study of non-Abelian gerbes, see [8] for details.
6. Many examples of finite and discrete crossed modules are known in algebraic topology where crossed modules are a standard tool. For a recent survey, see, for example [18].

Summarizing this section, we can say that the notion of a strict Lie 2-group is useful in order make the categorical structure transparent while the notion of a Lie crossed module provides us with particular examples and allows us to perform calculations.

### 3.4. Suitable 2-categories

We have already seen that Lie 2-groups provide two compositions with the required identities such that the relevant diagrams (cf. (3.9)) can be drawn and such

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<sup>2</sup> Triviality of  $t$  already implies that  $H$  is Abelian by the Eckmann–Hilton argument. This example nevertheless provides a non-trivial generalization of 2-form electrodynamics because it involves an interplay of  $G$  and  $H$  via the action  $\alpha$ .

that the Eckmann–Hilton argument can be avoided. Before we can generalize the constructions of Section 2, however, we need higher level analogues of category, functor and natural equivalence. We therefore have to climb up by one level and introduce the notion of 2-categories.

A 2-category  $\mathcal{C}$  is a category ‘enriched in  $\mathbf{Cat}$ ’, i.e., for each pair of objects  $(x, y)$ , we have now a category  $\mathcal{C}(x, y)$  rather than just the set of all morphisms  $x \rightarrow y$ . The objects in  $\mathcal{C}(x, y)$  are the morphisms of  $\mathcal{C}$  and the morphisms of  $\mathcal{C}(x, y)$  are new data. They are called *2-morphisms*. The definition of MacLane [13] is of this type. We restrict ourselves to the *strict* case, i.e., equalities of morphisms are satisfied exactly and are not *weakened* to hold only up to 2-isomorphism. For more details on 2-categories, see [13, Chapter XII.3; 19]. In the following, we remove one level of abstraction from this definition and write down the conditions in detail.

**Definition 3.8.** A *small strict 2-category* consists of sets  $\mathcal{C}_0$  (*objects*),  $\mathcal{C}_1$  (*morphisms*), and  $\mathcal{C}_2$  (*2-morphisms*) together with various maps satisfying axioms as follows.

1. Maps  $s^{(1)} : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ ,  $t^{(1)} : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ ,  $\text{id}^{(1)} : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ , and  $\cdot : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$  such that  $(\mathcal{C}_0, \mathcal{C}_1, s^{(1)}, t^{(1)}, \text{id}^{(1)}, \cdot)$  forms a small category (Definition 2.1). We have denoted by

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 := \{(g_1, g_2) \in \mathcal{C}_1 \times \mathcal{C}_1 : s^{(1)}(g_2) = t^{(1)}(g_1)\} \quad (3.22)$$

the pairs of horizontally composable morphisms. Compositions in this category are visualized by diagrams such as (3.5).

2. Maps  $s^{(2)} : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ ,  $t^{(2)} : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ ,  $\text{id}^{(2)} : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ , and  $\circ : \mathcal{C}_2 \times_{\mathcal{C}_1} \mathcal{C}_2 \rightarrow \mathcal{C}_2$  such that  $(\mathcal{C}_1, \mathcal{C}_2, s^{(2)}, t^{(2)}, \text{id}^{(2)}, \circ)$  forms a small category. We have denoted by

$$\mathcal{C}_2 \times_{\mathcal{C}_1} \mathcal{C}_2 := \{(f_1, f_2) \in \mathcal{C}_2 \times \mathcal{C}_2 : s^{(2)}(f_2) = t^{(2)}(f_1)\} \quad (3.23)$$

the pairs of vertically composable 2-morphisms.

3. Axioms

$$t^{(1)}(s^{(2)}(f)) = t^{(1)}(t^{(2)}(f)), \quad s^{(1)}(s^{(2)}(f)) = s^{(1)}(t^{(2)}(f)) \quad (3.24)$$

stating that source  $s^{(2)}(f)$  and target  $t^{(2)}(f)$  of any 2-morphism  $f \in \mathcal{C}_2$  are parallel morphisms, i.e., that 2-morphisms are bi-gons as in (3.6). The composition  $(\circ)$  listed under item (2.) is therefore visualized by diagrams such as (3.8).

4. A map  $\cdot : \mathcal{C}_2 \times_{\mathcal{C}_0} \mathcal{C}_2 \rightarrow \mathcal{C}_2$  (horizontal composition of 2-morphisms), where

$$\mathcal{C}_2 \times_{\mathcal{C}_0} \mathcal{C}_2 := \{(f_1, f_2) \in \mathcal{C}_2 \times \mathcal{C}_2 : t^{(1)}(s^{(2)}(f_1)) = s^{(1)}(s^{(2)}(f_2))\} \quad (3.25)$$

denotes the set of all pairs of horizontally composable 2-morphisms. This composition is shown in diagram (3.7).

5. Further axioms,<sup>3</sup>

$$s^{(2)}(f_1 \cdot f_2) = s^{(2)}(f_1) \cdot s^{(2)}(f_2), \quad (3.26)$$

<sup>3</sup> In more abstract terms they state that  $(\mathcal{C}_0, \mathcal{C}_2, s^{(2)} \circ s^{(1)}, t^{(2)} \circ t^{(1)}, \text{id}^{(1)} \circ \text{id}^{(2)}, \cdot)$  forms a small category and that  $s^{(2)}$ ,  $t^{(2)}$ , and  $\text{id}^{(2)}$  give rise to functors between this category and the one formed by  $\mathcal{C}_0$  and  $\mathcal{C}_1$ .

$$t^{(2)}(f_1 \cdot f_2) = t^{(2)}(f_1) \cdot t^{(2)}(f_2), \quad (3.27)$$

$$\mathrm{id}^{(2)}(\mathrm{id}^{(1)}(s^{(1)}(s^{(2)}(f)))) \cdot f = f = f \cdot \mathrm{id}^{(2)}(\mathrm{id}^{(1)}(t^{(1)}(t^{(2)}(f))))), \quad (3.28)$$

$$(f_1 \cdot f_2) \cdot f_3 = f_1 \cdot (f_2 \cdot f_3) \quad (3.29)$$

for any  $f \in \mathcal{C}_2$  and horizontally composable  $f_1, f_2, f_3 \in \mathcal{C}_2$ , as well as,

$$\mathrm{id}^{(2)}(g_1) \cdot \mathrm{id}^{(2)}(g_2) = \mathrm{id}^{(2)}(g_1 \cdot g_2), \quad (3.30)$$

$$(f_1 \circ f'_1) \cdot (f_2 \circ f'_2) = (f_1 \cdot f_2) \circ (f'_1 \cdot f'_2) \quad (3.31)$$

for any  $g_1, g_2 \in \mathcal{C}_1$  and  $f_1, f'_1, f_2, f'_2 \in \mathcal{C}_2$  whenever they are composable.

