Higher Yang–Mills Theory

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Abstract

Electromagnetism can be generalized to Yang–Mills theory by replacing the group U(1) by a nonabelian Lie group. This raises the question of whether one can similarly generalize 2-form electromagnetism to a kind of 'higher-dimensional Yang–Mills theory'. It turns out that to do this, one should replace the Lie group by a 'Lie 2-group', which is a category C where the set of objects and the set of morphisms are Lie groups, and the source, target, identity and composition maps are homomorphisms. We show that this is the same as a 'Lie crossed module': a pair of Lie groups G, H with a homomorphism $t \colon H \to G$ and an action of G on H satisfying two compatibility conditions. Following Breen and Messing's ideas on the geometry of nonabelian gerbes, one can define 'principal 2-bundles' for any Lie 2-group C and do gauge theory in this new context. Here we only consider trivial 2-bundles, where a connection consists of a $\mathfrak{g}$-valued 1-form together with an $\mathfrak{h}$-valued 2-form, and its curvature consists of a $\mathfrak{g}$-valued 2-form together with a $\mathfrak{h}$-valued 3-form. We generalize the Yang–Mills action for this sort of connection, and use this to derive 'higher Yang–Mills equations'. Finally, we show that in certain cases these equations admit self-dual solutions in five dimensions.

1   Introduction

Describing electromagnetism in terms of the 1-form A is very natural, since to compute the action of a charged particle we merely integrate this 1-form over the particle's path and add the result to the action for a neutral particle moving along the same path. A 2-form B called the Kalb–Ramond field plays a similar role in the theory of strings: we integrate this over the string worldsheet to get a term in the action. Extending this analogy, one can write down equations for '2-form electromagnetism' that are formally identical to Maxwell's equations.

Just as the electromagnetic field $F = dA$ automatically satisfies $dF = 0$, so that the only nontrivial Maxwell equation is

$$*d*F = J,$$

the field $G = dB$ automatically satisfies $dG = 0$, but one can impose an additional equation

$$*d*G = K$$

in which the current K is now a 2-form. Note that just as the worldline of charged point particle defines a distributional current 1-form, the worldsheet of a charged string defines a current 2-form, so the whole scheme is consistent.

Since we can generalize Maxwell's equations to Yang–Mills theory by replacing A by a 1-form taking values in a nonabelian Lie algebra, it is interesting to see if we can push the above analogy further and set up some sort of '2-form Yang–Mills theory'. There are indications from string theory that something like this should exist [11], but naive attempts to write down such a theory are already known to fail [18]. Fundamentally, the problem is that there is no good way to define a notion of holonomy assigning elements of a nonabelian group to the surfaces traced out by the motion of a 1-dimensional extended object.

To see the problem, first recall that it is very natural for holonomies to assign elements of a nonabelian group to paths:
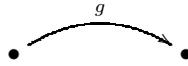

The reason is that composition of paths then corresponds to multiplication in the group:

while reversing the direction of a path corresponds to taking inverses:
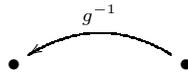
and the associative law makes the holonomy along a triple composite unambiguous:
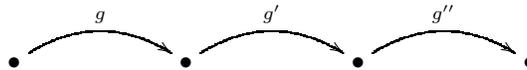
In short, the geometry dictates the algebra.

Now suppose we wish to do something similar for surfaces. Naively we might wish our holonomy to assign a group element to each surface like this:
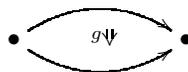
There are two obvious ways to compose surfaces of this sort, vertically:
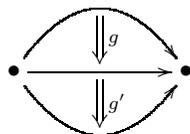
and horizontally:

Suppose that both of these correspond to multiplication in the group G. Then to obtain well-defined holonomy for this surface regardless of whether we do vertical or horizontal composition first:
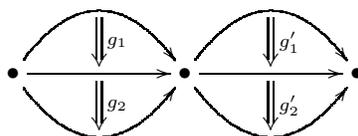
we must have

$$(g_1 g_2)(g'_1 g'_2) = (g_1 g'_1)(g_2 g'_2).$$

This forces G to be abelian!

In fact, this argument goes back to a classic paper by Eckmann and Hilton [12]. Moreover, they showed that even if we allow G to be equipped with two products, say $g \circ g'$ for vertical composition and gg' for horizontal, so long as both products share the same unit and satisfy this 'exchange law':

$$(g_1 \circ g'_1)(g_2 \circ g'_2) = (g_1 g_2) \circ (g'_1 g'_2)$$

then in fact they must agree — so by the previous argument, both are abelian. The proof is very easy:

$$gg' = (g \circ 1)(1 \circ g') = (g1) \circ (1g') = g \circ g'.$$

The way around this roadblock is to consider a sort of connection that allows us to define holonomies *both for paths and for surfaces.* Assume that for each path we have a holonomy taking values in some group G:

$$\bullet \xrightarrow{\ \ g\ \ } \bullet$$
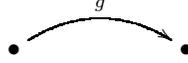
where composition of paths corresponds to multiplication in G. Assume also that for each 1-parameter family of paths with fixed endpoints we have a holonomy taking values in some other group H:

$$\bullet \overset{h\Downarrow}{\rightrightarrows} \bullet$$
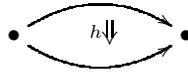
where vertical composition corresponds to multiplication in H:

$$\bullet\ \begin{matrix} \Downarrow h \\ \longrightarrow \\ \Downarrow h' \end{matrix}\ \bullet$$
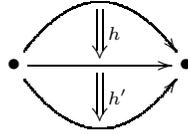
Next, assume that we can parallel transport an element $g \in G$ along a 1-parameter family of paths to get a new element $g' \in G$:

$$\bullet\ \begin{matrix} g \\ h\Downarrow \\ g' \end{matrix}\ \bullet$$
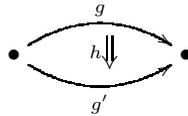
Here a caveat is in order: if g is the holonomy along the initial path, we do not assume that g' is the holonomy along the final path. To require this would impose a kind of 'flatness' condition on our connection. In the formalism we are heading toward, there will be a $\mathfrak{g}$-valued curvature 2-form F that measures the deviation from this sort of flatness. There will also be an $\mathfrak{h}$-valued 3-form that measures how the H-valued holonomy along a 1-parameter family of paths changes as we vary this family while keeping its initial and final paths fixed.

