Quantum Field Theory

Part II: Spin One Half

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Part II: Spin One Half

33: Representations of the Lorentz Group

Prerequisite: 2

In section 2, we saw that we could define a unitary operator $U(\Lambda)$ that implemented a Lorentz transformation on a scalar field $\varphi(x)$ via

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \ . \tag{1}$$

As shown in section 2, this implies that the derivative of the field transforms as

$$U(\Lambda)^{-1}\partial^\mu\varphi(x)U(\Lambda) = \Lambda^\mu{}_\rho\bar\partial^\rho\varphi(\Lambda^{-1}x) \ , \tag{2}$$

where the bar on he derivative means that it is with respect to the argument $\bar x = \Lambda^{-1}x$.

Eq. (2) suggests that we could define a *vector field* $A^\mu(x)$ which would transform as

$$U(\Lambda)^{-1}A^\rho(x)U(\Lambda) = \Lambda^\mu{}_\rho A^\rho(\Lambda^{-1}x) \ , \tag{3}$$

or a *tensor field* $B^{\mu\nu}(x)$ which would transform as

$$U(\Lambda)^{-1}B^{\mu\nu}(x)U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma B^{\rho\sigma}(\Lambda^{-1}x) \ . \tag{4}$$

Note that if $B^{\mu\nu}$ is either symmetric, $B^{\mu\nu}(x) = B^{\nu\mu}(x)$, or antisymmetric, $B^{\mu\nu}(x) = -B^{\nu\mu}(x)$, then this symmetry property is preserved by the Lorentz transformation. Also, if we take the trace to get $T(x) \equiv g_{\mu\nu}B^{\mu\nu}(x)$, then, using $g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$, we find that $T(x)$ transforms like a scalar field,

$$U(\Lambda)^{-1}T(x)U(\Lambda) = T(\Lambda^{-1}x) \ . \tag{5}$$

Thus, given a tensor field $B^{\mu\nu}(x)$ with no particular symmetry, we can write

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \tfrac{1}{4}g^{\mu\nu}T(x) \ , \tag{6}$$

where $A^{\mu\nu}$ is antisymmetric ($A^{\mu\nu} = -A^{\nu\mu}$) and $S^{\mu\nu}$ is symmetric ($S^{\mu\nu} = S^{\nu\mu}$) and traceless ($g_{\mu\nu}S^{\mu\nu} = 0$). The key point is that the fields $A^{\mu\nu}$, $S^{\mu\nu}$, and T do not mix with each other under Lorentz transformations.

Is it possible to further break apart these fields into still smaller sets that do not mix under Lorentz transformations? How do we make this decomposition into *irreducible representations* of the Lorentz group for a field carrying n vector indices? Are there any other kinds of indices we could consistently assign to a field? If so, how do these behave under a Lorentz transformation?

The answers to these questions are to be found in the theory of *group representations*. Let us see how this works for the Lorentz group in four spacetime dimensions.

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we can write

$$U(1{+}\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \ , \tag{7}$$

where $M^{\mu\nu} = -M^{\nu\mu}$ is a set of hermitian operators, the *generators of the Lorentz group.* As shown in section 2, these obey the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\Big(g^{\mu\rho}M^{\nu\sigma} - (\mu{\leftrightarrow}\nu)\Big) - (\rho{\leftrightarrow}\sigma) \ . \tag{8}$$

We can identify the components of the angular momentum operator $\vec{J}$ as $J_i \equiv \frac{1}{2}\varepsilon_{ijk}M^{jk}$ and the components of the boost operator $\vec{K}$ as $K_i \equiv M^{i0}$. We then find from eq. (8) that

$$[J_i, J_j] = +i\varepsilon_{ijk}J_k \ , \tag{9}$$

$$[J_i, K_j] = +i\varepsilon_{ijk}K_k \ , \tag{10}$$

$$[K_i, K_j] = -i\varepsilon_{ijk}J_k \ . \tag{11}$$

We would now like to find all the *representations* of eqs. (9–11). A representation is a set of finite-dimensional matrices with the same commutation relations. For example, if we restrict our attention to eq. (9) alone, we know (from standard results in the quantum mechanics of angular momentum) that we can find three $(2j{+}1) \times (2j{+}1)$ hermitian matrices $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ that obey eq. (9), and that the eigenvalues of (say) $\mathcal{J}_3$ are $-j, -j{+}1, \ldots, +j$, where j has the possible values $0, \frac{1}{2}, 1, \ldots$. We further know that these matrices constitute all of the inequivalent, irreducible representations of SO(3), the rotation group in three dimensions. (*Inequivalent* means not related by a unitary transformation; *irreducible* means cannot be made block-diagonal by a unitary transformation.) We would like to extend these conclusions to encompass the full set of eqs. (9–11).

In order to do so, it is helpful to define some nonhermitian operators whose physical significance is obscure, but which simplify the commutation relations. These are

$$N_i \equiv \tfrac{1}{2}(J_i - iK_i) \ , \tag{12}$$

$$N_i^\dagger \equiv \tfrac{1}{2}(J_i + iK_i) \ . \tag{13}$$

In terms of N_i and $N_i^\dagger$, eqs. (9–11) become

$$[N_i, N_j] = i\varepsilon_{ijk} N_k \ , \tag{14}$$

$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger \ , \tag{15}$$

$$[N_i, N_j^\dagger] = 0 \ . \tag{16}$$

We recognize these as the commutation relations of two independent SO(3) groups [or, equivalently, SU(2); see section 32]. Thus the Lorentz group in four dimensions is equivalent to SO(3)×SO(3). And, as just discussed, we are already familiar with the representation theory of SO(3). We therefore conclude that the representations of the Lorentz group in four spacetime dimensions are specified by two numbers n and n', each a nonnegative integer or half-integer.

This turns out to be correct, but there is a complication. To derive the usual representation theory of SO(3), as is done in any text on quantum mechanics, we need to use the fact that the components J_i of the angular momentum operator are hermitian. The components N_i of eq. (13), on the other hand, are not. This means that we have to redo the usual derivation of the representations of SO(3), and see what changes.

As we have already noted, the final result is the naive one, that the representations of the Lorentz group in four dimensions are the same as the representations of SO(3)×SO(3). Those uninterested in the (annoyingly complicated) details can skip ahead all the way ahead to the last four paragraphs of this section.

We begin by noting that $\vec{N}^2$ commutes with N_i; this is easily derived from eq. (14). Similarly, $\vec{N}^{\dagger 2}$ commutes with $N_i^\dagger$. Eq. (16) then implies that $\vec{N}^2$, N_3, $\vec{N}^{\dagger 2}$, and $N_3^\dagger$ are all mutually commuting. Therefore, we can define a set of simultaneous eigenkets $|n,m;n',m'\rangle$, where the eigenvalues of $\vec{N}^2$, N_3, $\vec{N}^{\dagger 2}$, and $N_3^\dagger$ are $f(n)$, m, $f(n')$, and m, respectively. [Later we will see that n and n' must be nonnegative integers or half-integers, and that $f(n) = n(n{+}1)$, as expected.] We also define a set of bra states $\langle n,m;n',m'|$ that, *by definition*, obey

$$\langle n_2,m_2;n_2',m_2'|n_1,m_1;n_1',m_1'\rangle = \delta_{n_{ß2}n_{ß1}}\delta_{m_{ß2}m_{ß1}}\delta_{n'_{ß2}n'_{ß1}}\delta_{m'_{ß2}m'_{ß1}} \equiv \Delta_{21} \tag{17}$$

and

$$\sum |n,m;n',m'\rangle\langle n,m;n',m'| = 1 \; . \tag{18}$$

In eq. (18), the sum is over all allowed values of n, m, n', and m'; our goal is to determine these allowed values.

From the discussion so far, we can conclude that

$$\langle n_2,m_2;n_2',m_2'|\,\vec{N}^2\,|n_1,m_1;n_1',m_1'\rangle = f(n_1)\,\Delta_{21} \; , \tag{19}$$

$$\langle n_2,m_2;n_2',m_2'|\,N_3\,|n_1,m_1;n_1',m_1'\rangle = m_1\,\Delta_{21} \; , \tag{20}$$

$$\langle n_2,m_2;n_2',m_2'|\,\vec{N}^{\dagger 2}\,|n_1,m_1;n_1',m_1'\rangle = f(n_1')\,\Delta_{21} \; , \tag{21}$$

$$\langle n_2,m_2;n_2',m_2'|\,N_3\,|n_1,m_1;n_1',m_1'\rangle = m_1'\,\Delta_{21} \; . \tag{22}$$

Note that we have not yet made any assumptions about the properties of the states under hermitian conjugation. From eqs. (14) and (15), we see that hermitian conjugation exchanges the two SO(3) groups. Therefore, we must have

$$|n,m;n',m'\rangle^\dagger = \langle n',m';n,m| \; , \tag{23}$$

$$\langle n,m;n',m'|^\dagger = |n',m';n,m\rangle \; , \tag{24}$$

up to a possible phase factor that turns out to be irrelevant. Compare the ordering of the labels in eqs. (23) and (24) with those in eqs. (17) and (18); a state $|n,m;n',m'\rangle$ has zero inner product with its own hermitian conjugate if $n \neq n'$ or $m \neq m'$.

Next, take the hermitian conjugates of eqs. (19) and (20), using eqs. (23) and (24). We get

$$\langle n_1', m_1'; n_1, m_1 | \vec{N}^{\dagger 2} | n_2', m_2'; n_2, m_2 \rangle = [f(n_1)]^* \, \Delta_{21} \ , \tag{25}$$

$$\langle n_1', m_1'; n_1, m_1 | N_3^\dagger | n_2', m_2'; n_2, m_2 \rangle = m_1^* \, \Delta_{21} \ , \tag{26}$$

Comparing eqs. (25) and (26) with eqs. (21) and (22), we find that the allowed values of $f(n)$ and m are real.