A 2-morphism  $f : g_1 \Rightarrow g_2$  is called *2-isomorphism* if it has a two-sided inverse, i.e., if there exists a 2-morphism  $f^* : g_2 \Rightarrow g_1$  such that

$$f \circ f^* = \mathrm{id}_{g_1}^{(2)}, \quad f^* \circ f = \mathrm{id}_{g_2}^{(2)}. \quad (3.32)$$

A strict 2-category, in which all morphisms and all 2-morphisms are isomorphisms, is called a *strict 2-groupoid*.

Recall that a group gives rise to a groupoid with one object (Example 2.2). In a similar way, a strict Lie 2-group provides us with a strict 2-groupoid with one object. In the following, we use the language of Lie crossed modules in order to explicitly describe this strict Lie 2-group.

**Example 3.9.** Let  $(G, H, t, \alpha)$  be a Lie crossed module with the definitions used in Theorem 3.5, item (2.). Then there is a small strict 2-groupoid  $\mathcal{G}^{G,H}$  defined as follows.<sup>4</sup> Set  $\mathcal{G}_0^{G,H} := \{*\}$ ,  $\mathcal{G}_1^{G,H} := G$ , and  $\mathcal{G}_2^{G,H} := H \rtimes G$ , the semi-direct product given by (3.12).

The maps  $s^{(1)}$  and  $t^{(1)}$  are trivial,  $\mathrm{id}_*^{(1)} = 1 \in G$  and the composition of morphisms  $(\cdot)$  is the multiplication in  $G$ . This agrees precisely with Example 2.2.

The other maps are defined as follows:

$$s^{(2)} : H \rtimes G \rightarrow G, \quad (h, g) \mapsto g, \quad (3.33)$$

$$t^{(2)} : H \rtimes G \rightarrow G, \quad (h, g) \mapsto t(h)g, \quad (3.34)$$

$$\mathrm{id}^{(2)} : G \rightarrow H \rtimes G, \quad g \mapsto (1, g), \quad (3.35)$$

and, whenever composable,

$$(h, g) \circ (h', g') := (h'h, g) \quad (3.36)$$

for all  $g, g' \in G$  and  $h, h' \in H$ . The horizontal composition of 2-morphisms  $(\cdot)$  is the product in  $H \rtimes G$ , cf. (3.12).

<sup>4</sup> Strictly speaking, we should call it  $\mathcal{G}^{G,H,t,\alpha}$ .

This 2-groupoid with one object, obtained from a strict Lie 2-group, is our generalized notion of gauge group. In order to specify configurations and local gauge transformations, we have to introduce structure preserving maps between 2-categories, called *2-functors*, and suitable natural equivalences.

**Definition 3.10.** Let  $\mathcal{C}$  and  $\mathcal{C}'$  be small strict 2-categories. A *strict 2-functor*  $F : \mathcal{C} \rightarrow \mathcal{C}'$  is a triple of maps  $F_0 : \mathcal{C}_0 \rightarrow \mathcal{C}'_0$ ,  $F_1 : \mathcal{C}_1 \rightarrow \mathcal{C}'_1$ , and  $F_2 : \mathcal{C}_2 \rightarrow \mathcal{C}'_2$ , sending objects to objects, morphisms to morphisms, and 2-morphisms to 2-morphisms, such that the following conditions hold:

1.  $(F_0, F_1)$  is a functor  $(\mathcal{C}_0, \mathcal{C}_1, s^{(1)}, t^{(1)}, \text{id}^{(1)}, \cdot) \rightarrow (\mathcal{C}'_0, \mathcal{C}'_1, s'^{(1)}, t'^{(1)}, \text{id}'^{(1)}, \cdot)$ ,
2.  $(F_1, F_2)$  is a functor  $(\mathcal{C}_1, \mathcal{C}_2, s^{(2)}, t^{(2)}, \text{id}^{(2)}, \circ) \rightarrow (\mathcal{C}'_1, \mathcal{C}'_2, s'^{(2)}, t'^{(2)}, \text{id}'^{(2)}, \circ')$ ,
3.  $F_2(f_1 \cdot f_2) = F_2 f_1 \cdot' F_2 f_2$  for any  $(f_1, f_2) \in \mathcal{C}_2 \times_{\mathcal{C}_0} \mathcal{C}_2$ .

Given any two parallel 2-functors between 2-categories, we seek the notion of a natural transformation in order to compare these 2-functors. There are various flavours of these defined in the literature. We need what is usually called *pseudo-natural transformation*. It is a quasi-natural transformation in the terminology of [19] in which all 2-morphisms are isomorphisms.

**Definition 3.11.** Let  $\mathcal{C}, \mathcal{C}'$  be small strict 2-categories and  $F, \tilde{F} : \mathcal{C} \rightarrow \mathcal{C}'$  be parallel strict 2-functors. A *pseudo-natural transformation*  $\eta : F \Rightarrow \tilde{F}$  is a pair of maps  $\eta : \mathcal{C}_0 \rightarrow \mathcal{C}'_1$  and  $\eta : \mathcal{C}_1 \rightarrow \mathcal{C}'_2$  associating a morphism  $\eta_x : Fx \rightarrow \tilde{F}x$  to each object  $x \in \mathcal{C}_0$  and a 2-isomorphism  $\eta_g : Fg \cdot \eta_{t(g)} \Rightarrow \eta_{s(g)} \cdot \tilde{F}g$  to each morphism  $g \in \mathcal{C}_1$ , such that the following three conditions hold:

1. For any 2-morphism  $f : g \Rightarrow g'$  in  $\mathcal{C}$  between morphisms  $g, g' : x \rightarrow y$ , the diagram

$$\begin{array}{ccc}
 & \xrightarrow{Fg} & \\
 Fx & \Downarrow Ff & Fy \\
 & \xrightarrow{Fg'} & \\
 \eta_x \downarrow & \eta_g \swarrow & \searrow \eta_{g'} \downarrow \eta_y \\
 & \xrightarrow{\tilde{F}g} & \\
 \tilde{F}x & \Downarrow \tilde{F}f & \tilde{F}y \\
 & \xrightarrow{\tilde{F}g'} &
 \end{array} \quad (3.37)$$

2-commutes (this means it commutes for 2-morphisms), i.e., we have the following equality of 2-morphisms:

$$(Ff \cdot \text{id}_{\eta_y}^{(2)}) \circ \eta_{g'} = \eta_g \circ (\text{id}_{\eta_x}^{(2)} \cdot \tilde{F}f). \quad (3.38)$$

In (3.37), the 2-morphism  $\eta_{g'} : Fg' \cdot \eta_y \Rightarrow \eta_x \cdot \tilde{F}g'$  is located at the front face of the diagram while  $\eta_g : Fg \cdot \eta_y \Rightarrow \eta_x \cdot \tilde{F}g$  is at the back.