Now, the picture above suggests that we should think of h as a kind of 'arrow' or 'morphism' going from g to g', and use category theory to formalize the situation. However, in category theory, when a morphism goes from an object x to an object y, we think of the morphism as determining both its source x and its target y. The group element h does not determine g or g'. However, the pair (h, g) does. Thus it is useful to create a category C where the set C_0 of objects is just G, while the set C_1 of morphisms consists of pairs $f = (h, g) \in H \times G$. Switching our notation to reflect this, we rewrite the above

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picture as
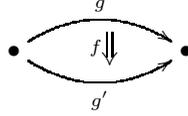
and write $f\colon g \to g'$ for short. We have source and target maps

$$s, t\colon C_1 \to C_0$$

with $s(f) = g$ and $t(f) = g'$.

In this new notation we can vertically compose $f\colon g \to g'$ and $f'\colon g \to g'$ to get $f \circ f'\colon g \to g''$, as follows:
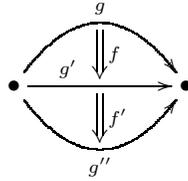
This is just composition of morphisms in the category C. However, we can also horizontally compose $f_1\colon g_1 \to g_1'$ and $f_2\colon g_2 \to g_2'$ to get $f_1 f_2\colon g_1 g_2 \to g_1' g_2'$, as follows:
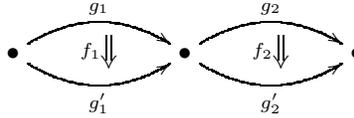
We assume this operation makes C_1 into a group with the pair $(1, 1) \in H \times G$ as its multiplicative unit.

The good news is that now we can assume an exchange law saying this holonomy is well-defined:
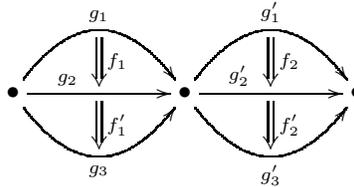
namely:

$$(f_1 \circ f_1')(f_2 \circ f_2') = (f_1 f_2) \circ (f_1' f_2'),$$

without forcing either C_0 or C_1 to be abelian! Instead, C_1 is forced to be a semidirect product of the groups G and H.

The structure we are rather roughly describing here is in fact already known to mathematicians under the name of a 'categorical group' [13, 16]. The reason is that C turns out to be a category whose set of objects C_0 is a group, whose

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set of morphisms C_1 is a group, and where all the usual category operations are group homomorphisms. To keep the terminology succinct and to hint at generalizations to still higher-dimensional holonomies, we prefer to call this sort of structure a '2-group'. Moreover, we shall focus most of our attention on 'Lie 2-groups', where C_0 and C_1 are Lie groups and all the operations are smooth.

As we shall see, a Lie 2-group C amounts to the same thing as a 'Lie crossed module': a pair of Lie groups G and H together with a homomorphism $t \colon H \to G$ and an action α of G on H satisfying a couple of compatibility conditions. The idea is to let $G = C_0$, let H be the subgroup of C_1 consisting of morphisms with source equal to 1, and let t be the map sending each such morphism to its target. The action α is defined by letting $\alpha(g)f$ be this horizontal composite:

It appears that one can develop a full-fledged theory of bundles, connections, curvature, the Yang–Mills equations, and so on with a Lie 2-group taking the place of a Lie group. So far most work has focused on the special case when G is trivial and H is abelian [8, 10, 9, 14, 15, 17], but here we really want H to be nonabelian. Some important progress in this direction can be found in Breen and Messing's paper on the differential geometry of nonabelian gerbes [5], and also Attal's work on a combinatorial version of this theory [1]. While they use different terminology, their work basically develops the theory of connections and curvature for Lie 2-groups where H is an arbitrary Lie group, $G = \mathrm{Aut}(H)$, t sends each element of H to the corresponding inner automorphism, and the action of G on H is the obvious one. We call this sort of Lie 2-group the 'automorphism 2-group' of H. Luckily, it is easy to extrapolate the whole theory from this case.

In particular, for any Lie 2-group C one can define the notion of a 'principal 2-bundle' having C as its gauge 2-group, and then define connections and curvature for these principal 2-bundles. However, to keep things simple, we do this here only for *trivial* principal 2-bundles. This renders the whole apparatus of 2-bundles unnecessary, since a connection then amounts to just a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on the base manifold M. The curvature consists of a $\mathfrak{g}$-valued 2-form:

$$F = dA + \frac{1}{2} A \wedge A - \underline{B}$$

together with a $\mathfrak{h}$-valued 3-form:

$$G = dB + \underline{A \wedge B}$$

where the $\mathfrak{g}$-valued 2-form $\underline{B}$ is obtained from B using the map $dt \colon \mathfrak{h} \to \mathfrak{g}$, and the $\mathfrak{h}$-valued 3-form $\underline{A \wedge B}$ is defined using the action $d\alpha$ of $\mathfrak{g}$ on $\mathfrak{h}$. The

suprising new $\underline{B}$ term in the curvature 2-form F comes from the fact that we use 2-dimensional parallel transport to compare holonomies along nearby paths, instead of merely taking their difference.