We now define the raising and lowering operators

$$N_\pm \equiv N_1 \pm i N_2 \ , \tag{27}$$

$$(N^\dagger)_\pm \equiv N_1^\dagger \pm i N_2^\dagger \ ; \tag{28}$$

note that

$$(N_\pm)^\dagger = (N^\dagger)_\mp \ . \tag{29}$$

The commutation relations (14) become

$$[N_3, N_\pm] = \pm N_\pm \ , \tag{30}$$

$$[N_+, N_-] = 2N_3 \ , \tag{31}$$

plus the equivalent with $N \to N^\dagger$. By inserting a complete set of states into eq. (30), and mimicking the usual procedure in quantum mechanics, it is possible to show that

$$\langle n_2, m_2{+}1; n_2', m_2' | N_+ | n_1, m_1; n_1', m_1' \rangle = \lambda_+(n_1, m_1) \, \Delta_{21} \ , \tag{32}$$

$$\langle n_1, m_1; n_1', m_1' | N_- | n_2, m_2{+}1; n_2', m_2' \rangle = \lambda_-(n_1, m_1) \, \Delta_{21} \ , \tag{33}$$

where $\lambda_+(n,m)$ and $\lambda_-(n,m)$ are functions to be determined. By inserting a complete set of states into eq. (31), and using eqs. (32) and (33), we can show that

$$\lambda_+(n, m{-}1)\lambda_-(n, m{-}1) - \lambda_+(n,m)\lambda_-(n,m) = 2m \ . \tag{34}$$

The solution of this recursion relation is

$$\lambda_+(n,m)\lambda_-(n,m) = C(n) - m(m{+}1) \ , \tag{35}$$

where $C(n)$ is an arbitrary function of n.

Next we need the *parity operator* P, introduced in section 23. From the discussion there, we can conclude that

$$P^{-1}M^{\mu\nu}P = \mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}M^{\rho\sigma} \ , \tag{36}$$

where

$$\mathcal{P}^{\mu}{}_{\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \tag{37}$$

Eq. (36) implies

$$P^{-1}J_iP = +J_i \ , \tag{38}$$
$$P^{-1}K_iP = -K_i \ , \tag{39}$$

or, equivalently,

$$P^{-1}N_iP = N_i^{\dagger} \ , \tag{40}$$
$$P^{-1}N_i^{\dagger}P = N_i \ . \tag{41}$$

Since P exchanges N_i and $N_i^{\dagger}$, it must be that

$$P|n,m;n',m'\rangle = |n',m';n,m\rangle \ , \tag{42}$$
$$P^{-1}|n,m;n',m'\rangle = |n',m';n,m\rangle \ , \tag{43}$$

up to a possible phase factor that turns out to be irrelevant. Taking the hermitian conjugate of eqs. (42) and (43), we get

$$\langle n',m';n,m|P^{-1} = \langle n,m;n',m'| \ , \tag{44}$$
$$\langle n',m';n,m|P \quad = \langle n,m;n',m'| \ , \tag{45}$$

where we have used the fact that P is unitarity: $P^{\dagger} = P^{-1}$.

Now we can take eq. (32) and insert PP^{-1} on either side of N_+ to get

$$\begin{aligned} \lambda_+(n_1,m_1)\,\Delta_{21} &= \langle n_2,m_2{+}1;n_2',m_2'|\,PP^{-1}N_+PP^{-1}\,|n_1,m_1;n_1',m_1'\rangle \\ &= \langle n_2',m_2';n_2,m_2{+}1|\,P^{-1}N_+P\,|n_1',m_1';n_1,m_1\rangle \\ &= \langle n_2',m_2';n_2,m_2{+}1|\,(N^{\dagger})_+\,|n_1',m_1';n_1,m_1\rangle \ . \end{aligned} \tag{46}$$

In the second line, we used eqs. (43) and (45). In the third, we used eq. (40). Now taking the hermitian conjugate of eq. (46), and using eqs. (23), (24), and (29), we find

$$\langle n_1, m_1; n_1', m_1'| \, N_- \, |n_2, m_2{+}1; n_2', m_2'\rangle = [\lambda_+(n_1, m_1)]^* \, \Delta_{21} \; . \tag{47}$$

Comparing eq. (47) with eq. (33), we see that

$$\lambda_-(n, m) = [\lambda_+(n, m)]^* \; . \tag{48}$$

This is the final ingredient. Putting eq. (48) into eq. (35), we get

$$|\lambda_+(n, m)|^2 = C(n) - m(m{+}1) \; . \tag{49}$$

From here, everything can be done by mimicking the usual procedure in the quantum mechanics of angular momentum. We see that the left-hand side of eq. (49) is real and nonnegative, while the right-hand side becomes negative for sufficiently large $|m|$. This is not a problem if there are two values of m, differing by an integer, for which $\lambda_+(n, m)$ is zero. From this we can deduce that the allowed values of m are real integers or half-integers, and that if we choose $C(n) = n(n{+}1)$, then n is an integer or half-integer such that the allowed values of m are $-n$, $-n{+}1$, $\ldots$, $+n$. We can also show that $f(n) = C(n) = n(n{+}1)$. Thus the representations of the Lorentz group in four dimensions are just the same as those of SO(3)$\times$SO(3).

We will label these representations as $(2n{+}1, 2n'{+}1)$; the number of components of a representation is then $(2n{+}1)(2n'{+}1)$. Different components within a representation can also be labeled by their angular momentum representations. To do this, we first note that, from eqs. (12) and (13), we have $\vec{J} = \vec{N} + \vec{N}^\dagger$. Thus, deducing the allowed values of j given n and n' becomes a standard problem in the addition of angular momenta. The general result is that the allowed values of j are $|n{-}n'|, |n{-}n'|{+}1, \ldots, n{+}n'$, and each of these values appears exactly once.

The four simplest and most often encountered representations are $(1, 1)$, $(2, 1)$, $(1, 2)$, and $(2, 2)$. These are given special names:

$$(1, 1) = \textit{Scalar} \text{ or } \textit{singlet}$$

$$
\begin{aligned}
(2,1) &= \textit{Left-handed spinor} \\
(1,2) &= \textit{Right-handed spinor} \\
(2,2) &= \textit{Vector}
\end{aligned}
\tag{50}
$$

It may seem a little surprising that $(2,2)$ is to be identified as the vector representation. To see that this must be the case, we first note that the vector representation is irreducible: all the components of a four-vector mix with each other under a general Lorentz transformation. Secondly, the vector representation has four components. Therefore, the only candidate irreducible representations are $(4,1)$, $(1,4)$, and $(2,2)$. The first two of these contain angular momenta $j=\frac{3}{2}$ only, whereas $(2,2)$ contains $j=0$ and $j=1$. This is just right for a four-vector, whose time component is a scalar under spatial rotations, and whose space components are a three-vector.

In order to gain a better understanding of what it means for $(2,2)$ to be the vector representation, we must first investigate the spinor representations $(1,2)$ and $(2,1)$, which contain angular momenta $j=\frac{1}{2}$ only.

Problems

33.1) Express $A^{\mu\nu}(x)$, $S^{\mu\nu}(x)$, and $T(x)$ in terms of $B^{\mu\nu}(x)$.

33.2) Verify that eqs. (14–16) follow from eqs. (9–11).

34: Left- and Right-Handed Spinor Fields

Prerequisite: 33

Consider a *left-handed spinor field* $\psi_a(x)$, also known as a *left-handed Weyl field*, which is in the $(2,1)$ representation of the Lorentz group. Here the index a is a *left-handed spinor index* that takes on two possible values. Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x)\ , \tag{51}$$

where $L_a{}^b(\Lambda)$ is a matrix in the $(2,1)$ representation. These matrices satisfy the group composition rule

$$L_a{}^b(\Lambda_1)L_b{}^c(\Lambda_2) = L_a{}^c(\Lambda_1\Lambda_2)\ . \tag{52}$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we can write

$$L_a{}^b(1{+}\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^b\ , \tag{53}$$

where $(S^{\mu\nu}_{\rm L})_a{}^b = -(S^{\nu\mu}_{\rm L})_a{}^b$ is a set of 2×2 matrices that obey the same commutation relations as the generators $M^{\mu\nu}$, namely

$$[S^{\mu\nu}_{\rm L}, S^{\rho\sigma}_{\rm L}] = i\Big(g^{\mu\rho}S^{\nu\sigma}_{\rm L} - (\mu{\leftrightarrow}\nu)\Big) - (\rho{\leftrightarrow}\sigma)\ . \tag{54}$$

Eq. (51) becomes

$$[\psi_a(x), M^{\mu\nu}] = -i(x^\mu\partial^\nu{-}x^\nu\partial^\mu)\psi_a(x) + (S^{\mu\nu}_{\rm L})_a{}^b\psi_b(x)\ , \tag{55}$$

where $U(1{+}\delta\omega) = I + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$. The first term on the right-hand side of eq. (55) would also be present for a scalar field, and is not the focus of our current interest; we will suppress it by evaluating the fields at the spacetime origin, $x^\mu = 0$. Recalling that $M^{ij} = \varepsilon^{ijk} J_k$, where J_k is the angular momentum operator, we have

$$\varepsilon^{ijk}[\psi_a(0), J_k] = (S_{\rm L}^{ij})_a{}^b \psi_b(0) \; . \tag{56}$$

Recall that the $(2,1)$ representation of the Lorentz group includes angular momentum $j = \frac{1}{2}$ only. For a spin-one-half operator, the standard convention is that the matrix on the right-hand side of eq. (56) is $\frac{1}{2}\varepsilon^{ijk}\sigma_k$, where σ_k is a Pauli matrix:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{57}$$

We therefore conclude that

$$(S_{\rm L}^{ij})_a{}^b = \tfrac{1}{2}\varepsilon^{ijk}\sigma_k \; . \tag{58}$$

Thus, for example, setting i=1 and j=2 yields $(S_{\rm L}^{12})_a{}^b = \frac{1}{2}\varepsilon^{12k}\sigma_k = \frac{1}{2}\sigma_3$, where the subscript a is the row index (and the superscript b is the column index) of the matrix $\frac{1}{2}\sigma_3$. Therefore, $(S_{\rm L}^{12})_1{}^1 = +\frac{1}{2}$, $(S_{\rm L}^{12})_2{}^2 = -\frac{1}{2}$, and $(S_{\rm L}^{12})_1{}^2 = (S_{\rm L}^{12})_2{}^1 = 0$.