2. For any two composable morphisms  $g_1 : x \rightarrow y$  and  $g_2 : y \rightarrow z$  in  $\mathcal{C}$ , the diagram

$$\begin{array}{ccccc}
 & & Fy & & \\
 & \nearrow Fg_1 & & \nwarrow Fg_2 & \\
 Fx & \xrightarrow{F(g_1 \cdot g_2)} & Fz & & \\
 \downarrow \eta_x & & \downarrow \eta_y & & \downarrow \eta_z \\
 & \nwarrow \eta_{g_1} & & \nearrow \eta_{g_2} & \\
 & & \tilde{F}y & & \\
 \tilde{F}x & \xrightarrow{\tilde{F}(g_1 \cdot g_2)} & \tilde{F}z & & \\
 & \nearrow \tilde{F}g_1 & & \nwarrow \tilde{F}g_2 &
 \end{array}
 \quad (3.39)$$

which has the equalities  $F(g_1 \cdot g_2) = Fg_1 \cdot Fg_2$  at the top and  $\tilde{F}(g_1 \cdot g_2) = \tilde{F}g_1 \cdot \tilde{F}g_2$  at the bottom, 2-commutes, i.e.,

$$\eta_{g_1 \cdot g_2} = \left( \text{id}_{Fg_1}^{(2)} \cdot \eta_{g_2} \right) \circ \left( \eta_{g_1} \cdot \text{id}_{Fg_2}^{(2)} \right). \quad (3.40)$$

3. For each object  $x \in \mathcal{C}_0$ , we have  $\eta_{\text{id}_x^{(1)}} = \text{id}_{\eta_x}^{(2)}$ .

## 4. 2-Form lattice gauge theory

In the previous section, we have generalized the notions of category, functor, and natural transformation to the next level and we have seen how one can construct explicit examples using Lie crossed modules. Let us now generalize the ideas of Section 2 step by step in order to obtain a higher level lattice gauge theory, *lattice 2-gauge theory*, to be precise. Some key ideas of this section are taken from [9] which deals with 2-categories constructed from monoidal categories rather than from Lie 2-groups.

### 4.1. Lattice

First we have to define a small strict 2-category which represents the lattice. It is most convenient to do this for a triangulation rather than for a cubic lattice.<sup>5</sup>

**Definition 4.1.** A simplicial 2-complex  $(V, E, F)$  consists of sets  $V$  (vertices),  $E$  (edges), and  $F$  (faces) together with maps  $s : E \rightarrow V$ ,  $t : E \rightarrow V$  and  $\partial_1, \partial_2, \partial_3 : F \rightarrow E$  such that  $(V, E)$  with  $s$  and  $t$  forms a directed graph (Example 2.3). The maps  $\partial_1, \partial_2, \partial_3$  indicate the three edges in the boundary of each triangular face,

<sup>5</sup> The material presented here can be seen as a simplified version of a special case of Street's construction [20].

$$\begin{array}{ccc}
 v_1 & \xrightarrow{\partial_1 f} & v_2 \\
 & \searrow f & \downarrow \partial_2 f \\
 & & v_3 \\
 & \nearrow \partial_3 f &
 \end{array}
 \quad (4.1)$$

and are required to satisfy for each  $f \in F$ ,

$$s(\partial_1(f)) = t(\partial_3(f)), \quad s(\partial_2(f)) = t(\partial_1(f)), \quad s(\partial_3(f)) = t(\partial_2(f)). \quad (4.2)$$

**Example 4.2.** Let  $(V, E, F)$  be a simplicial 2-complex. Then there is a small strict 2-category  $\mathcal{C}^{V,E,F}$  defined as follows. The sets  $\mathcal{C}_0^{V,E,F}$  of objects and  $\mathcal{C}_1^{V,E,F}$  of morphisms are defined as in Example 2.3. The maps  $s^{(1)}$ ,  $t^{(1)}$ ,  $\text{id}^{(1)}$ , and  $\cdot$  are the same as  $s$ ,  $t$ ,  $\text{id}$ , and  $\circ$  in Example 2.3. This defines a small category which describes the edges and their compositions. The set  $\mathcal{C}_2^{V,E,F}$  of 2-morphisms consists of

1. All faces  $f \in F$ . We set  $s^{(2)}(f) = (\partial_1 f) \cdot (\partial_2 f)$  and  $t^{(2)}(f) = (\partial_3 f)^*$ ,
2. For each face  $f \in F$  another face  $f^*$  with the double arrow reversed,

$$\begin{array}{ccc}
 v_1 & \xrightarrow{\partial_1 f} & v_2 \\
 & \searrow f^* & \downarrow \partial_2 f \\
 & & v_3 \\
 & \nearrow (\partial_3 f)^* &
 \end{array}
 \quad (4.3)$$

such that  $s^{(2)}(f^*) = (\partial_2 f)^*$  and  $t^{(2)}(f^*) = (\partial_1 f) \cdot (\partial_2 f)$ ,

3. For each face  $f \in F$  another face  $\bar{f}$  with all single arrows reversed,

$$\begin{array}{ccc}
 v_1 & \xleftarrow{(\partial_1 f)^*} & v_2 \\
 & \searrow \bar{f} & \uparrow (\partial_2 f)^* \\
 & & v_3 \\
 & \nearrow \partial_3 f &
 \end{array}
 \quad (4.4)$$

such that  $s^{(2)}(\bar{f}) = (\partial_2 f)^* \cdot (\partial_1 f)^*$  and  $t^{(2)}(\bar{f}) = \partial_3 f$ .

4. For each edge  $e \in E$  a 2-morphism  $\text{id}_e^{(2)}$  with  $s^{(2)}(\text{id}_e^{(2)}) = e = t^{(2)}(\text{id}_e^{(2)})$ .
5. All formal horizontal  $(\cdot)$  and vertical  $(\circ)$  compositions of faces, subject to the following relations:
  - 5.1.  $f \circ f^* = \text{id}_{s^{(2)}(f)}^{(2)}$  and  $f^* \circ f = \text{id}_{t^{(2)}(f)}^{(2)}$ ,
  - 5.2.  $\text{id}_{s^{(2)}(f)}^{(2)} \circ f = f = f \circ \text{id}_{t^{(2)}(f)}^{(2)}$ ,
  - 5.3.  $f \circ \bar{f} = \text{id}_{s^{(2)}(\text{id}^{(1)}(s^{(1)}(s^{(2)}(f))))}^{(2)}$  and  $\bar{f} \circ f = \text{id}_{t^{(2)}(\text{id}^{(1)}(t^{(1)}(s^{(2)}(f))))}^{(2)}$ ,
  - 5.4.  $\text{id}^{(2)}(\text{id}^{(1)}(s^{(1)}(s^{(2)}(f)))) \cdot f = f \cdot \text{id}^{(2)}(\text{id}^{(1)}(t^{(1)}(t^{(2)}(f))))$ ,
  - 5.5.  $\text{id}_{e_1 \cdot e_2}^{(2)} = \text{id}_{e_1}^{(2)} \cdot \text{id}_{e_2}^{(2)}$  for all composable edges  $e_1, e_2 \in E$ ,
  - 5.6. the exchange law,  $(f_1 \cdot f_2) \circ (f'_1 \cdot f'_2) = (f_1 \circ f'_1) \cdot (f_2 \circ f'_2)$ , whenever faces  $f_1, f'_1, f_2, f'_2$  are composable.