The curvature automatically satisfies these 'higher Bianchi identities':

$$\begin{array}{rcl} dF + A \wedge F & = & -\underline{G}, \\ dG + \underline{A \wedge G} & = & \underline{F \wedge B}. \end{array}$$

Moreover, we can develop a generalization of Yang–Mills theory whenever the base manifold M is oriented and equipped with a metric, and $\mathfrak{g}$ and $\mathfrak{h}$ are equipped with invariant inner products. To do this, we start with a generalization of the Yang–Mills action:

$$S(A, B) = \int_M \operatorname{tr}(F \wedge *F) + \operatorname{tr}(G \wedge *G).$$

Extremizing this action we obtain the 'higher Yang–Mills equations':

$$\begin{array}{rcl} d * F + A \wedge *F & = & \underline{*G \wedge B}. \\ d * G + \underline{A \wedge *G} & = & -*\underline{F} \end{array}$$

Here the underlines again refer to certain natural maps involving $\mathfrak{g}$ and $\mathfrak{h}$. In the special case when H is trivial, these equations reduce to the ordinary Yang–Mills equations with gauge group G. When G is trivial and $H = \mathrm{U}(1)$, the higher Yang–Mills equations reduce to those of 2-form electromagnetism.

In short, by replacing the concept of 'Lie group' by the concept of 'Lie 2-group', we arrive at equations that constitute a simultaneous generalization of Yang–Mills theory and 2-form electromagnetism. It remains to see how useful these equations are, either in physics or in pure mathematics. However, we shall show that under certain conditions on the Lie 2-group C, they admit 'self-dual' solutions in 5-dimensional spacetime. Since the self-dual solutions of the ordinary Yang–Mills equations are important in 4-dimensional topology, it is tempting to hope that the higher Yang–Mills equations will play an interesting role in topology one dimension up.

The plan of the paper is as follows. In Section 2 we carefully define Lie 2-groups, show they are the same as Lie crossed modules, and give a number of examples. In Section 2 we define Lie 2-algebras, show they are the same as 'differential crossed modules', and show how to get a Lie 2-algebra from a Lie 2-group. In Section 4 we define connections and curvature for trivial 2-bundles, and prove the higher Bianchi identities in this case. Finally, in Section 5 we derive the higher Yang–Mills equations from an action principle, and show they have self-dual solutions in 5 dimensions. There is much we leave undone: for example, we do not develop the general theory of 2-bundles, nor do we work out the theory of holonomies for the connections we consider, even though it serves as a vital motivation for the concept of Lie 2-group.

2 Lie 2-Groups

The concept of 'Lie 2-group' is a kind of blend of the concepts of Lie group and category. A small category C has a set C_0 of objects, a set C_1 of morphisms, functions

$$s, t \colon C_1 \to C_0$$

assigning to each morphism $f \colon x \to y$ its source x and target y, a function

$$i \colon C_0 \to C_1$$

assigning to each object its identity morphism, and finally, a function

$$\circ \colon C_1 \times_{C_0} C_1 \to C_1$$

describing composition of morphisms, where

$$C_1 \times_{C_0} C_1 = \{(f, g) \in C_1 \times C_1 \colon t(f) = s(g)\}$$

is the set of composable pairs of morphisms. If we now take the words 'set' and 'function' and replace them by 'Lie group' and 'homomorphism', we get the definition of a Lie 2-group. Here and in all that follows, we require that homomorphisms between Lie groups be smooth:

Definition 1. *A **Lie 2-group** is a category C where the set C_0 of objects and the set C_1 of morphisms are Lie groups, the functions $s, t \colon C_1 \to C_0$, $i \colon C_0 \to C_1$ are homomorphisms, $C_1 \times_{C_0} C_1$ is a Lie subgroup of $C_1 \times C_1$, and $\circ \colon C_1 \times_{C_0} C_1 \to C_1$ is a homomorphism.*

The fact that composition is a homomorphism implies the exchange law

$$(f_1 \circ f_1')(f_2 \circ f_2') = (f_1 f_2) \circ (f_1' f_2')$$

whenever we have a situation of this sort:
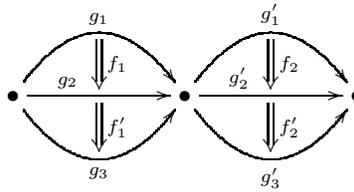
A homomorphism between Lie 2-groups is a kind of blend of the concepts of homomorphism between Lie groups and functor between categories:

Definition 2. *Given Lie 2-groups C and C', a **homomorphism** $F \colon C \to C'$ is a functor which acts on objects and morphisms to give homomorphisms $F_0 \colon C_0 \to C_0'$ and $F_1 \colon C_1 \to C_1'$, respectively.*

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A Lie 2-group with only identity morphisms is the same thing as a Lie group. To get more interesting examples it is handy to think of a Lie 2-group as special sort of crossed module. To do this, start with a Lie 2-group C and form the pair of Lie groups

$$G = C_0, \qquad H = \ker s \subseteq C_1.$$

The target map restricts to a homomorphism

$$t\colon H \to G.$$

Besides the usual action of G on itself by conjugation, there is also an action of G on H,

$$\alpha\colon G \to \mathrm{Aut}(H),$$

given by

$$\alpha(g)(h) = i(g)\, h\, i(g)^{-1}.$$

The target map is equivariant with respect to this action:

$$t(\alpha(g)(h) = g\, t(h)\, g^{-1}.$$

and we also have

$$\alpha(t(h))(h') = hh'h^{-1}$$

for all $h, h' \in H$. A setup like this with groups rather than Lie groups is called a 'crossed module', so here we are getting a 'Lie crossed module':

Definition 3. *A **Lie crossed module** is a quadruple (G, H, t, α) consisting of Lie groups G and H, a homomorphism $t\colon H \to G$, and an action of G on H (that is, a homomorphism $\alpha\colon G \to \mathrm{Aut}(H)$) satisfying*

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and

$$\alpha(t(h))(h') = hh'h^{-1}$$

for all $g \in G$ and $h, h' \in H$.