Once we have the $(2,1)$ representation matrices for the angular momentum operator J_i, we can easily get them for the boost operator $K_k = M^{k0}$. This is because $J_k = N_k + N_k^\dagger$ and $K_k = i(N_k - N_k^\dagger)$, and, in the $(2,1)$ representation, $N_k^\dagger$ is zero. Therefore, the representation matrices for K_k are simply i times those for J_k, and so

$$(S_{\rm L}^{k0})_a{}^b = \tfrac{1}{2} i\sigma_k \; . \tag{59}$$

Now consider taking the hermitian conjugate of the left-handed spinor field $\psi_a(x)$. Recall that hermitian conjugation swaps the two SO(3) factors in the Lorentz group. Therefore, the hermitian conjugate of a field in the $(2,1)$ representation should be a field in the $(1,2)$ representation; such a field is called a *right-handed spinor field*, or a *right-handed Weyl field*. We will distinguish the indices of the $(1,2)$ representation from those of the $(2,1)$ representation by putting dots over them. Thus, we write

$$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot a}(x) \; . \tag{60}$$

Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}\psi^\dagger_{\dot a}(x)U(\Lambda) = R_{\dot a}{}^{\dot b}(\Lambda)\psi^\dagger_{\dot b}(\Lambda^{-1}x) \ , \tag{61}$$

where $R_{\dot a}{}^{\dot b}(\Lambda)$ is a matrix in the $(1,2)$ representation. These matrices satisfy the group composition rule

$$R_{\dot a}{}^{\dot b}(\Lambda_1)R_{\dot b}{}^{\dot c}(\Lambda_2) = R_{\dot a}{}^{\dot c}(\Lambda_1\Lambda_2) \ . \tag{62}$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we can write

$$R_{\dot a}{}^{\dot b}(1{+}\delta\omega) = \delta_{\dot a}{}^{\dot b} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b} \ , \tag{63}$$

where $(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b} = -(S^{\nu\mu}_{\rm R})_{\dot a}{}^{\dot b}$ is a set of 2×2 matrices that obey the same commutation relations as the generators $M^{\mu\nu}$. We then have

$$[\psi^\dagger_{\dot a}(0), M^{\mu\nu}] = (S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b}\psi^\dagger_{\dot b}(0) \ . \tag{64}$$

Taking the hermitian conjugate of this equation, we get

$$[M^{\mu\nu}, \psi_a(0)] = [(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b}]^*\psi_b(0) \ . \tag{65}$$

Comparing this with eq. (55), we see that

$$(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b} = -[(S^{\mu\nu}_{\rm L})_a{}^b]^* \ . \tag{66}$$

In the previous section, we examined the Lorentz-transformation properties of a field carrying two vector indices. To help us get better acquainted with the properties of spinor indices, let us now do the same for a field that carries two $(2,1)$ indices. Call this field $C_{ab}(x)$. Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}C_{ab}(x)U(\Lambda) = L_a{}^c(\Lambda)L_b{}^d(\Lambda)C_{cd}(\Lambda^{-1}x) \ . \tag{67}$$

The question we wish to address is whether or not the four components of C_{ab} can be grouped into smaller sets that do not mix with each other under Lorentz transformations.

To answer this question, recall from quantum mechanics that two spin-one-half particles can be in a state of total spin zero, or total spin one.

Furthermore, the single spin-zero state is the unique *antisymmetric* combination of the two spin-one-half states, and the three spin-one states are the three *symmetric* combinations of the two spin-one-half states. We can write this schematically as $2 \otimes 2 = 1_\mathrm{A} \oplus 3_\mathrm{S}$, where we label the representation of SO(3) by the number of its components, and the subscripts S and A indicate whether that representation appears in the symmetric or antisymmetric combination of the two 2's. For the Lorentz group, the relevant relation is $(2,1) \otimes (2,1) = (1,1)_\mathrm{A} \oplus (3,1)_\mathrm{S}$. This implies that we should be able to write

$$C_{ab}(x) = \varepsilon_{ab} D(x) + G_{ab}(x) \ , \tag{68}$$

where $D(x)$ is a scalar field, $\varepsilon_{ab} = -\varepsilon_{ba}$ is an antisymmetric set of constants, and $G_{ab}(x) = G_{ba}(x)$. The symbol ε_{ab} is uniquely determined by its symmetry properties up to an overall constant; we will choose $\varepsilon_{21} = -\varepsilon_{12} = +1$.

Given that $D(x)$ is a Lorentz scalar, eq. (68) is consistent with eq. (67) only if

$$L_a{}^c(\Lambda) L_b{}^d(\Lambda) \varepsilon_{cd} = \varepsilon_{ab} \ . \tag{69}$$

This means that ε_{ab} is an *invariant symbol* of the Lorentz group: it does not change under a Lorentz transformation that acts on all of its indices. In this way, ε_{ab} is analogous to the metric $g_{\mu\nu}$, which is also an invariant symbol, since

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu} \ . \tag{70}$$

We use $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ to raise and lower vector indices, and we can use ε_{ab} and and its inverse ε^{ab} to raise and lower left-handed spinor indices. Here we define ε^{ab} via

$$\varepsilon^{12} = \varepsilon_{21} = +1 \ , \qquad \varepsilon^{21} = \varepsilon_{12} = -1 \ . \tag{71}$$

With this definition, we have

$$\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c \ , \qquad \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c \ . \tag{72}$$

We can then define

$$\psi^a(x) \equiv \varepsilon^{ab}\psi_b(x) \ . \tag{73}$$

We also have (suppressing the spacetime argument of the field)

$$\psi_a = \varepsilon_{ab}\psi^b = \varepsilon_{ab}\varepsilon^{bc}\psi_c = \delta_a{}^c\psi_c \ , \tag{74}$$

as we would expect. However, the antisymmetry of ε^{ab} means that we must be careful with minus signs; for example, eq. (73) can be written in various ways, such as

$$\psi^a = \varepsilon^{ab}\psi_b = -\varepsilon^{ba}\psi_b = -\psi_b\varepsilon^{ba} = \psi_b\varepsilon^{ab} \ . \tag{75}$$

We must also be careful about signs when we contract indices, since

$$\psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a = -\varepsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b \ . \tag{76}$$

In section 35, we will (mercifully) develop an index-free notation that automatically keeps track of these essential (but annoying) minus signs.

An exactly analogous discussion applies to the second SO(3) factor; from the group-theoretic relation $(1,2)\otimes(1,2)=(1,1)_{\rm A}\oplus(1,3)_{\rm S}$, we can deduce the existence of an invariant symbol $\varepsilon_{\dot a\dot b} = -\varepsilon_{\dot b\dot a}$. We will normalize $\varepsilon^{\dot a\dot b}$ according to eq. (71). Then eqs. (72–76) hold if *all* the undotted indices are replaced by dotted indices.

Now consider a field carrying one undotted and one dotted index, $A_{a\dot a}(x)$. Such a field is in the $(2,2)$ representation, and in section 33 we concluded that the $(2,2)$ representation was the vector representation. We would more naturally write a field in the vector representation as $A^\mu(x)$. There must, then, be a dictionary that gives us the components of $A_{a\dot a}(x)$ in terms of the components of $A^\mu(x)$; we can write this as

$$A_{a\dot a}(x) = \sigma^\mu_{a\dot a}A_\mu(x) \ , \tag{77}$$

where $\sigma^\mu_{a\dot a}$ is another invariant symbol. That such a symbol must exist can be deduced from the group-theoretic relation

$$(2,1)\otimes(1,2)\otimes(2,2) = (1,1)\oplus\ldots \ . \tag{78}$$

As we will see in section 35, it turns out to be consistent with our already established conventions for $S_{\rm L}^{\mu\nu}$ and $S_{\rm R}^{\mu\nu}$ to choose

$$\sigma^\mu_{a\dot a} = (I,\vec\sigma) \ . \tag{79}$$

Thus, for example, $\sigma^3_{1\dot 1}=+1$, $\sigma^3_{2\dot 2}=-1$, $\sigma^3_{1\dot 2}=\sigma^3_{2\dot 1}=0$.