## 4.2. Configurations

Let us now study the configurations of our generalized lattice gauge theory. By analogy with Section 2, these are the 2-functors from the generalized lattice  $\mathcal{C}^{V,E,F}$  to the generalized gauge group  $\mathcal{G}^{G,H}$ .

**Example 4.3.** A strict 2-functor  $F : \mathcal{C}^{V,E,F} \rightarrow \mathcal{G}^{G,H}$  is a triple of maps

$$F_0 : V \rightarrow \{*\}, \quad (4.5)$$

$$F_1 : E \rightarrow G, \quad (4.6)$$

$$F_2 : F \rightarrow H \rtimes G, \quad (4.7)$$

i.e., the edges are coloured by group elements of  $G$  while the triangular faces are coloured by elements of  $H \rtimes G$ .

Let us assume there is such a strict 2-functor which is used to label the triangle  $(1, 2, 3)$ ,



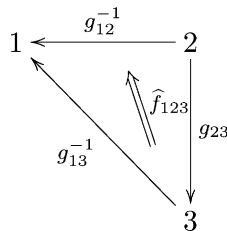
$$(4.8)$$

We denote by  $f_{123} := F_2(f) \in H \rtimes G$  the group element associated with the triangle and by  $g_{12} := \partial_1 f$ ,  $g_{23} := \partial_2 f$  and  $g_{13} := (\partial_3 f)^{-1}$  the elements of  $G$  associated with the edges. Then we can derive a number of useful properties of such a labeled triangle.

1. Write  $f_{123} = (h, g) \in H \rtimes G$ . We know that  $s^{(2)}(f) = g_{12} \cdot g_{23}$  and  $t^{(2)}(f) = g_{13}$  so that  $t(h) = g_{13} g_{23}^{-1} g_{12}^{-1}$  which is just the holonomy around the triangle.
2. We can horizontally compose the 2-morphism  $f_{123}$  with identities

$$\hat{f}_{123} := \text{id}_{g_{12}^{-1}}^{(2)} \cdot f_{123} \cdot \text{id}_{g_{13}^{-1}}^{(2)} : g_{23} \cdot g_{13}^{-1} \Rightarrow g_{12}^{-1}, \quad (4.9)$$

which can be visualized as follows:



$$(4.10)$$

We have thus obtained another triangle on which the arrow for the 2-morphism has been ‘rotated.’ On our generalized lattice  $\mathcal{C}^{V,E,F}$  (Example 4.2), this new ‘rotated’ triangle is considered different from the original one. At first sight, it seems

that triangles proliferate if we ‘rotate’ them in this way. In our generalized gauge group  $\mathcal{G}^{G,H}$ , one can show, however, that ‘triple rotation’ does not change the associated 2-morphism because

$$\mathrm{id}_{g_{13}}^{(2)} \cdot \mathrm{id}_{g_{23}^{-1}}^{(2)} \cdot \mathrm{id}_{g_{12}^{-1}}^{(2)} \cdot f_{123} \cdot \mathrm{id}_{g_{13}^{-1}}^{(2)} \cdot \mathrm{id}_{g_{12}}^{(2)} \cdot \mathrm{id}_{g_{23}}^{(2)} = f_{123}. \quad (4.11)$$

3. From the conditions on the identities in Example 4.2 and the properties of the strict 2-functor, we know that for each vertex  $v \in V$ , the identity edge  $\mathrm{id}_v^{(1)}$  is mapped to the unit in  $G$

$$F_1 \mathrm{id}_v^{(1)} = \mathrm{id}_*^{(1)} = 1 \in G, \quad (4.12)$$

while for any edge  $e$ , labeled by  $g := F_1 e$ , the unit  $\mathrm{id}_e^{(2)}$  is mapped to

$$F_2 \mathrm{id}_e^{(2)} = \mathrm{id}_g^{(2)} = (1, g) \in H \rtimes G. \quad (4.13)$$

These conditions imply that for each 2-morphism  $f_{123}$ , the 2-morphism  $f_{123}^*$  (Example 4.2) is given by its inverse with respect to vertical composition,

$$f_{123}^* = (h, g)^* = (h^{-1}, t(h)g), \quad (4.14)$$

while the other 2-morphism  $\bar{f}_{123}$  is the inverse with respect to horizontal composition,

$$\bar{f}_{123} = (h, g)^{-1} = (\alpha(g^{-1})[h^{-1}], g^{-1}). \quad (4.15)$$

4. Finally, these two ways of reversing the orientation of triangles are related in the following way:

$$\mathrm{id}_{g_{23}}^{(2)} \cdot \bar{f}_{123}^* \cdot \mathrm{id}_{g_{13}}^{(2)} = \mathrm{id}_{g_{12}^{-1}}^{(2)} \cdot f. \quad (4.16)$$

A careful analysis shows that this is exactly what one expects from the geometry of the triangle if these operations are combined.

Starting from the labeled triangle (4.8), we can obtain other 2-morphisms for the same triangle by ‘rotating’ or reversing the orientation using either  $f \mapsto f^*$  or  $f \mapsto \bar{f}$ . The relations listed above make sure that one obtains only six distinct 2-morphisms by combining these operations. The 2-groupoid  $\mathcal{G}^{G,H}$  has therefore all the properties which one expects from the combinatorics of the triangle (4.8). While edges come in two different orientations, there are six different versions of each triangle.

### 4.3. Local gauge transformations

Given two configurations of our generalized lattice gauge theory, represented by strict 2-functors from  $\mathcal{C}^{V,E,F}$  to  $\mathcal{G}^{G,H}$ , the analogy with Section 2 suggests that the local gauge transformations are given by pseudo-natural transformations. In our situation, this reads as follows.

**Example 4.4.** Let  $F, \tilde{F} : \mathcal{C}^{V,E,F} \rightarrow \mathcal{G}^{G,H}$  be parallel strict 2-functors. A pseudo-natural transformation  $\eta : F \Rightarrow \tilde{F}$  is a pair of maps

$$\eta : V \rightarrow G, \quad v \mapsto \eta_v, \quad (4.17)$$

$$\eta : E \rightarrow H \rtimes G, \quad e \mapsto \eta_e : F_1 e \cdot \eta_{t(e)} \Rightarrow \eta_{s(e)} \cdot \tilde{F}_1 e, \quad (4.18)$$

visualized for an edge  $e : v \rightarrow w$  by

$$\begin{array}{ccc} * & \xrightarrow{Fe} & * \\ \eta_v \downarrow & \swarrow \eta_e & \downarrow \eta_w \\ * & \xrightarrow{\tilde{F}e} & * \end{array} \quad (4.19)$$

The special case in which  $\eta_e = \text{id}_e^{(2)} = (1, 1)$  corresponds to an ordinary local gauge transformation because source and target of  $\eta_e$  agree and therefore the diagram commutes for morphisms. Compare this with Example 2.7. The appearance of a non-trivial 2-morphism  $\eta_e$  can be viewed as a way of parameterizing how ‘non-commuting’ the diagram is. This is the way in which the local gauge symmetry is generalized here. We call the pseudo-natural transformations of the above example the *local 2-gauge transformations*.