This definition becomes a bit more memorable if allow ourselves to write $\alpha(g)(h)$ as ghg^{-1}; then the equations above become

$$t(ghg^{-1}) = g\, t(h)\, g^{-1}$$

and

$$t(h)\, h'\, t(h)^{-1} = hh'h^{-1}.$$

Definition 4. *A **homomorphism** from the Lie crossed module (G, H, t, α) to the Lie crossed module (G', H', t', α') is a commutative square of homomorphisms*

$$
\begin{array}{ccc}
H & \xrightarrow{\ \tilde{f}\ } & H' \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\
G & \xrightarrow{\ f\ } & G'
\end{array}
$$

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such that

$$f(\alpha(g)(h)) = \alpha'(\tilde{f}(g))(f(h))$$

for all $g \in G$, $h \in H$.

Proposition 5. *The category of Lie 2-groups is equivalent to the category of Lie crossed modules.*

Proof — This follows easily from the well-known equivalence between crossed modules and 2-groups, also known as categorical groups [7] or sometimes gr-categories [3]. A nice exposition of this can be found in a paper by Forrester-Barker [13], but for the convenience of the reader we sketch how to recover a Lie 2-group from a Lie crossed module.

Suppose we have a Lie crossed module (G, H, t, α). Let

$$C_0 = G, \qquad C_1 = H \rtimes G$$

where the semidirect product is formed using the action of G on H, so that multiplication in C_1 is given by

$$(h, g)(h', g') = (h\alpha(g)(h'), gg').$$

We make this into a Lie 2-group where the source and target maps $s, t \colon C_1 \to C_0$ are given by

$$s(h, g) = g, \qquad t(h, g) = t(h)g,$$

the map $i \colon C_0 \to C_1$ is given by

$$i(g) = (1, g),$$

and the composite of the morphisms

$$(h, g) \colon g \to g', \qquad (h', g') \colon g' \to g''$$

is

$$(hh', g) \colon g \to g''.$$

One can check that this construction indeed gives a Lie 2-group, and that together with the previous construction it sets up an equivalence between the categories of Lie 2-groups and Lie crossed modules. □

Crossed modules are important in homotopy theory [6], and the reader who is fonder of crossed modules than categories is free to think of Lie 2-groups as an idiosyncratic way of talking about Lie crossed modules. Both perspectives are useful, but one advantage of Lie crossed modules is that they allow us to quickly describe some examples:

Example 6. Given any Lie group G, abelian Lie group H, and homomorphism $\alpha \colon G \to \mathrm{Aut}(H)$, there is a Lie crossed module with $t \colon G \to H$ the trivial homomorphism and G acting on H via ρ. Because t is trivial, the corresponding Lie 2-group C is 'skeletal', meaning that any two isomorphic objects are equal. It is easy to see that conversely, all skeletal Lie 2-groups are of this form.

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Example 7. Given any Lie group G, we can form a Lie crossed module as in Example 6 by taking $H = \mathfrak{g}$, thought of as an abelian Lie group, and letting α be the adjoint representation of G on $\mathfrak{g}$. If C is the corresponding Lie 2-group, we have

$$C_1 = \mathfrak{g} \rtimes G \cong TG$$

where TG is the tangent bundle of G, which becomes a Lie group with product

$$dm: TG \times TG \to TG,$$

obtained by differentiating the product

$$m: G \times G \to G.$$

We call C the **tangent 2-group** of G.

Example 8. Similarly, given any Lie group G, we can form a Lie crossed module as in Example 6 by letting α be the coadjoint representation on $H = \mathfrak{g}^*$. If C is the corresponding Lie 2-group, we have

$$C_1 = \mathfrak{g}^* \rtimes G \cong T^*G$$

where T^*G is the cotangent bundle of G, and we call C the **cotangent 2-group** of G.

Example 9. If G is the Lorentz group $\mathrm{SO}(n,1)$, we can form a Lie crossed module as in Example 6 by letting α be the defining representation of $\mathrm{SO}(n,1)$ on $H = \mathbb{R}^{n+1}$. If C is the corresponding Lie 2-group, we have

$$C_1 = \mathbb{R}^{n+1} \rtimes \mathrm{SO}(n,1) \cong \mathrm{ISO}(n,1)$$

where $\mathrm{ISO}(n,1)$ is the Poincaré group. We call C the **Poincaré 2-group**.

Example 10. Given any Lie group H, there is a Lie crossed module with $G = \mathrm{Aut}(H)$, $t: H \to G$ the homomorphism assigning to each element of H the corresponding inner automorphism, and the obvious action of G as automorphisms of H. We call the corresponding Lie 2-group C the **automorphism 2-group** of H.

Example 11. If we take $H = \mathrm{SU}(2)$ and form the automorphism 2-group of H, we get a Lie 2-group with $G = \mathrm{SO}(3)$. Since $\mathrm{SU}(2)$ can be thought of as the unit quaternions, this example is closely related to a larger Lie 2-group implicit in Thompson's work on 'quaternionic gerbes' [19]. This larger 2-group is just the automorphism 2-group of the multiplicative group of nonzero quaternions.