In general, whenever the product of a set of representations includes the singlet, there is a corresponding invariant symbol. For example, we can deduce the existence of $g_{\mu\nu} = g_{\nu\mu}$ from

$$(2,2) \otimes (2,2) = (1,1)_{\rm S} \oplus (1,3)_{\rm A} \oplus (3,1)_{\rm A} \oplus (3,3)_{\rm S} \; . \tag{80}$$

Another invariant symbol, the *Levi-Civita symbol*, follows from

$$(2,2) \otimes (2,2) \otimes (2,2) \otimes (2,2) = (1,1)_{\rm A} \oplus \ldots \; , \tag{81}$$

where the subscript A denotes the completely antisymmetric part. The Levi-Civita symbol is $\varepsilon^{\mu\nu\rho\sigma}$, which is antisymmetric on exchange of any pair of its indices, and is normalized via $\varepsilon^{0123} = +1$. To see that $\varepsilon^{\mu\nu\rho\sigma}$ is invariant, we note that $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \varepsilon^{\alpha\beta\gamma\delta}$ is antisymmetric on exchange of any two of its uncontracted indices, and therefore must be proportional to $\varepsilon^{\mu\nu\rho\sigma}$. The constant of proportionality works out to be $\det \Lambda$, which is $+1$ for a proper Lorentz transformation.

We are finally ready to answer a question we posed at the beginning of section 33. There we considered a field $B^{\mu\nu}(x)$ carrying two vector indices, and we decomposed it as

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \tfrac{1}{4} g^{\mu\nu} T(x) \; , \tag{82}$$

where $A^{\mu\nu}$ is antisymmetric ($A^{\mu\nu} = -A^{\nu\mu}$) and $S^{\mu\nu}$ is symmetric ($S^{\mu\nu} = S^{\nu\mu}$) and traceless ($g_{\mu\nu} S^{\mu\nu} = 0$). We asked whether further decomposition into still smaller irreducible representations was possible. The answer to this question can be found in eq. (80). Obviously, $T(x)$ corresponds to $(1,1)$, and $S^{\mu\nu}(x)$ to $(3,3)$. [Note that a symmetric traceless tensor has three independent diagonal components, and six independent off-diagonal components, for a total of nine, the number of components of the $(3,3)$ representation.] But, according to eq. (80), the antisymmetric field $A^{\mu\nu}(x)$ should correspond to $(3,1) \oplus (1,3)$. A field in the $(3,1)$ representation carries a symmetric pair of left-handed (undotted) spinor indices; its hermitian conjugate is a field in the $(1,3)$ representation that carries a symmetric pair of right-handed (dotted) spinor indices. We should, then, be able to find a mapping, analogous to eq. (77), that gives $A^{\mu\nu}(x)$ in terms of a field $G_{ab}(x)$ and its hermitian conjugate $G^\dagger_{\dot a \dot b}(x)$.

This mapping is provided by the generator matrices $S_{\rm L}^{\mu\nu}$ and $S_{\rm R}^{\mu\nu}$. We first note that the Pauli matrices are traceless, and so eqs. (58) and (59) imply that $(S_{\rm L}^{\mu\nu})_a{}^a = 0$. Using eq. (73), we can rewrite this as $\varepsilon^{ab}(S_{\rm L}^{\mu\nu})_{ab} = 0$. Since ε^{ab} is antisymmetric, $(S_{\rm L}^{\mu\nu})_{ab}$ must be symmetric on exchange of its two spinor indices. An identical argument shows that $(S_{\rm R}^{\mu\nu})_{\dot a\dot b}$ must be symmetric on exchange of its two spinor indices. Furthermore, according to eqs. (58) and (59), we have

$$(S_{\rm L}^{10})_a{}^b = -i(S_{\rm L}^{23})_a{}^b \; . \tag{83}$$

This can be written covariantly with the Levi-Civita symbol as

$$(S_{\rm L}^{\mu\nu})_a{}^b = -\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(S_{{\rm L}\,\rho\sigma})_a{}^b \; . \tag{84}$$

Similarly,

$$(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b} = +\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(S_{{\rm R}\,\rho\sigma})_{\dot a}{}^{\dot b} \; . \tag{85}$$

Eq. (85) follows from taking the complex conjugate of eq. (84) and using eq. (66).

Now, given a field $G_{ab}(x)$ in the $(3,1)$ representation, we can map it into a *self-dual antisymmetric tensor* $G^{\mu\nu}(x)$ via

$$G^{\mu\nu}(x) \equiv (S_{\rm L}^{\mu\nu})^{ab}G_{ab}(x) \; . \tag{86}$$

By *self-dual*, we mean that $G^{\mu\nu}(x)$ obeys

$$G^{\mu\nu}(x) = -\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}(x) \; . \tag{87}$$

Taking the hermitian conjugate of eq. (86), and using eq. (66), we get

$$G^{\dagger\mu\nu}(x) = -(S_{\rm R}^{\mu\nu})^{\dot a\dot b}G^\dagger_{\dot a\dot b}(x) \; , \tag{88}$$

which is *anti-self-dual*,

$$G^{\dagger\mu\nu}(x) = +\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}G^\dagger_{\rho\sigma}(x) \; . \tag{89}$$

Given a hermitian antisymmetric tensor field $A^{\mu\nu}(x)$, we can extract its self-dual and anti-self-dual parts via

$$G^{\mu\nu}(x) = \tfrac{1}{2}A^{\mu\nu}(x) - \tfrac{i}{4}\varepsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}(x) \; , \tag{90}$$

$$G^{\dagger\mu\nu}(x) = \tfrac{1}{2}A^{\mu\nu}(x) + \tfrac{i}{4}\varepsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}(x) \; . \tag{91}$$

Then we have

$$A^{\mu\nu}(x) = G^{\mu\nu}(x) + G^{\dagger\mu\nu}(x) \; . \tag{92}$$

The field $G^{\mu\nu}(x)$ is in the $(3,1)$ representation, and the field $G^{\dagger\mu\nu}(x)$ is in the $(1,3)$ representation; these do not mix under Lorentz transformations.

Problems

34.1) Verify that eq. (55) follows from eq. (51).

34.2) Verify that eqs. (58) and (59) obey eq. (54).

34.3) Consider a field $C^{a\ldots c\,\dot a\ldots\dot c}(x)$, with N undotted indices and M dotted indices, that is furthermore symmetric on exchange of any pair of undotted indices, and also symmetric on exchange of any pair of dotted indices. Show that this field corresponds to a single irreducible representation $(2n{+}1, 2n'{+}1)$ of the Lorentz group, and identify n and n'.

35: Manipulating Spinor Indices

Prerequisite: 34

In section 34 we introduced the invariant symbols ε_{ab}, ε^{ab}, $\varepsilon_{\dot{a}\dot{b}}$, and $\varepsilon^{\dot{a}\dot{b}}$, where

$$\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = +1 \;, \qquad \varepsilon^{21} = \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1 \;. \tag{93}$$

We use the ε symbols to raise and lower spinor indices, contracting the second index on the ε. (If we contract the first index instead, then there is an extra minus sign).

Another invariant symbol is

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma}) \;, \tag{94}$$

where I is the 2×2 identity matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{95}$$

are the Pauli matrices.

Now let's consider some combinations of invariant symbols with some indices contracted, such as $g_{\mu\nu}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}}$. This object must also be invariant. Then, since it carries two undotted and two dotted spinor indices, it must be proportional to $\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}$. Using eqs. (93) and (94), we can laboriously check this; it turns out to be correct. [If it wasn't, then eq. (94) would not be a tenable choice of numerical values for this symbol.] The proportionality constant works out to be minus two:

$$\sigma^{\mu}_{a\dot{a}}\sigma_{\mu b\dot{b}} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}} \;. \tag{96}$$

Similarly, $\varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^\mu_{a\dot a}\sigma^\nu_{b\dot b}$ must be proportional to $g^{\mu\nu}$, and the proportionality constant is again minus two:

$$\varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^\mu_{a\dot a}\sigma^\nu_{b\dot b} = -2g^{\mu\nu} \; . \tag{97}$$

Next, let's see what we can learn about the generator matrices $(S^{\mu\nu}_{\rm L})_a{}^b$ and $(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b}$ from the fact that ε_{ab}, $\varepsilon_{\dot a\dot b}$, and $\sigma^\mu_{a\dot a}$ are all invariant symbols. Begin with

$$\varepsilon_{ab} = L(\Lambda)_a{}^c L(\Lambda)_b{}^d \varepsilon_{cd} \; , \tag{98}$$

which expresses the Lorentz invariance of ε_{ab}. For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we have

$$L_a{}^b(1{+}\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^b \; , \tag{99}$$

and eq. (98) becomes

$$\begin{aligned} \varepsilon_{ab} &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[\,(S^{\mu\nu}_{\rm L})_a{}^c\varepsilon_{cb} + (S^{\mu\nu}_{\rm L})_b{}^d\varepsilon_{ad}\Big] + O(\delta\omega^2) \\ &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[-(S^{\mu\nu}_{\rm L})_{ab} + (S^{\mu\nu}_{\rm L})_{ba}\Big] + O(\delta\omega^2) \; . \end{aligned} \tag{100}$$

Since eq. (100) holds for any choice of $\delta\omega_{\mu\nu}$, it must be that the factor in square brackets vanishes. Thus we conclude that $(S^{\mu\nu}_{\rm L})_{ab} = (S^{\mu\nu}_{\rm L})_{ba}$, which we had already deduced in section 34 by a different method. Similarly, starting from the Lorentz invariance of $\varepsilon_{\dot a\dot b}$, we can show that $(S^{\mu\nu}_{\rm R})_{\dot a\dot b} = (S^{\mu\nu}_{\rm R})_{\dot b\dot a}$.