Let us now visualize how a generic local 2-gauge transformation acts on the labeled triangle (4.8),

$$\begin{array}{ccccc} & & * & & \\ & g_{12} & \nearrow f_{123} & \searrow g_{23} & \\ * & & & & * \\ & g_{13} & \searrow \eta_2 & \nearrow \eta_{13} & \\ \eta_1 \downarrow & & & & \downarrow \eta_3 \\ * & & & & * \\ & \tilde{g}_{12} & \nearrow \tilde{f}_{123} & \searrow \tilde{g}_{23} & \\ & & * & & \\ & \tilde{g}_{13} & \nwarrow \eta_2 & \nearrow \eta_{13} & \end{array} \quad (4.20)$$

Here we have denoted by  $f_{123}$ ,  $g_{12}$ , etc. the face and edges labeled by the strict 2-functor  $F$  and by  $\tilde{f}_{123}$ ,  $\tilde{g}_{12}$ , etc. the face and edges labeled by  $\tilde{F}$ . The three squares in (4.20) are labeled by the 2-morphisms  $\eta_{ij} := \eta_{e_{ij}} : g_{ij} \cdot \eta_j \Rightarrow \eta_i \cdot \tilde{g}_{ij}$ . By conditions (1)–(3) of Definition 3.11, the diagram (4.20) 2-commutes. Therefore, we can calculate  $\tilde{f}_{123}$  from

$$\tilde{f}_{123} = \text{id}_{\eta_1}^{(2)} \cdot \left( \left( \eta_{12}^* \cdot \text{id}_{g_{23}}^{(2)} \right) \circ \left( \text{id}_{g_{12}}^{(2)} \cdot \eta_{23}^* \right) \circ f_{123} \circ \eta_{13} \right). \quad (4.21)$$

Observe that the right-hand side involves both  $g_{ij}$  and  $\tilde{g}_{ij}$ .

#### 4.4. Gauge invariant expressions

In standard lattice gauge theory, gauge invariant quantities can be constructed from the holonomy around closed loops, namely by evaluating a group character. In order to find expressions that are invariant under local 2-gauge transformations, we consider the vertical composition of 2-morphisms over a closed surface, in the simplest case a tetrahedron. While the holonomy around the loop was based at a point, our vertical composition is now based at an edge of the tetrahedron.

Fig. 3 shows a tetrahedron  $(1, 2, 3, 4)$ . We label its triangles  $(i, j, k)$ ,  $i < j < k$ , as in (4.8). A 2-morphism around the surface of the tetrahedron can be calculated as follows.

**Definition 4.5.** Let  $F : \mathcal{C}^{V,E,F} \rightarrow \mathcal{G}^{G,H}$  be a strict 2-functor. For each tetrahedron  $(1, 2, 3, 4)$ , the 2-holonomy is the 2-morphism  $\Phi_{1234} : g_{14} \Rightarrow g_{14}$  in  $\mathcal{G}^{G,H}$ , given by

$$\Phi_{1234} := f_{124}^* \circ (\text{id}_{g_{12}}^{(2)} \cdot f_{234}^*) \circ (f_{123} \cdot \text{id}_{g_{34}}^{(2)}) \circ f_{134}. \quad (4.22)$$

Since any 2-morphism associated with a closed surface has the same source and target,  $\Phi_{1234} = (h_{1234}, g_{14})$  is characterized by an element  $h_{1234} \in \ker t \trianglelefteq H$ . It is therefore sufficient to define our gauge invariant actions as functions on  $\ker t$ .

Notice that we have labeled the 1- and 2-cells (edges and faces) of the lattice with data from the 2-groupoid  $\mathcal{G}^{G,H}$ , but that the 2-action is defined one level higher, namely for the 3-cells (tetrahedra). So far we have not formally defined the notion of a tetrahedron. For our purposes it is sufficient that the preceding definition can be used whenever we have a collection of four triangles whose edges match as shown in Fig. 3.

In standard lattice gauge theory, the action is a character of the holonomy around a loop. In our generalized setting, it turns out that, for our 2-holonomy, we need the following two invariances in order to obtain a locally 2-gauge invariant action.

**Definition 4.6.** Let  $(G, H, t, \alpha)$  be a Lie crossed module. A 2-action is a function  $S : H \rtimes G \rightarrow \mathbb{R}$  which is the composition  $S(\Phi) = s_0(\pi(\Phi))$  of a function  $s_0 : \ker t \rightarrow \mathbb{R}$  with the projection  $\pi : H \rtimes G \rightarrow H$ . We define the function  $s_0$  only on  $\ker t \trianglelefteq H$ . It is required to be a class function of  $H$ , i.e.,

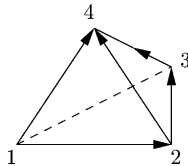


Fig. 3. A tetrahedron with vertices labeled 1, 2, 3, 4. Each triangle  $(i, j, k)$ ,  $i < j < k$ , is coloured as in (4.8).

$$s_0(h'hh^{-1}) = s_0(h) \quad (4.23)$$

for all  $h \in \ker t$ ,  $h' \in H$ , and to be constant on the orbits of  $G$ , i.e.,

$$s_0(\alpha(g)[h]) = s_0(h) \quad (4.24)$$

for all  $h \in \ker t$  and  $g \in G$ .

We present examples and discuss possible physical applications in Section 5. We have required the two invariances for the following purpose.

**Lemma 4.7.** *Let  $(G, H, t, \alpha)$  be a Lie crossed module and  $S : H \rtimes G \rightarrow \mathbb{R}$  be a 2-action. Let  $f = (h, g) \in H \rtimes G$  be any 2-morphism  $f : g \Rightarrow h$ , i.e.,  $h \in \ker t$ . Then  $S(f)$  is invariant under horizontal composition with identities because for any  $\text{id}_g^{(2)} = (1, \tilde{g})$ ,  $\tilde{g} \in G$ ,*

$$(h, g) \cdot (1, \tilde{g}) = (h, g\tilde{g}) \quad \text{and} \quad (1, \tilde{g}) \cdot (h, g) = (\alpha(\tilde{g})[h], \tilde{g}g), \quad (4.25)$$

and we have  $s_0(\alpha(\tilde{g})[h]) = s_0(h)$ . Furthermore,  $S(f)$  is invariant under vertical conjugation because for any  $(\tilde{h}, \tilde{g}) \in H \rtimes G$ ,

$$(\tilde{h}, \tilde{g})^* \circ (h, g) \circ (\tilde{h}, \tilde{g}) = (\tilde{h}^{-1}h\tilde{h}, \tilde{g}), \quad (4.26)$$

em and  $s_0(\tilde{h}^{-1}h\tilde{h}) = s_0(h)$ .