Example 10 is important in the theory of nonabelian gerbes. Any Lie 2-group C becomes a strict monoidal category if we define a tensor product

$$\otimes: C \times C \to C$$

by

$$g \otimes g' = gg'$$

for all objects $g, g' \in C_0$, and

$$f \otimes f' = ff'$$

for all morphisms $f, f' \in C_1$, where ff' is defined using the product in the group C_1, not composition of morphisms. When C is the automorphism 2-group of H, this monoidal category is equivalent to the monoidal category of 'H-bitorsors'. An H-**bitorsor** X is a manifold with commuting left and right actions of H, both of which are free and transitive. A morphism between H-bitorsors $f: X \to Y$ is a smooth map which is equivariant with respect to both these actions. The 'tensor product' of H-bitorsors X and Y is defined to be the space

$$X \otimes Y = X \times Y / ((xh, y) \sim (x, hy)),$$

which inherits a left H-action from X and a right H-action from Y. Accompanying this there is an obvious notion of the tensor product of morphisms between bitorsors, making H-bitorsors into a weak monoidal category H-Bitors. Every H-bitorsor is isomorphic to H with its usual right action on itself but with the left action twisted by an automorphism of H. Using this fact, it is not hard to check that H-Bitors is equivalent to the automorphism 2-group of H.

One can describe H-gerbes in terms of H-bitorsors [1, 4, 5]. Alternatively, one can construct an H-gerbe by taking the stack of sections of a 'principal 2-bundle' where the 'gauge 2-group' is the automorphism 2-group of H. By the above remarks these viewpoints are not really that different, although the latter one seems not to have been developed in the literature.

Finally, for the reader familiar with categorification [2], it is worth mentioning that what we are calling Lie 2-groups should really be called 'strict' Lie 2-groups. Ultimately more interesting will be the more general 'weak' ones, which will be weak rather than strict monoidal categories, and where the inverse for an object g satisfies the laws $gg^{-1} = g^{-1}g = 1$ only up to isomorphism. Our goal here is not to develop higher Yang–Mills theory in its full generality, but merely to show that it exists.

3   Lie 2-Algebras

Just as a Lie 2-group is a category built from Lie groups, a Lie 2-algebra is a category built from Lie algebras:

Definition 12. *A* **Lie 2-algebra** *is a category $\mathfrak{c}$ where the set $\mathfrak{c}_0$ of objects and the set $\mathfrak{c}_1$ of morphisms are Lie algebras, and the functions $s, t: \mathfrak{c}_1 \to \mathfrak{c}_0$, $i: \mathfrak{c}_0 \to \mathfrak{c}_1$ and $\circ: \mathfrak{c}_1 \times_{\mathfrak{c}_0} \mathfrak{c}_1 \to \mathfrak{c}_1$ are Lie algebra homomorphisms.*

Definition 13. *Given Lie 2-algebras $\mathfrak{c}$ and $\mathfrak{c}'$, a* **homomorphism** *$F: \mathfrak{c} \to \mathfrak{c}'$ is a functor which acts on objects and morphisms to give Lie algebra homomorphisms $F_0: \mathfrak{c}_0 \to \mathfrak{c}'_0$ and $F_1: \mathfrak{c}_1 \to \mathfrak{c}'_1$, respectively.*

Just as Lie groups have Lie algebras, Lie 2-groups have Lie 2-algebras:

Proposition 14. *Any Lie 2-group C has a Lie 2-algebra $\mathfrak{c}$ for which $\mathfrak{c}_0$ is the Lie algebra of the Lie group of objects C_0, $\mathfrak{c}_1$ is the Lie algebra of the Lie group of morphisms C_1, and the maps $s, t\colon \mathfrak{c}_1 \to \mathfrak{c}_0$, $i\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ and $\circ\colon \mathfrak{c}_1 \times_{\mathfrak{c}_0} \mathfrak{c}_1 \to \mathfrak{c}_1$ are the differentials of those for C. Similarly, any homomorphism of Lie 2-groups $F\colon C \to C'$ gives rise to a homomorphism of Lie 2-algebras, which we denote by $dF\colon \mathfrak{c} \to \mathfrak{c}'$. We obtain a functor from the category of Lie 2-groups to the category of Lie 2-algebras.*

Proof — The proof is straightforward.   □

The equivalence between Lie 2-groups and Lie crossed modules has an analogue for Lie 2-algebras. In this analogue, the replacement for the Lie group $\mathrm{Aut}(H)$ is the Lie algebra $\mathrm{Der}(\mathfrak{h})$ consisting of all 'derivations' of $\mathfrak{h}$, that is, linear maps $f\colon \mathfrak{h} \to \mathfrak{h}$ such that

$$f([y, y']) = [f(y), y'] + [y, f(y')].$$

Definition 15. *A **differential crossed module** is a quadruple $(\mathfrak{g}, \mathfrak{h}, t, \alpha)$ consisting of Lie algebras $\mathfrak{g}, \mathfrak{h}$, a homomorphism $t\colon \mathfrak{h} \to \mathfrak{g}$, and an action α of $\mathfrak{g}$ as derivations of $\mathfrak{h}$ (that is, a homomorphism $\alpha\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$) satisfying*

$$t(\alpha(x)(y)) = [x, t(y)]$$

and

$$\alpha(t(y))(y') = [y, y']$$

for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$.