Next, start from

$$\sigma^\rho_{a\dot a} = \Lambda^\rho{}_\tau L(\Lambda)_a{}^b R(\Lambda)_{\dot a}{}^{\dot b}\sigma^\tau_{b\dot b} \; , \tag{101}$$

which expresses the Lorentz invariance of $\sigma^\rho_{a\dot a}$. For an infinitesimal transformation, we have

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau + \tfrac{1}{2}\delta\omega_{\mu\nu}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau) \; , \tag{102}$$

$$L_a{}^b(1{+}\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^b \; , \tag{103}$$

$$R_{\dot a}{}^{\dot b}(1{+}\delta\omega) = \delta_{\dot a}{}^{\dot b} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b} \; . \tag{104}$$

Substituting eqs. (102–104) into eq. (101) and isolating the coefficient of $\delta\omega_{\mu\nu}$ yields

$$(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)\sigma^\tau_{a\dot a} + i(S^{\mu\nu}_{\rm L})_a{}^b\sigma^\rho_{b\dot a} + i(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b}\sigma^\rho_{a\dot b} = 0 \; . \tag{105}$$

Now multiply by $\sigma_{\rho c\dot{c}}$ to get

$$\sigma^{\mu}_{c\dot{c}}\sigma^{\nu}_{a\dot{a}} - \sigma^{\nu}_{c\dot{c}}\sigma^{\mu}_{a\dot{a}} + i(S^{\mu\nu}_{\mathrm{L}})_a{}^b\sigma^{\rho}_{b\dot{a}}\sigma_{\rho c\dot{c}} + i(S^{\mu\nu}_{\mathrm{R}})_{\dot{a}}{}^{\dot{b}}\sigma^{\rho}_{a\dot{b}}\sigma_{\rho c\dot{c}} = 0 \ . \tag{106}$$

Next use eq. (96) in each of the last two terms to get

$$\sigma^{\mu}_{c\dot{c}}\sigma^{\nu}_{a\dot{a}} - \sigma^{\nu}_{c\dot{c}}\sigma^{\mu}_{a\dot{a}} + 2i(S^{\mu\nu}_{\mathrm{L}})_{ac}\varepsilon_{\dot{a}\dot{c}} + 2i(S^{\mu\nu}_{\mathrm{R}})_{\dot{a}\dot{c}}\varepsilon_{ac} = 0 \ . \tag{107}$$

If we multiply eq. (107) by $\varepsilon^{\dot{a}\dot{c}}$, and remember that $\varepsilon^{\dot{a}\dot{c}}(S^{\mu\nu}_{\mathrm{R}})_{\dot{a}\dot{c}} = 0$ and that $\varepsilon^{\dot{a}\dot{c}}\varepsilon_{\dot{a}\dot{c}} = -2$, we get a formula for $(S^{\mu\nu}_{\mathrm{L}})_{ac}$, namely

$$(S^{\mu\nu}_{\mathrm{L}})_{ac} = \tfrac{i}{4}\varepsilon^{\dot{a}\dot{c}}(\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{c\dot{c}} - \sigma^{\nu}_{a\dot{a}}\sigma^{\mu}_{c\dot{c}}) \ . \tag{108}$$

Similarly, if we multiply eq. (107) by ε^{ac}, we get

$$(S^{\mu\nu}_{\mathrm{R}})_{\dot{a}\dot{c}} = \tfrac{i}{4}\varepsilon^{ac}(\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{c\dot{c}} - \sigma^{\nu}_{a\dot{a}}\sigma^{\mu}_{c\dot{c}}) \ . \tag{109}$$

These formulae can be made to look a little nicer if we define

$$\bar{\sigma}^{\mu\dot{a}a} \equiv \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{b\dot{b}} \ . \tag{110}$$

Numerically, it turns out that

$$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma}) \ . \tag{111}$$

Using $\bar{\sigma}^{\mu}$, we can write eqs. (108) and (109) as

$$(S^{\mu\nu}_{\mathrm{L}})_a{}^b = +\tfrac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_a{}^b \ , \tag{112}$$

$$(S^{\mu\nu}_{\mathrm{R}})^{\dot{a}}{}_{\dot{b}} = -\tfrac{i}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{a}}{}_{\dot{b}} \ . \tag{113}$$

In eq. (113), we have suppressed a contracted pair of undotted indices arranged as ${}^c{}_c$, and in eq. (112), we have suppressed a contracted pair of dotted indices arranged as ${}_{\dot{c}}{}^{\dot{c}}$.

We will adopt this as a general convention: a missing pair of contracted, undotted indices is understood to be written as ${}^c{}_c$, and a missing pair of contracted, dotted indices is understood to be written as ${}_{\dot{c}}{}^{\dot{c}}$. Thus, if χ and ψ are two left-handed Weyl fields, we have

$$\chi\psi = \chi^a\psi_a \quad \text{and} \quad \chi^{\dagger}\psi^{\dagger} = \chi^{\dagger}_{\dot{a}}\psi^{\dagger\dot{a}} \ . \tag{114}$$

We expect Weyl fields to describe spin-one-half particles, and (by the spin-statistics theorem) these particles must be *fermions*. Therefore the correspoding fields must *anticommute*, rather than commute. That is, we should have

$$\chi_a(x)\psi_b(y) = -\psi_b(y)\chi_a(x)\ . \tag{115}$$

Thus we can rewrite eq. (114) as

$$\chi\psi = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \psi\chi\ . \tag{116}$$

The second equality follows from anticommutation of the fields, and the third from switching ${}_a{}^a$ to ${}^a{}_a$ (which introduces an extra minus sign). Eq. (116) tells us that $\chi\psi = \psi\chi$, which is a nice feature of this notation. Furthermore, if we take the hermitian conjugate of $\chi\psi$, we get

$$(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = (\psi_a)^\dagger(\chi^a)^\dagger = \psi^\dagger_{\dot a}\chi^{\dagger\dot a} = \psi^\dagger\chi^\dagger\ . \tag{117}$$

That $(\chi\psi)^\dagger = \psi^\dagger\chi^\dagger$ is just what we would expect if we ignored the indices completely. Of course, by analogy with eq. (116), we also have $\psi^\dagger\chi^\dagger = \chi^\dagger\psi^\dagger$.

In order to tell whether a spinor field is left-handed or right-handed when its spinor index is suppressed, we will adopt the convention that a right-handed field is always written as the hermitian conjugate of a left-handed field. Thus, a right-handed field is always written with a dagger, and a left-handed field is always written without a dagger.

Let's try computing the hermitian conjugate of something a little more complicated:

$$\psi^\dagger\bar\sigma^\mu\chi = \psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a c}\,\chi_c\ . \tag{118}$$

This behaves like a vector field under Lorentz transformations,

$$U(\Lambda)^{-1}[\psi^\dagger\bar\sigma^\mu\chi]U(\Lambda) = \Lambda^\mu{}_\nu[\psi^\dagger\bar\sigma^\nu\chi]\ . \tag{119}$$

(To avoid clutter, we suppressed the spacetime argument of the fields; as usual, it is x on the left-hand side and $\Lambda^{-1}x$ on the right.)

The hermitian conjugate of eq. (118) is

$$[\psi^\dagger\bar\sigma^\mu\chi]^\dagger = [\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a c}\,\chi_c]^\dagger$$

$$
\begin{aligned}
&= \chi^{\dagger}_{\dot c}\,(\bar\sigma^{\mu a\dot c})^*\,\psi_a \\
&= \chi^{\dagger}_{\dot c}\,\bar\sigma^{\mu\dot c a}\,\psi_a \\
&= \chi^{\dagger}\bar\sigma^{\mu}\psi\ . \qquad (120)
\end{aligned}
$$

In the third line, we used the hermiticity of the matrices $\bar\sigma^\mu = (I, -\vec\sigma)$.

We will get considerably more practice with this notation in the following sections.

Problems

35.1) Verify that eq. (111) follows from eqs. (94) and (110).

35.2) Verify that eq. (112) is consistent with eqs. (58) and (59).

35.3) Verify that eq. (113) is consistent with eq. (66).

35.4) Verify eq. (97).

Hint for all problems: write everything in "matrix multiplication" order, and note that, numerically, $\varepsilon^{ab} = -\varepsilon_{ab} = i\sigma_2$. Then make use of the properties of the Pauli matrices.

36: Lagrangians for Spinor Fields

Prerequisite: 4, 22, 35

Suppose we have a left-handed spinor field ψ_a. We would like to find a suitable lagrangian for it. This lagrangian must be Lorentz invariant, and it must be hermitian. We would also like it to be quadratic in ψ and its hermitian conjugate $\psi^\dagger_{\dot a}$, because this will lead to a linear equation of motion, with plane-wave solutions. We want plane-wave solutions because these describe free particles, the starting point for a theory of interacting particles.

Let us begin with terms with no derivatives. The only possibility is $\psi\psi = \psi^a\psi_a = \varepsilon^{ab}\psi_b\psi_a$, plus its hermitian conjugate. Alas, $\psi\psi$ appears to be zero, since $\psi_b\psi_a = \psi_a\psi_b$, while $\varepsilon^{ab} = -\varepsilon^{ba}$.

However, from the spin-statistics theorem, we expect that spin-one-half particles must be *fermions*, described by fields that *anticommute*. Therefore, we should have $\psi_b\psi_a = -\psi_a\psi_b$ rather than $\psi_b\psi_a = +\psi_a\psi_b$. Then $\psi\psi$ does not vanish, and we can use it as a term in Ł.