**Theorem 4.8.** *Let  $(V, E, F)$  be a simplicial 2-complex and  $(G, H, t, \alpha)$  be a Lie crossed module. Let  $F, \tilde{F} : \mathcal{C}^{V,E,F} \rightarrow \mathcal{G}^{G,H}$  be parallel strict 2-functors and  $S : H \rtimes G \rightarrow \mathbb{R}$  be some 2-action. If there exists a pseudo-natural transformation  $\eta : F \Rightarrow \tilde{F}$ , then the 2-action, evaluated on any tetrahedron  $(1, 2, 3, 4)$  in  $(V, E, F)$  agrees for  $F$  and  $\tilde{F}$ , i.e.,*

$$S(\Phi_{1234}) = S(\tilde{\Phi}_{1234}), \quad (4.27)$$

where  $\tilde{\Phi}_{1234}$  is the 2-holonomy of Definition 4.5 using the strict 2-functor  $F$  and  $\tilde{\Phi}_{1234}$  using  $\tilde{F}$ .

**Proof.** According to Definition 4.5,

$$\tilde{\Phi}_{1234} = \tilde{f}_{124}^* \circ (\text{id}_{g_{12}}^{(2)} \cdot \tilde{f}_{234}^*) \circ (\tilde{f}_{123} \cdot \text{id}_{g_{34}}^{(2)}) \circ \tilde{f}_{134}, \quad (4.28)$$

and

$$\Phi_{1234} = f_{124}^* \circ (\text{id}_{g_{12}}^{(2)} \cdot f_{234}^*) \circ (f_{123} \cdot \text{id}_{g_{34}}^{(2)}) \circ f_{134}. \quad (4.29)$$

The two coloured tetrahedra corresponding to  $\Phi_{1234}$  and  $\tilde{\Phi}_{1234}$  with the pseudo-natural transformation  $\eta$  are shown in Fig. 4. On the faces of this diagram, there are 2-morphisms from the strict 2-functors  $F$  and  $\tilde{F}$  and also from the pseudo-natural transformation  $\eta$  which we have suppressed in order to keep the drawing transparent. The four prisms attached to the triangular faces of the inner tetrahedron are of the form (4.20) and therefore 2-commute. We read off from the picture that

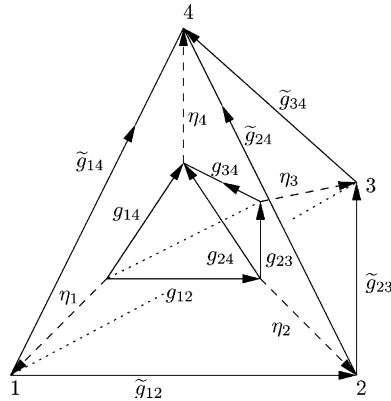


Fig. 4. The inner tetrahedron is labeled by a strict 2-functor  $F : \mathcal{C}^{V,E,F} \rightarrow \mathcal{G}^{G,H}$ , the outer one by some strict 2-functor  $\tilde{F}$  (notation as in (4.8)). Both 2-functors are related by a pseudo-natural transformation  $\eta : F \Rightarrow \tilde{F}$ . For simplicity, we have not drawn the double arrows on the faces.

$$\tilde{\Phi}_{1234} = \text{id}_{\eta_1^{-1}}^{(2)} \cdot \left( \eta_{14}^* \circ (\Phi_{1234} \cdot \text{id}_{\eta_4}^{(2)}) \circ \eta_{14} \right), \quad (4.30)$$

so that  $\Phi_{1234}$  and  $\tilde{\Phi}_{1234}$  are related by horizontal composition with identities and by vertical conjugation. Therefore, the value of  $S$  agrees according to Lemma 4.7.  $\square$

**Remark 4.9.** It can be shown that the edge, here  $e_{14}$ , on which the 2-morphism  $\Phi_{1234}$  is based, does not matter. By horizontal composition with identity 2-morphisms we can obtain an analogous 2-morphism based on any other edge which yields the same value of the action.

Of course, we can calculate locally 2-gauge invariant expressions for any closed surface by calculating appropriate compositions of the 2-morphisms. Pasting theorems, see, for example [21], guarantee that the 2-holonomy is well defined and independent of choices.

The gauge invariant expressions of the standard formulation of lattice gauge theory are in general not invariant under local 2-gauge symmetry transformations unless all 2-morphisms  $\eta_e$  associated with the edges have the same source and target.

**Remark 4.10.** As  $\ker t$  corresponds to the set of all 2-morphisms whose source and target coincide, the Eckmann–Hilton argument implies that  $\ker t$  is always Abelian (in fact, it is always contained in the centre of  $H$ ). Even though both  $G$  and  $H$  can be non-Abelian and there is a non-trivial interplay between the two via  $t$  and  $\alpha$ , the quantities on which the 2-actions depend, are therefore always Abelian.

This completes the construction of our generalization of lattice gauge theory. The generalized lattice and the generalized gauge group are both described by 2-categories. The configurations are given by strict 2-functors, the local 2-gauge symmetries by pseudo-natural transformations. In the last step, we have found actions that are invariant under this generalized local gauge symmetry.



#### 4.5. Partition function

It is now straightforward to write down a path integral for a quantum version of our lattice 2-gauge theory. Integrate over  $G$  for each edge and over  $H \rtimes G$  for each triangle, subject to the condition ( $\delta$ -functions on  $G$ ) that source and target of the 2-morphisms associated to the faces match. The integrand is the product over  $w(\Phi_{jklm}) := \exp(-S(\Phi_{jklm}))$  or  $\exp(iS(\Phi_{jklm}))$  for all tetrahedra  $(jklm)$  depending on whether Euclidean or real time is used. Here  $S$  denotes some 2-action. The partition function therefore reads

$$Z = \left( \prod_{e \in E} \int_G dg_e \right) \left( \prod_{t \in F} \int_{H \rtimes G} df_t \right) \times \left( \prod_{t \in F} \delta_G(s^{(2)}(f_t) \cdot (g_{\partial_1 t} g_{\partial_2 t})^{-1}) \delta_G(t^{(2)}(f_t) \cdot g_{\partial_3 t}) \right) \left( \prod_{\sigma \in T} w(\Phi_\sigma) \right). \quad (4.31)$$

Here we have denoted the triangles by  $t \in F$  and the tetrahedra by  $\sigma \in T$ . We already know some observables of this theory, namely the expectation values of the 2-gauge invariant expressions constructed in Theorem 4.8.