This definition becomes easier to remember if we allow ourselves to write $\alpha(x)(y)$ as $[x, y]$. Then the fact that α is an action of $\mathfrak{g}$ as derivations of $\mathfrak{h}$ simply means that $[x, y]$ is linear in each argument and the following 'Jacobi identities' hold:

$$[[x, x'], y] = [x, [x', y]] - [x', [x, y]], \tag{1}$$

$$[x, [y, y']] = [[x, y], y'] - [[x, y'], y] \tag{2}$$

for all $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$. Furthermore, the two equations in the above definition become

$$t([x, y]) = [x, t(y)] \tag{3}$$

and

$$[t(y), y'] = [y, y']. \tag{4}$$

Proposition 16. *The category of Lie 2-algebras is equivalent to the category of differential crossed modules.*

Proof — The proof is just like that of Proposition 5.   □

Any Lie crossed module gives rise to a differential crossed module; this is just a less elegant way of saying that any Lie 2-group has a Lie 2-algebra:

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Proposition 17. *Any Lie crossed module (G, H, t, α) has a differential crossed module $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{h}$ is the Lie algebra of H, $dt: \mathfrak{h} \to \mathfrak{g}$ is the differential of $t: H \to G$, and the action of $\mathfrak{g}$ as derivations of $\mathfrak{h}$ is the differential $d\alpha: \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ of the homomorphism $\alpha: G \to \mathrm{Aut}(H)$. We obtain a functor from the category of Lie crossed modules to the category of differential crossed modules.*

Proof — This can easily be seen directly, but it also follows from combining Propositions 5, 14, and 16.   □

4   Connections and Curvature

The concepts of connections and curvature can be generalized by replacing Lie groups by Lie 2-groups and Lie algebras by Lie 2-algebras. In particular, Breen and Messing [5] have defined these notions in the special case when the 'gauge 2-group' C is of the special sort described in Example 10: namely, the automorphism 2-group of a Lie group. Given this, generalizing to arbitrary Lie 2-groups is not very hard. However, to bring out the basic ideas in their simplest form, we do this here only for 'trivial principal 2-bundles', where connections and curvature can be thought of as Lie-algebra-valued differential forms:

Definition 18. *Suppose C is a Lie 2-group and (G, H, t, α) is the corresponding Lie crossed module. A* **connection** *on the manifold M is a pair (A, B) consisting of a $\mathfrak{g}$-valued 1-form A on M together with a $\mathfrak{h}$-valued 2-form B.*

Now suppose that C is Lie 2-group, (G, H, t, α) the corresponding Lie crossed module, and $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ the differential crossed module obtained via Proposition 17.

Definition 19. *Suppose (A, B) is a connection on M. Then the* **curvature** *of this connection is the pair (F, G) where F is the $\mathfrak{g}$-valued 2-form*

$$F = dA + \frac{1}{2} A \wedge A - \underline{B}$$

and G is the $\mathfrak{h}$-valued 3-form

$$G = dB + \underline{A \wedge B}.$$

Here we define the wedge product of $\mathfrak{g}$-valued differential forms by

$$(x \otimes \mu) \wedge (x' \otimes \mu') = [x, x'] \otimes (\mu \wedge \mu')$$

where $x, x' \in \mathfrak{g}$ and μ, μ' are diferential forms; this extends by linearity in each argument to all $\mathfrak{g}$-valued differential forms. Similarly, given a $\mathfrak{g}$-valued differential form $X = x \otimes \mu$ and an $\mathfrak{h}$-valued differential form $Y = y \otimes \nu$, we define

$$\underline{X \wedge Y} = [x, y] \otimes (\mu \wedge \nu)$$

where for simplicity of notation we write $[x, y]$ as an abbrevation for $d\alpha(x)(y)$. Also, given an $\mathfrak{h}$-valued differential form $Y = y \otimes \nu$, we define

$$\underline{Y} = dt(y) \otimes \nu$$

and extend this to arbitrary $\mathfrak{h}$-valued differential forms by linearity.

We can streamline the definition of the curvature 3-form G and the statement of the Bianchi identities if we define the exterior covariant derivative of a $\mathfrak{g}$-valued differential form X by

$$d_A X = dX + A \wedge X,$$

and of an $\mathfrak{h}$-valued differential form Y by

$$d_A Y = dY + \underline{A \wedge Y}.$$

It is then well-known that

$$d_A^2 X = F \wedge X, \qquad d_A^2 Y = \underline{F \wedge Y}.$$

In this notation we have
$$G = d_A B.$$

Proposition 20. *If (A, B) is any connection on M, then its curvature (F, G) satisfies the* **higher Bianchi identities***:*

$$\begin{array}{rcl} d_A F & = & -\underline{G} \\ d_A G & = & \underline{F \wedge B}. \end{array} \qquad (5)$$

Proof — For the first equation, note that

$$\begin{array}{rcl} d_A F & = & dF + A \wedge F \\ & = & d(dA + \tfrac{1}{2} A \wedge A - \underline{B}) + A \wedge (dA + \tfrac{1}{2} A \wedge A - \underline{B}) \\ & = & -d_A \underline{B} \\ & = & -\underline{d_A B} = -\underline{G} \end{array}$$

where we use the fact that for any $\mathfrak{h}$-valued form Y we have

$$d_A \underline{Y} = \underline{d_A Y}$$

thanks to equation (3). For the second equation, note that $d_A G = d_A^2 B = \underline{F \wedge B}$. $\square$

As we shall see, the higher Yang–Mills equations have a very similar appearance to the higher Bianchi identities.

5 Higher Yang–Mills Equations

To set up Yang–Mills theory one assumes the spacetime manifold is equipped with a metric, and for convenience usually an orientation as well. Also, one needs the gauge group to have a nondegenerate invariant symmetric bilinear form on its Lie algebra. Here we wish to set up a version of 'higher Yang–Mills theory', so we will need similar extra structure on our manifold M and our Lie 2-group C.