Next we need a term with derivatives. The obvious choice is $\partial^\mu\psi\partial_\mu\psi$, plus its hermitian conjugate. This, however, yields a hamiltonian that is unbounded below, which is unacceptable. To get a bounded hamiltonian, the kinetic term must involve both ψ and $\psi^\dagger$. A candidate is $i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$. This not hermitian, but

$$
\begin{aligned}
(i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi)^\dagger &= (i\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c)^\dagger \\
&= -i\partial_\mu\psi^\dagger_{\dot c}\,(\bar\sigma^{\mu a\dot c})^*\psi_a \\
&= -i\partial_\mu\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a \\
&= i\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\partial_\mu\psi_a - i\partial_\mu(\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a). \\
&= i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - i\partial_\mu(\psi^\dagger\bar\sigma^\mu\psi)\ . \qquad (121)
\end{aligned}
$$

In the third line, we used the hermiticity of the matrices $\bar\sigma^\mu = (I, -\vec\sigma)$. In the fourth line, we used $A\partial B = -(\partial A)B + \partial(AB)$. In the last line, the second term is a total divergence, and vanishes (with suitable boundary conditions on the fields at infinity) when we integrate it over d^4x to get the action S. Thus $i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$ has the hermiticity properties necessary for a term in $\mathcal{L}$.

Our complete lagrangian for ψ is then

$$\mathcal{L} = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}m\psi\psi - \tfrac{1}{2}m^*\psi^\dagger\psi^\dagger \ , \tag{122}$$

where m is a complex parameter with dimensions of mass. The phase of m is actually irrelevant: if $m = |m|e^{i\alpha}$, we can set $\psi = e^{-i\alpha/2}\,\tilde\psi$ in eq. (122); then we get a lagrangian for $\tilde\psi$ that is identical to eq. (122), but with m replaced by $|m|$. So we can, without loss of generality, take m to be real and positive in the first place, and that is what we will do, setting $m^* = m$ in eq. (122).

The equation of motion for ψ is then

$$0 = -\frac{\delta S}{\delta\psi^\dagger} = -i\bar\sigma^\mu\partial_\mu\psi + m\psi^\dagger \ , \tag{123}$$

Restoring the spinor indices, this reads

$$0 = -i\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c + m\psi^{\dagger\dot a} \ . \tag{124}$$

Taking the hermitian conjugate (or, equivalently, computing $-\delta S/\delta\psi$), we get

$$\begin{aligned} 0 &= +i(\bar\sigma^{\mu a\dot c})^*\,\partial_\mu\psi^\dagger_{\dot c} + m\psi^a \\ &= +i\bar\sigma^{\mu\dot c a}\partial_\mu\psi^\dagger_{\dot c} + m\psi^a \\ &= -i\sigma^\mu_{a\dot c}\,\partial_\mu\psi^{\dagger\dot c} + m\psi_a \ . \end{aligned} \tag{125}$$

In the second line, we used the hermiticity of the matrices $\bar\sigma^\mu = (I, -\vec\sigma)$. In the third, we lowered the undotted index, and switched ${}^{\dot c}{}_{\dot c}$ to ${}_{\dot c}{}^{\dot c}$, which gives an extra minus sign.

Eqs. (125) and (124) can be combined to read

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot c}\,\partial_\mu \\ -i\bar\sigma^{\mu\dot a c}\,\partial_\mu & m\delta^{\dot a}{}_{\dot c} \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot c} \end{pmatrix} = 0 \ . \tag{126}$$

We can write this more compactly by introducing the 4×4 *gamma matrices*

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix} . \tag{127}$$

Using the sigma-matrix relations,

$$\begin{aligned} (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_a{}^c &= -2g^{\mu\nu} \delta_a{}^c \,, \\ (\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu)^{\dot{a}}{}_{\dot{c}} &= -2g^{\mu\nu} \delta^{\dot{a}}{}_{\dot{c}} \,, \end{aligned} \tag{128}$$

which are most easily derived from the numerical formulae $\sigma^\mu_{a\dot{a}} = (I, \vec{\sigma})$ and $\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$, we see that the gamma matrices obey

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \,, \tag{129}$$

where $\{A, B\} \equiv AB + BA$ denotes the anticommutator, and there is an understood 4×4 identity matrix on the right-hand side. We also introduce a four-component *Majorana field*

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix} . \tag{130}$$

Then eq. (126) becomes

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0 \,. \tag{131}$$

This is the *Dirac equation.* We first encountered it in section 1, where the gamma matrices were given different names ($\beta = \gamma^0$ and $\alpha^k = \gamma^0 \gamma^k$). Also, in section 1 we were trying (and failing) to interpret Ψ as a wave function, rather than as a quantum field.

Now consider a theory of two left-handed spinor fields with an SO(2) symmetry,

$$Ł = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2} m \psi_i \psi_i - \tfrac{1}{2} m \psi_i^\dagger \psi_i^\dagger \,, \tag{132}$$

where the spinor indices are suppressed and $i = 1, 2$ is implicitly summed. As in the analogous case of two scalar fields discussed in sections 22 and 23, this lagrangian is invariant under the SO(2) transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{133}$$

We can write the lagrangian so that the SO(2) symmetry appears as a U(1) symmetry instead; let

$$\chi = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \ , \tag{134}$$

$$\xi = \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \ . \tag{135}$$

In terms of these fields, we have

$$Ł = i\chi^\dagger\bar\sigma^\mu\partial_\mu\chi + i\xi^\dagger\bar\sigma^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger \ . \tag{136}$$

Eq. (136) is invariant under the U(1) version of eq. (133),

$$\begin{aligned} \chi &\to e^{-i\alpha}\chi \ , \\ \xi &\to e^{+i\alpha}\xi \ , \\ \chi^\dagger &\to e^{+i\alpha}\chi^\dagger \ , \\ \xi^\dagger &\to e^{-i\alpha}\xi^\dagger \ . \end{aligned} \tag{137}$$

Next, let us derive the equations of motion that we get from eq. (136), following the same procedure that ultimately led to eq. (126). The result is

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot c}\,\partial_\mu \\ -i\bar\sigma^{\mu\dot a c}\,\partial_\mu & m\delta^{\dot a}{}_{\dot c} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot c} \end{pmatrix} = 0 \ . \tag{138}$$

We can now define a four-component *Dirac field*

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot c} \end{pmatrix} \ , \tag{139}$$

which obeys the *Dirac equation*, eq. (131). (We have annoyingly used the same symbol Ψ to denote both a Majorana field and a Dirac field; these are different objects, and so we must always announce which is meant when we write Ψ.)

We can also write the lagrangian, eq. (136), in terms of the Dirac field Ψ, eq. (139). First we take the hermitian conjugate of Ψ to get

$$\Psi^\dagger = (\chi^\dagger_{\dot a}\,,\ \xi^a) \ . \tag{140}$$

Introduce the matrix

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot a}{}_{\dot c} \\ \delta_a{}^c & 0 \end{pmatrix} \ . \tag{141}$$

Numerically, $\beta = \gamma^0$. However, the spinor index structure of β and γ^0 is different, and so we will distinguish them. Given β, we define

$$\overline{\Psi} \equiv \Psi^\dagger \beta = (\xi^a,\, \chi^\dagger_{\dot a}) \ . \tag{142}$$

Then we have

$$\overline{\Psi}\Psi = \xi^a \chi_a + \chi^\dagger_{\dot a} \xi^{\dagger \dot a} \ . \tag{143}$$

Also,

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi = \xi^a \sigma^\mu_{a\dot c}\, \partial_\mu \xi^{\dagger \dot c} + \chi^\dagger_{\dot a}\, \bar\sigma^{\mu \dot a c}\, \partial_\mu \chi_c \ . \tag{144}$$

Using $A\partial B = -(\partial A)B + \partial(AB)$, the first term on the right-hand side of eq. (144) can be rewritten as

$$\xi^a \sigma^\mu_{a\dot c}\, \partial_\mu \xi^{\dagger \dot c} = -(\partial_\mu \xi^a)\sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c} + \partial_\mu(\xi^a \sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c}) \ . \tag{145}$$

The first term on the right-hand side of eq. (145) can be rewritten as

$$-\,(\partial_\mu \xi^a)\sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c} = +\xi^{\dagger \dot c} \sigma^\mu_{a\dot c}\, \partial_\mu \xi^a = +\xi^\dagger_{\dot c}\, \bar\sigma^{\mu \dot c a} \partial_\mu \xi_a \ . \tag{146}$$

Here we used anticommutation of the fields to get the first equality, and switched ${}^{\dot c}{}_{\dot c}$ to ${}_{\dot c}{}^{\dot c}$ and ${}_a{}^a$ to ${}^a{}_a$ (thus generating two minus signs) to get the second. Combining eqs. (144–146), we get

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi = \chi^\dagger \bar\sigma^\mu \partial_\mu \chi + \xi^\dagger \bar\sigma^\mu \partial_\mu \xi + \partial_\mu(\xi \sigma^\mu \xi^\dagger) \ . \tag{147}$$

Therefore, up to an irrelevant total divergence, we have

$$Ł = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi \ . \tag{148}$$

This form of the lagrangian is invariant under the U(1) transformation

$$\begin{aligned} \Psi &\to e^{-i\alpha}\, \Psi \ , \\ \overline{\Psi} &\to e^{+i\alpha}\, \overline{\Psi} \ , \end{aligned} \tag{149}$$

which, given eq. (139), is the same as eq. (137). The Noether current associated with this symmetry is

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar\sigma^\mu\chi - \xi^\dagger\bar\sigma^\mu\xi \; . \tag{150}$$

In quantum electrodynamics, the electromagnetic current is $e\overline{\Psi}\gamma^\mu\Psi$, where e is the charge of the electron.

As in the case of a complex scalar field with a U(1) symmetry, there is an additional discrete symmetry, called *charge conjugation*, that enlarges SO(2) to O(2). Charge conjugation simply exchanges χ and ξ. We can define a unitary *charge conjugation operator* C that implements this,

$$\begin{aligned} C^{-1}\chi_a(x)C &= \xi_a(x) \; , \\ C^{-1}\xi_a(x)C &= \chi_a(x) \; , \end{aligned} \tag{151}$$

where, for the sake of precision, we have restored the spinor index and spacetime argument. We then have $C^{-1}\text{Ł}(x)C = \text{Ł}(x)$.