For any configuration given by  $g_e \in G$  for each edge  $e \in E$  and  $f_t \in H \rtimes G$  for each triangle  $t \in F$ , we obtain a locally 2-gauge equivalent configuration by applying the pseudo-natural transformation (4.21).

### 5. Examples and physical applications

In this section, we come back to some examples of strict Lie 2-groups (Example 3.7), illustrate what the admissible gauge invariant 2-actions are and sketch possible applications.

**Example 5.1.** The trivial 2-group (see Example 3.7(1)). In this case the assignment of variables reduces to conventional lattice gauge theory. There are no labels at the faces, and the local gauge transformations (Example 4.4) reduce to the ordinary ones (Example 2.7). As there are no variables associated with the faces, 2-actions are useless in this case.

**Example 5.2.** The purely Abelian 2-group (see Example 3.7(2)). In this case, the edges are not labeled while the faces are labeled with elements of some Abelian group  $H$ . We have  $\ker t = H$ . If we choose  $H = U(1)$ , we recover 2-form electrodynamics. Possible 2-actions are real characters of  $H$ , i.e., for  $G = U(1)$  they are of the form  $s_0(e^{i\varphi}) = \cos(k\varphi)$  for some  $k \in \mathbb{Z}$ , and our model agrees with that of [4,5] for  $p = 2$  except that it is defined on triangulations rather than on hyper-cubic lattices.

**Example 5.3.** The Euclidean 2-groups (see Example 3.7(4)). Again we have  $\ker t = H$ . The allowed 2-actions are functions of the  $SO(V, \eta)$ -invariant norm, i.e., of the form  $s_0(h) = f(\eta(h, h))$  for some function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . This resembles the expanders used in [17] in the case of the Poincaré 2-group.

Observe that this example is a non-trivial generalization of 2-form electrodynamics even though  $H$  is Abelian. There is a non-trivial interplay with  $G$  which requires the 2-action to be constant on the orbits of  $G$  on  $H$ . For the Poincaré 2-group, this example suggests the following more ambitious conjecture.

**Conjecture 5.4.** *Lattice 2-gauge theory using the Poincaré 2-group in four dimensions with a suitable action is equivalent to the refined Barrett–Crane–Yetter state sum proposed in [17] in the same fashion as standard lattice gauge theory with  $\delta$ -function Boltzmann weight is related to the Ooguri model, see, for example [22,23].*

*Lattice 2-gauge theory should therefore be formulated on the 2-complex dual to the triangulation used in the state sum model so that we can apply a suitable harmonic analysis and obtain sums over representations of  $H$  for the edges and sums over representations of  $G$  for the faces of the triangulation. This way it might be possible to obtain further conditions on the measure used in the state sum model and to relate it to a classical action with constraints at the classical level.*

**Example 5.5.** The automorphism 2-group of a simple compact Lie group  $H$  (see Example 3.7(5)). In this case  $\ker t = Z(H)$  is the centre of  $H$  and any function  $s_0 : Z(H) \rightarrow \mathbb{R}$  gives rise to a 2-action. This example can be understood as a lattice model of the connections on non-Abelian gerbes. We notice some coincidences which suggest the following conjecture.

**Conjecture 5.6.** *Lattice 2-gauge theory with the automorphism 2-group for  $H = SU(3)$ , i.e.,  $G = SU(3)/\mathbb{Z}_3$  and  $\ker t = \mathbb{Z}_3$ , describes some aspects of the collective phenomena of strongly coupled pure QCD. In fact, the 2-gauge invariant expressions of Theorem 4.8 resemble the observables that detect centre monopoles and vortices which seem to play a key role in the understanding of the confinement of static quarks in pure lattice QCD.*

## 6. Discussion and outlook

### 6.1. Technical questions

At the technical level, there are a number of natural questions to ask. What is the most general gauge invariant expression? In standard lattice gauge theory, these are *spin networks* [24], generalizations of Wilson loops that include branchings of the lines with intertwiners of the gauge group at the branching points. In order to fully understand the possible spin networks, one has to study the representation category of the gauge group. In our lattice 2-gauge theory, the expressions  $S(\Phi_{1234})$  of Theorem 4.8 are the analogues of Wilson loops. The most general gauge invariant expres-

sions will be given by coloured branched surfaces, i.e., by some sort of *spin foams* [25]. In order to understand these spin foams, we have to study the representation 2-category of a Lie 2-group. Work on the representation theory of 2-groups is in progress [26,27]. It is desirable to find a gauge invariant expression which combines both holonomies of morphisms around loops and 2-holonomies around closed surfaces so that this expression reduces to the Wilson action if we choose the trivial 2-group. At the moment we have either the old observables (characters of holonomies) which are not 2-gauge invariant or very special new ones (2-actions of 2-holonomies) which disappear for the trivial 2-group.

A related question is that of a maximal gauge fixing. In standard lattice gauge theory, one can gauge fix all edges of a spanning tree to be labeled by the group unit. In lattice 2-gauge theory, the gauge fixing will make use of suitable surfaces.

A further technical observation is that for 2-categories, there is one more natural type of maps besides functors and pseudo-natural transformations. These are called *modifications* and relate two natural transformations.

**Definition 6.1.** Let  $\mathcal{C}, \mathcal{C}'$  be small strict 2-categories,  $F, \tilde{F} : \mathcal{C} \rightarrow \mathcal{C}'$  be strict 2-functors and  $\eta, \vartheta : F \Rightarrow \tilde{F}$  be pseudo-natural transformations. A *quasi-modification*  $\mu : \eta \Rightarrow \vartheta$  is a map  $\mathcal{C}_0 \rightarrow \mathcal{C}'_2$  assigning to each object  $x \in \mathcal{C}_0$  a 2-morphism  $\mu_x : \eta_x \Rightarrow \vartheta_x$  in  $\mathcal{C}'$  such that

$$(Ff \cdot \mu_y) \circ \vartheta_{g'} = \eta_g \circ (\mu_x \cdot \tilde{F}f) \quad (6.1)$$

holds for each 2-morphism  $f : g \Rightarrow g'$  in  $\mathcal{C}$  where  $g, g' : x \rightarrow y$  are morphisms in  $\mathcal{C}$ . This is illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & Fg & & \\
 & \swarrow & \Downarrow Ff & \searrow & \\
 Fx & & & & Fy \\
 & \searrow & Fg' & \swarrow & \\
 & & & & \\
 \vartheta_x & \xleftarrow{\mu_x} & \eta_x & & \eta_y & \xrightarrow{\mu_y} & \vartheta_y \\
 & \searrow & & \swarrow & & & \\
 & & \tilde{F}g & & & & \\
 \tilde{F}x & & \Downarrow \tilde{F}f & & \tilde{F}y & & \\
 & \swarrow & \tilde{F}g' & \searrow & & & 
 \end{array} \quad (6.2)$$

We have suppressed the double arrows for the following 2-morphisms in order to keep the diagram simple:

$$\eta_g : Fg \cdot \eta_y \Rightarrow \eta_x \cdot \tilde{F}g, \quad (6.3)$$

$$\eta_{g'} : Fg' \cdot \eta_y \Rightarrow \eta_x \cdot \tilde{F}g', \quad (6.4)$$

$$\vartheta_g : Fg \cdot \vartheta_y \Rightarrow \vartheta_x \cdot \tilde{F}g, \quad (6.5)$$

$$\vartheta_{g'} : F_{g'} \cdot \vartheta_y \Rightarrow \vartheta_x \cdot \tilde{F}_{g'}. \quad (6.6)$$

It is open what modifications mean physically. Are some gauge equivalent configurations ‘more equal’ than others? An explanation why we are not forced to use this additional structural level may be the fact that our action is a map into the real numbers as opposed to a map into some 2-category. The requirement to use real numbers is, of course, imposed by the physical framework, but the 2-categorical treatment might indicate that one should try to categorify the action or the path integral.