In particular, suppose that M is an oriented manifold equipped with a semi-Riemannian metric. Suppose also that C is a Lie 2-group whose differential crossed module $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ has the following extra structure: $\mathfrak{g}$ is equipped with a nondegenerate symmetric bilinear form that is invariant under the adjoint action of G, and $\mathfrak{h}$ is equipped with a nondegenerate symmetric bilinear form that is invariant under the adjoint action of H and also the action of G coming from $\alpha: G \to \mathrm{Aut}(H)$. If we write both bilinear forms as $\langle \cdot, \cdot \rangle$, these invariance conditions imply that

$$\langle [x, x'], x'' \rangle = -\langle x', [x, x''] \rangle, \tag{6}$$

$$\langle [y, y'], y'' \rangle = -\langle y', [y, y''] \rangle, \tag{7}$$

and

$$\langle [x, y], y' \rangle = -\langle y, [x, y'] \rangle \tag{8}$$

for all $x, x', x'' \in \mathfrak{g}$ and $y, y', y'' \in \mathfrak{h}$. Note that in the last equation we are using the bracket $[x, y]$ of $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$ to stand for $d\alpha(x)(y)$.

Copying the usual formula for the Yang–Mills action, we write down the following action as a function of the connection (A, B):

$$S(A, B) = \int_M \mathrm{tr}(F \wedge *F) + \mathrm{tr}(G \wedge *G). \tag{9}$$

Here, as usual, we define

$$\mathrm{tr}((x \otimes \mu) \wedge (x' \otimes \nu)) = \langle x, y \rangle \, \mu \wedge \nu$$

whenever $x, x' \in \mathfrak{g}$ and μ, ν are differential forms on M; this extends by linearity in each argument to all $\mathfrak{g}$-valued differential forms, and we use the same sort of formula for $\mathfrak{h}$-valued differential forms. Also as usual, we define the Hodge dual of a $\mathfrak{g}$-valued differential form by

$$*(x \otimes \mu) = x \otimes *\mu$$

and similarly for $\mathfrak{h}$-valued differential forms.

Starting with this action we can obtain the equations of motion by taking the variational derivative δS and setting $\delta S = 0$. To prepare for the calculation note that

$$\delta F = d_A \delta A - \underline{\delta B}$$

and

$$\delta G = d_A \delta B + \underline{\delta A \wedge B}.$$

We thus have

$$
\begin{aligned}
\delta S &= 2 \int_M \operatorname{tr}(\delta F \wedge F) + \operatorname{tr}(\delta G \wedge *G) \\
&= 2 \int_M \operatorname{tr}\left((d_A \delta A - \underline{\delta B}) \wedge *F\right) + \operatorname{tr}\left((d_A \delta B + \underline{\delta A \wedge B}) \wedge *G\right) \\
&= 2 \int_M \operatorname{tr}(\delta A \wedge d_A * F - \underline{\delta B} \wedge *F) + \operatorname{tr}(-\delta B \wedge d_A * G + (\underline{\delta A \wedge B}) \wedge *G)
\end{aligned}
$$

where in the last step we do integration by parts with the help of equations (6) and (7). We ignore boundary terms by assuming the variations δA and δB are compactly supported.

To go further with the last term in the above expression, note that there is a unique linear map

$$\sigma \colon \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{g}$$

such that

$$\langle x, \sigma(y, y') \rangle = \langle [x, y], y' \rangle$$

for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$. Given $\mathfrak{h}$-valued differential forms $Y = y \otimes \nu$ and $Y' = y' \otimes \nu'$, we define

$$\underline{Y \wedge Y'} = \sigma(y, y') \otimes (\nu \wedge \nu'),$$

and we extend this to arbitrary $\mathfrak{h}$-valued differential forms by linearity. Using this notation it follows that

$$\delta S = 2 \int_M \operatorname{tr}(\delta A \wedge d_A * F - \underline{\delta B} \wedge *F) + \operatorname{tr}(-\delta B \wedge d_A * G + \delta A \wedge (\underline{B \wedge *G})).$$

Furthermore, note that the map σ is skew-symmetric, since

$$\langle x, \sigma(y', y) \rangle = \langle [x, y'], y \rangle = -\langle y', [x, y] \rangle = \langle x, \sigma(y, y') \rangle$$

for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$, by equation (8). It follows that

$$\underline{B \wedge *G} = -\underline{*G \wedge B},$$

and it will be convenient to use this to rewrite the above formula as

$$\delta S = 2 \int_M \operatorname{tr}(\delta A \wedge d_A * F - \underline{\delta B} \wedge *F) - \operatorname{tr}(\delta B \wedge d_A * G + \delta A \wedge (\underline{*G \wedge B})).$$

Next we rewrite the second term in δS by defining a map

$$dt^* \colon \mathfrak{h} \to \mathfrak{g}$$

such that

$$\langle x, dt^*(y) \rangle = \langle dt(x), y \rangle$$

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for all $x \in \mathfrak{g}$, $y \in \mathfrak{h}$. Given a $\mathfrak{g}$-valued differential form $X = x \otimes \mu$, we define

$$\underline{X} = dt^*(x) \otimes \mu$$

and extend this to arbitrary $\mathfrak{g}$-valued differential forms by linearity. Using this we obtain

$$
\begin{aligned}
\delta S &= 2 \int_M \text{tr}(\delta A \wedge d_A * F - \delta B \wedge *\underline{F}) - \text{tr}(\delta B \wedge d_A * G + \delta A \wedge (\underline{*G \wedge B})) \\[4pt]
&= 2 \int_M \text{tr}\left(\delta A \wedge (d_A * F - \underline{*G \wedge B})\right) - \text{tr}\left(\delta B \wedge (d_A * G + *\underline{F})\right).
\end{aligned}
$$

From this we see:

Theorem 21. *Suppose that M is an oriented semi-Riemannian manifold, and suppose that the Lie 2-group C has an associated Lie crossed module (G, H, t, α) for which $\mathfrak{g}$ and $\mathfrak{h}$ are equipped with nondegenerate invariant symmetric bilinear forms. Then the connection (A, B) on M satisfies $\delta S = 0$ if and only if the* **higher Yang–Mills equations** *hold:*

$$
\begin{aligned}
d_A * F &= \underline{*G \wedge B} \\
d_A * G &= -*\underline{F}
\end{aligned}
\tag{10}
$$