To express eq. (151) in terms of the Dirac field, eq. (139), we first introduce the *charge conjugation matrix*

$$\mathcal{C} \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot a\dot c} \end{pmatrix}. \tag{152}$$

Next we notice that, if we take the transpose of $\overline{\Psi}$, eq. (142), we get

$$\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi^a \\ \chi^\dagger_{\dot a} \end{pmatrix} \; . \tag{153}$$

Then, if we multiply by $\mathcal{C}$, we get a field that we will call Ψ^{C}, the *charge conjugate* of Ψ,

$$\Psi^{\mathrm{C}} \equiv \mathcal{C}\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot a} \end{pmatrix} \; . \tag{154}$$

We see that Ψ^{C} is the same as the original field Ψ, eq. (139), except that the roles of χ and ξ have been switched.

The charge conjugation matrix has a number of useful properties. As a numerical matrix, it obeys

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C} \; , \tag{155}$$

and we can also write it as

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix} . \tag{156}$$

A result that we will need later is

$$\begin{aligned} \mathcal{C}^{-1}\gamma^\mu\mathcal{C} &= \begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu_{b\dot{c}} \\ \bar{\sigma}^{\mu\dot{b}c} & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{ce} & 0 \\ 0 & \varepsilon^{\dot{c}\dot{e}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varepsilon^{ab}\sigma^\mu_{b\dot{c}}\varepsilon^{\dot{c}\dot{e}} \\ \varepsilon_{\dot{a}\dot{b}}\bar{\sigma}^{\mu\dot{b}c}\varepsilon_{ce} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\bar{\sigma}^{\mu a\dot{e}} \\ -\sigma^\mu_{\dot{a}e} & 0 \end{pmatrix} . \end{aligned} \tag{157}$$

The minus signs in the last line come from raising or lowering an index by contracting with the first (rather than the second) index of an ε symbol. Comparing with

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{e\dot{a}} \\ \bar{\sigma}^{\mu\dot{e}a} & 0 \end{pmatrix} , \tag{158}$$

we see that

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}} . \tag{159}$$

Now let us return to the Majorana field, eq. (130). It is obvious that a Majorana field is its own charge conjugate, that is, $\Psi^{\mathrm{C}} = \Psi$. This condition is analogous to the condition $\varphi^\dagger = \varphi$ that is satisfied by a real scalar field. A Dirac field, with its U(1) symmetry, is analogous to a complex scalar field, while a Majorana field, which has no U(1) symmetry, is analogous to a real scalar field.

We can write the lagrangian for a single left-handed spinor field, eq. (122), in terms of a Majorana field, eq. (130), by retracing eqs. (140–148) with $\chi \to \psi$ and $\xi \to \psi$. The result is

$$\mathcal{L} = \tfrac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\overline{\Psi}\Psi . \tag{160}$$

However, this expression is not yet useful for deriving the equations of motion, because it does not yet incorporate the Majorana condition $\Psi^{\mathrm{C}} = \Psi$. To remedy this, we use eq. (155) to write the Majorana condition $\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$ as $\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$. Then we can replace $\overline{\Psi}$ in eq. (160) by $\Psi^{\mathrm{T}}\mathcal{C}$ to get

$$\text{Ł} = \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^{\mu}\partial_{\mu}\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi \; . \tag{161}$$

The equation of motion that follows from this lagrangian is once again the Dirac equation.

We can also recover the Weyl components of a Dirac or Majorana field by means of a suitable projection matrix. Define

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot a}{}_{\dot c} \end{pmatrix} , \tag{162}$$

where the subscript 5 is simply part of the traditional name of this matrix, rather than the value of some index. Then we can define left and right projection matrices

$$\begin{aligned} P_{\mathrm{L}} &\equiv \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix} , \\ P_{\mathrm{R}} &\equiv \tfrac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot a}{}_{\dot c} \end{pmatrix} . \end{aligned} \tag{163}$$

Thus we have, for a Dirac field,

$$\begin{aligned} P_{\mathrm{L}}\Psi &= \begin{pmatrix} \chi_c \\ 0 \end{pmatrix} , \\ P_{\mathrm{R}}\Psi &= \begin{pmatrix} 0 \\ \xi^{\dagger\dot c} \end{pmatrix} . \end{aligned} \tag{164}$$

The matrix γ_5 can also be expressed as

$$\begin{aligned} \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} \; , \end{aligned} \tag{165}$$

where $\varepsilon_{0123} = -1$.

Finally, let us consider the behavior of a Dirac or Majorana field under a Lorentz transformation. Recall that left- and right-handed spinor fields transform according to

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L(\Lambda)_a{}^c\,\psi_c(\Lambda^{-1}x)\ , \tag{166}$$

$$U(\Lambda)^{-1}\psi^\dagger_{\dot a}(x)U(\Lambda) = R(\Lambda)_{\dot a}{}^{\dot c}\,\psi^\dagger_{\dot c}(\Lambda^{-1}x)\ , \tag{167}$$

where, for an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$,

$$L(1{+}\delta\omega)_a{}^c = \delta_a{}^c + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^c\ , \tag{168}$$

$$R(1{+}\delta\omega)_{\dot a}{}^{\dot c} = \delta_{\dot a}{}^{\dot c} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot c}\ , \tag{169}$$

and where

$$(S^{\mu\nu}_{\rm L})_a{}^c = +\tfrac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_a{}^c\ , \tag{170}$$

$$(S^{\mu\nu}_{\rm R})^{\dot a}{}_{\dot c} = -\tfrac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot a}{}_{\dot c}\ . \tag{171}$$

From these formulae, and the definition of γ^μ, eq. (127), we can see that

$$\tfrac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} +(S^{\mu\nu}_{\rm L})_a{}^c & 0 \\ 0 & -(S^{\mu\nu}_{\rm R})^{\dot a}{}_{\dot c} \end{pmatrix} \equiv S^{\mu\nu}\ . \tag{172}$$

Then, for either a Dirac or Majorana field Ψ, we can write

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)\ , \tag{173}$$

where, for an infinitesimal transformation, the 4×4 matrix $D(\Lambda)$ is

$$D(1{+}\delta\omega) = 1 + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu}\ , \tag{174}$$

with $S^{\mu\nu}$ given by eq. (172). The minus sign in front of $S^{\mu\nu}_{\rm R}$ in eq. (172) is compensated by the switch from a ${}^{\dot c}{}_{\dot c}$ contraction in eq. (169) to a ${}_{\dot c}{}^{\dot c}$ contraction in eq. (173).

Problems

36.1) Using the results of problem 2.8, show that, for a rotation by an angle θ about the z axis, we have

$$D(\Lambda) = \exp(-i\theta S^{12}) \ , \tag{175}$$

and that, for a boost by rapidity η in the z direction, we have

$$D(\Lambda) = \exp(+i\eta S^{30}) \ . \tag{176}$$

36.2) Show that $\overline{D(\Lambda)}\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$.

37: Canonical Quantization of Spinor Fields I

Prerequisite: 36

Consider a left-handed Weyl field ψ with lagrangian

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger) \ . \tag{177}$$

The canonically conjugate momentum to the field $\psi_a(x)$ is then

$$\begin{aligned}\pi^a(x) &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_a(x))} \\ &= i\psi^\dagger_{\dot a}(x)\bar{\sigma}^{0\dot a a} \ .\end{aligned} \tag{178}$$

[Here we have glossed over a subtlety about differentiating with respect to an anticommuting object; we will take up this topic in section 44, and here simply assume that eq. (178) is correct.] The hamiltonian is

$$\begin{aligned}\mathcal{H} &= \pi^a\partial_0\psi_a - \mathcal{L} \\ &= i\psi^\dagger_{\dot a}\bar{\sigma}^{0\dot a a}\psi_a - \mathcal{L} \\ &= -i\psi^\dagger\bar{\sigma}^i\partial_i\psi + \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger) \ .\end{aligned} \tag{179}$$

The appropriate canonical *anticommutation* relations are

$$\begin{aligned}\{\psi_a(\mathbf{x},t), \psi_c(\mathbf{y},t)\} &= 0 \ , \\ \{\psi_a(\mathbf{x},t), \pi^c(\mathbf{y},t)\} &= i\delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y}) \ .\end{aligned} \tag{180}$$

Substituting in eq. (178) for π^c, we get

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot c}(\mathbf{y},t)\}\bar{\sigma}^{0\dot c c} = \delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{181}$$

Then, using $\bar{\sigma}^0 = \sigma^0 = I$, we have

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot c}(\mathbf{y},t)\} = \sigma^0_{a\dot c}\,\delta^3(\mathbf{x}-\mathbf{y}) \;, \tag{182}$$

or, equivalently,

$$\{\psi^a(\mathbf{x},t), \psi^{\dagger\dot c}(\mathbf{y},t)\} = \bar{\sigma}^{0\dot c a}\,\delta^3(\mathbf{x}-\mathbf{y}) \;. \tag{183}$$

We can also translate this into four-component notation for either a Dirac or a Majorana field. A Dirac field is defined in terms of two left-handed Weyl fields χ and ξ via

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot c} \end{pmatrix} \;. \tag{184}$$

We also define

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\, \chi^\dagger_{\dot a}) \;, \tag{185}$$

where

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot a}{}_{\dot c} \\ \delta_a{}^c & 0 \end{pmatrix} \;. \tag{186}$$

The lagrangian is

$$\begin{aligned} Ł &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m(\chi\xi + \xi^\dagger\chi^\dagger) \\ &= i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi \;. \end{aligned} \tag{187}$$

The fields χ and ξ each obey the canonical anticommutation relations of eq. (180). This translates into

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0 \;, \tag{188}$$

$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \;, \tag{189}$$

where α and β are four-component spinor indices, and

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot c} \\ \bar{\sigma}^{\mu\dot a c} & 0 \end{pmatrix} \;. \tag{190}$$

Eqs. (188) and (189) can also be derived directly from the four-component form of the lagrangian, eq. (187), by noting that the canonically conjugate momentum to the field Ψ is $\partial Ł/\partial(\partial_0\Psi) = i\overline{\Psi}\gamma^0$, and that $(\gamma^0)^2 = 1$.