Given a path integral formulation of lattice 2-gauge theory as sketched in Section 4.5, we have the *connection picture* of this theory which is given in terms of continuous variables. It is known that for lattice *BF*-theory (see, for example [22,23]) and also for standard lattice Yang–Mills theory [28,29], there is an equivalent *dual* formulation provided by the *representation picture* which is, in the case of Yang–Mills theory, the corresponding strong-weak dual theory. The dual formulation of standard lattice gauge theory is a spin foam model. How does the dual formulation of lattice 2-gauge theory look like?

It is already obvious that we have to understand the representation 2-category of the gauge 2-group in order to formulate such a model. If our lattice 2-gauge theory lives on the two-complex dual to some triangulation of a given four-manifold, the dual theory will involve sums over suitable vector spaces for all edges and for all faces of the original triangulation. As already mentioned, this is precisely the structure of the refined Barrett–Crane–Yetter state sum model of quantum gravity as proposed in [17]. An interesting project is therefore to perform the harmonic expansion (better: 2-harmonic expansion) of the Lie 2-group valued variables and, for generic 2-action, to generalize the transformation of [28,29] to lattice 2-gauge theory. We emphasize that the dual theory will involve one higher level of the given simplicial complex because the 2-action is associated with the 3-simplices.

Thinking about the representation picture of lattice 2-gauge theory, the entire program of the lattice gauge and state sum models is worth reconsidering: topological models, diagrammatic techniques generalizing the chain mail [30], and generalizing the ribbon diagrammatics of [31–33]. What is a suitable ‘non-commutative’ structure which generalizes the ribbon diagrams of [31–33] and maybe weakens the axioms of the 2-category used? For weak versions of 2-groups, see [15]. A general framework for state sum invariants of four-manifolds is provided by Mackaay’s construction [34].

## 6.2. A hierarchy of theories

One of the key ideas of higher category theory is that there is a hierarchy of structures (sets, categories, 2-categories, etc.). In this hierarchy, we call standard lattice gauge theory a *1-gauge theory* and the model constructed in the present article a *2-gauge theory*. Lattice 3-gauge theory is beyond the scope of this study, but we can still consider lattice 0-gauge theory in order to learn more about the hierarchy of models.

Consider the construction of Section 2. How does it collapse in the case of a 0-gauge theory? The lattice is a 0-category, i.e., just a set, a collection of points without structure. Similarly the set of labels ('0-gauge group'). A configuration is then a 0-functor which is just a map from the set of points to the 0-gauge group. There is no notion of natural equivalence and therefore no local symmetry. Such a model resembles a lattice spin model. These models include the lattice versions of non-linear sigma models. In total, we have the following information on the hierarchy of lattice  $n$ -gauge theories.

$n = 0$ . No local symmetry. The variables are associated with the vertices and the action terms with the edges. Gauge 'invariant' quantities are arbitrary functions. For models with a certain global symmetry, the strong coupling expansion [35] is an expansion in terms of spin networks (coloured graphs) and the dual formulation is a spin network model.

$n = 1$ . Standard lattice gauge theory. The variables are associated with the edges and the action terms with the faces. Gauge invariant quantities are spin networks. The strong coupling expansion [28,29] is an expansion in terms of spin foams (coloured 2-complexes) and the dual model is a spin foam model.

$n = 2$ . The model constructed in the present article. The generic gauge invariant quantities are certain spin foams. From the assignment of variables one can already see that the strong coupling expansion will lead to coloured 3-complexes.

It seems that we can minimally couple the model at level  $n$  with the model at  $n + 1$ . The classic example at  $n = 0$  is the Abelian Higgs model with frozen radial mode [36], but this construction can be extended to non-Abelian symmetry groups as well [35]. The next more general step would therefore be to couple a 2-gauge theory to standard lattice gauge theory.

### 6.3. General comments

It is an interesting question whether one can construct a Topological Quantum Field Theory (TQFT) from our 2-gauge theory. We should therefore require the higher level analogue of the flatness condition in order to obtain topological invariants. This means we should restrict the partition function of our quantum theory to those configurations for which the 2-holonomy  $\Phi_\sigma$  vanishes at every tetrahedron  $\sigma$ . This is the *zero 2-curvature condition*.

For certain finite crossed modules this TQFT was constructed by Yetter [37] using the language of categorical groups, i.e., group objects in the category of groupoids. In the topological case one can choose an arbitrary triangulation as the lattice and then construct the infinite refinement limit. This means in physical terms that the TQFT has a trivial renormalization.

The continuum counterpart of the higher lattice gauge theory was developed in [8] and, for non-Abelian gerbes, in [10]. In the continuum, it has turned out to be difficult to find fully gauge invariant actions. In the discrete approach, however, see also [9], invariant actions arise naturally from the tetrahedron diagram, Fig. 4. Our result is that the admissible 2-actions are sensitive only to  $\ker t \leq H$ , an observation which may help to better understand the continuum situation. In particular for the

automorphism 2-group,  $\ker t$  is often a discrete group. In this case, it will be impossible to write down a naive continuum expression for the 2-action which just uses Lie algebras instead of Lie groups. Continuum 2-actions will rather involve some kind of topological defect or singularity and will be given by non-local expressions similar to topological charges.

It should generally be possible to consider ‘infinitesimal’ simplices as in [10] in order to infer the continuum formulas corresponding to the given discrete expressions. It is open whether the converse is possible, i.e., to integrate the differential continuum expressions in order to recover the non-infinitesimal formulas. The problem is that there is no ‘surface ordered product’ available yet. Because of this obstacle, we favour the discrete approach, at least for now.

Finally, the model constructed here together with the continuum counterparts developed in [8,10] demonstrates that there exist theories with local symmetries beyond conventional gauge theory. Are the corresponding quantum field theories relevant in nature? We have indicated two possible areas of physics in which they might turn out to be useful, state sum models of quantum gravity and the low energy behaviour of QCD.

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