Collecting the higher Bianchi identities and higher Yang–Mills equations together, we have:

$$
\begin{array}{llll}
d_A F &= -\underline{G} \qquad & d_A * G &= -*\underline{F} \\
d_A G &= \underline{F \wedge B} \qquad & d_A * F &= \underline{*G \wedge B}
\end{array}
$$

These equations have a pleasant symmetry. In particular, the Bianchi identities would imply the Yang–Mills equations if we had

$$*F = G, \qquad *G = F.$$

These conditions are reminiscent of the self-dual Yang–Mills equations on a 4-dimensional Riemannian manifold. However, they only make sense under certain conditions. First, since F is a 2-form and G is a 3-form, we must be in 5 dimensions. Second, to obtain $*^2 = 1$ on 2-forms in 5-dimensional space we need the metric to have a signature with an even number of minus signs. Third, since F is $\mathfrak{g}$-valued and G is $\mathfrak{h}$-valued, we need an isomorphism $\mathfrak{g} \cong \mathfrak{h}$ in terms of which all the underline maps become identity maps. One way to achieve this is to let H be a semisimple Lie group and let C be its automorphism 2-group, as described in Example 10. In this case we have:

Theorem 22. *Suppose that M is a 5-dimensional oriented semi-Riemannian manifold of signature $(+ + + + +)$, $(+ + + - -)$ or $(+ - - - -)$. Let H be a semisimple Lie group, let C be the automorphism 2-group of H, and let (G, H, t, α) be the associated Lie crossed module. If we define the higher Yang–Mills action using the Killing forms on $\mathfrak{g}$ and $\mathfrak{h}$, then the connection (A, B) on M is a solution of the higher Yang–Mills equations if $*F = \underline{G}$.*

Proof — Since H is semisimple, the Killing form on $\mathfrak{h}$ is a nondegenerate symmetric bilinear form that is invariant under the adjoint action of H. Moreover, the connected component of $G = \text{Aut}(H)$ consists of inner automorphisms, so it is isomorphic to H modulo its center, which is discrete. This gives an isomorphism $\mathfrak{g} \cong \mathfrak{h}$. Using this isomorphism to identify $\mathfrak{g}$ with $\mathfrak{h}$, one can check that all the underline maps act trivially. The higher Bianchi identities thus become simply

$$
\begin{aligned}
d_A F &= -G \\
d_A G &= F \wedge B
\end{aligned}
$$

and these imply the higher Yang–Mills equations

$$
\begin{aligned}
d_A * G &= -*F \\
d_A * F &= *G \wedge B
\end{aligned}
$$

if we assume $*F = G$, and thus $*G = F$. $\qquad\square$

It would be interesting to study the moduli space of self-dual solutions of the higher Yang–Mills equations in 5 dimensions, and see if they have any implications for topology. Of course this involves the concept of 'gauge transformation' for connections on 2-bundles.

Acknowledgements

References

[1] R. Attal, Combinatorics of non-abelian gerbes with connection and curvature, available as math-ph/0203056.

[2] J. Baez and J. Dolan, Categorification, in *Higher Category Theory*, eds. E. Getzler and M. Kapranov, Contemporary Mathematics vol. 230, AMS, Providence,

[3] L. Breen, Théorie de Schreier supérieure, *Ann. Sci. École Norm. Sup.* **25** (1992), 465–514.

[4] L. Breen, Bitorseurs et cohomologie non-abélienne, in *The Grothendieck Festschrift*, ed. P. Cartier *et al*, Progress in Mathematics vol. **86**, Birkhäuser, Boston, 1990, pp. 401–476.

[5] L. Breen and W. Messing, Differential geometry of gerbes, available as math.AG/0106083.

[6] R. Brown, Groupoids and crossed objects in algebraic topology, *Homology, Homotopy and Applications* **1** (1999), 1–78. Available at `http://www.math.rutgers.edu/hha/volumes/1999/volume1-1.htm`

[7] R. Brown and C. B. Spencer, $\mathcal{G}$-groupoids, crossed modules, and the classifying space of a topological group, *Proc. Kon. Akad. v. Wet.* **79** (1976), 296–302.

[8] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhauser, Boston, 1993.

[9] M. Caicedo, I. Martín and A. Restuccia, Gerbes and duality, available as hep-th/0205002.

[10] A. Carey, S. Johnson and M. Murray, Holonomy on D-branes, available as hep-th-/0204199.

[11] R. Dijkgraaf, The mathematics of fivebranes, *Doc. Math.*, extra volume III (1998), 133–142.

[12] B. Eckmann and P. Hilton, Group-like structures in categories, *Math. Ann.* **145** (1962), 227–255.

[13] M. Forrester-Barker, Group objects and internal categories, available at `http://www.bangor.ac.uk/~map601/catgrp.abs.html`

[14] N. Hitchin, Lectures on special Lagrangian submanifolds, available as math.DG/9907034.

[15] A. Keurentjes, Flat connections from flat gerbes, available as hep-th/0201072.

[16] S. Mac Lane, *Categories for the Working Mathematician*, 2nd edition, Springer, Berlin, 1998.

[17] M. Mackaay and R. Picken, Holonomy and parallel transport for abelian gerbes, available as math.DG/0007053.

[18] C. Teitelboim, Gauge invariance for extended objects, Phys. Lett. **B167** (1986), 63–68.

[19] Finlay Thompson, Introducing quaternionic gerbes, available as math.DG/0009201.