A Majorana field is defined in terms of a single left-handed Weyl field ψ via

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix} . \tag{191}$$

We also define

$$\overline{\Psi} \equiv \Psi^\dagger \beta = (\psi^a,\, \psi^\dagger_{\dot{a}}) . \tag{192}$$

A Majorana field obeys the Majorana condition

$$\overline{\Psi} = \Psi^{\mathrm{T}} \mathcal{C} , \tag{193}$$

where

$$\mathcal{C} \equiv \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix} \tag{194}$$

is the charge conjugation matrix. The lagrangian is

$$\begin{aligned} Ł &= i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \tfrac{1}{2} m(\psi\psi + \psi^\dagger\psi^\dagger) \\ &= \tfrac{i}{2} \overline{\Psi} \gamma^\mu \partial_\mu \Psi - \tfrac{1}{2} m \overline{\Psi}\Psi \\ &= \tfrac{i}{2} \Psi^{\mathrm{T}} \mathcal{C} \gamma^\mu \partial_\mu \Psi - \tfrac{1}{2} m \Psi^{\mathrm{T}} \mathcal{C} \Psi . \end{aligned} \tag{195}$$

The field ψ obeys the canonical anticommutation relations of eq. (180). This translates into

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x}-\mathbf{y}) , \tag{196}$$

$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x}-\mathbf{y}) , \tag{197}$$

where α and β are four-component spinor indices. To derive eqs. (196) and (197) directly from the four-component form of the lagrangian, eq. (195), requires new formalism for the quantization of *constrained systems*. This is because the canonically conjugate momentum to the field Ψ is $\partial Ł/\partial(\partial_0\Psi) = \frac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^0$, and this is linearly related to Ψ itself; this relation constitutes a constraint that must be solved before imposition of the anticommutation relations. In this case, solving the constraint simply returns us to the Weyl formalism with which we began.

The equation of motion that follows from either eq. (187) or eq. (195) is the Dirac equation,

$$(-i\partial\!\!\!/ + m)\Psi = 0 \; . \tag{198}$$

Here we have introduced the *Feynman slash*: given any four-vector a^μ, we define

$$a\!\!\!/ \equiv a_\mu \gamma^\mu \; . \tag{199}$$

To solve the Dirac equation, we first note that if we act on it with $i\partial\!\!\!/ + m$, we get

$$\begin{aligned} 0 &= (i\partial\!\!\!/ + m)(-i\partial\!\!\!/ + m)\Psi \\ &= (\partial\!\!\!/\partial\!\!\!/ + m^2)\Psi \\ &= (-\partial^2 + m^2)\Psi \; . \end{aligned} \tag{200}$$

Here we have used

$$\begin{aligned} a\!\!\!/ a\!\!\!/ &= a_\mu a_\nu \gamma^\mu \gamma^\nu \\ &= a_\mu a_\nu \Big(\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\Big) \\ &= a_\mu a_\nu \Big(-g^{\mu\nu} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\Big) \\ &= -a_\mu a_\nu g^{\mu\nu} + 0 \\ &= -a^2 \; . \end{aligned} \tag{201}$$

From eq. (200), we see that Ψ obeys the Klein-Gordon equation. Therefore, the Dirac equation has plane-wave solutions. Let us consider a specific solution of the form

$$\Psi(x) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx} \; . \tag{202}$$

where $p^0 = \omega \equiv (\mathbf{p}^2 + m^2)^{1/2}$, and $u(\mathbf{p})$ and $v(\mathbf{p})$ are four-component constant spinors. Plugging eq. (202) into the eq. (198), we get

$$(p\!\!\!/ + m)u(\mathbf{p})e^{ipx} + (-p\!\!\!/ + m)v(\mathbf{p})e^{-ipx} = 0 \; . \tag{203}$$

Thus we require

$$\begin{aligned} (p\!\!\!/ + m)u(\mathbf{p}) &= 0 \; , \\ (-p\!\!\!/ + m)v(\mathbf{p}) &= 0 \; . \end{aligned} \tag{204}$$

Each of these equations has two linearly independent solutions that we will call $u_\pm(\mathbf{p})$ and $v_\pm(\mathbf{p})$; their detailed properties will be worked out in the next section. The general solution of the Dirac equation can then be written as

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right] , \tag{205}$$

where the integration measure is

$$\widetilde{dp} \equiv \frac{d^3p}{(2\pi)^3 2\omega} . \tag{206}$$

38: Spinor Technology

Prerequisite: 37

The four-component spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ obey the equations

$$\begin{aligned}(\not{p}+m)u_s(\mathbf{p}) &= 0\ ,\\ (-\not{p}+m)v_s(\mathbf{p}) &= 0\ .\end{aligned} \tag{207}$$

Each of these equations has two solutions, which we label via $s = +$ and $s = -$. For $m \neq 0$, we can go to the rest frame, $\mathbf{p} = \mathbf{0}$. We will then distinguish the two solutions by the eigenvalue of the spin matrix

$$S_z = \tfrac{i}{4}[\gamma^1,\gamma^2] = \tfrac{i}{2}\gamma^1\gamma^2 = \begin{pmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & \frac{1}{2}\sigma_3 \end{pmatrix} . \tag{208}$$

Specifically, we will require

$$\begin{aligned}S_z u_\pm(\mathbf{0}) &= \pm\tfrac{1}{2}u_\pm(\mathbf{0})\ ,\\ S_z v_\pm(\mathbf{0}) &= \mp\tfrac{1}{2}v_\pm(\mathbf{0})\ .\end{aligned} \tag{209}$$

The reason for the opposite sign for the v spinor is that this choice results in

$$\begin{aligned}[J_z, b^\dagger_\pm(\mathbf{0})] &= \pm\tfrac{1}{2}b^\dagger_\pm(\mathbf{0})\ ,\\ [J_z, d^\dagger_\pm(\mathbf{0})] &= \pm\tfrac{1}{2}d^\dagger_\pm(\mathbf{0})\ ,\end{aligned} \tag{210}$$

so that $b^\dagger_+(\mathbf{0})$ and $d^\dagger_+(\mathbf{0})$ each creates a particle with spin up along the z axis.

For $\mathbf{p} = \mathbf{0}$, we have $\not{p} = -m\gamma^0$, where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} . \tag{211}$$

Eqs. (207) and (209) are then easy to solve. Choosing (for later convenience) a specific normalization and phase for each of $u_\pm(\mathbf{0})$ and $v_\pm(\mathbf{0})$, we get

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}. \tag{212}$$

For later use we also compute the barred spinors

$$\begin{aligned}\overline{u}_s(\mathbf{p}) &\equiv u_s^\dagger(\mathbf{p})\beta \ , \\ \overline{v}_s(\mathbf{p}) &\equiv v_s^\dagger(\mathbf{p})\beta \ ,\end{aligned} \tag{213}$$

where

$$\beta = \gamma^0 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix} \tag{214}$$

satisfies

$$\beta^{\mathrm{T}} = \beta^\dagger = \beta^{-1} = \beta \ . \tag{215}$$

We get

$$\begin{aligned}\overline{u}_+(\mathbf{0}) &= \sqrt{m}\,(1,\,0,\,1,\,0) \ , \\ \overline{u}_-(\mathbf{0}) &= \sqrt{m}\,(0,\,1,\,0,\,1) \ , \\ \overline{v}_+(\mathbf{0}) &= \sqrt{m}\,(0,\,-1,\,0,\,1) \ , \\ \overline{v}_-(\mathbf{0}) &= \sqrt{m}\,(1,\,0,\,-1,\,0) \ .\end{aligned} \tag{216}$$

We can now find the spinors corresponding to an arbitrary three-momentum $\mathbf{p}$ by applying to $u_s(\mathbf{0})$ and $v_s(\mathbf{0})$ the matrix $D(\Lambda)$ that corresponds to an appropriate boost. This is given by

$$D(\Lambda) = \exp(i\eta\,\hat{\mathbf{p}}\!\cdot\!\mathbf{K}) \ , \tag{217}$$

where $\hat{\mathbf{p}}$ is a unit vector in the $\mathbf{p}$ direction, $K^j = \frac{i}{4}[\gamma^j, \gamma^0] = \frac{i}{2}\gamma^j\gamma^0$ is the boost matrix, and $\eta \equiv \sinh^{-1}(|\mathbf{p}|/m)$ is the *rapidity* (see problem 2.8). Thus we have

$$\begin{aligned} u_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0}) \;, \\ v_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0}) \;. \end{aligned} \tag{218}$$

We also have

$$\begin{aligned} \overline{u}_s(\mathbf{p}) &= \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K}) \;, \\ \overline{v}_s(\mathbf{p}) &= \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K}) \;. \end{aligned} \tag{219}$$

This follows from $\overline{K^j} = K^j$, where for any general combination of gamma matrices,

$$\overline{A} \equiv \beta A^\dagger \beta \;. \tag{220}$$

In particular, it turns out that

$$\begin{aligned} \overline{\gamma^\mu} &= \gamma^\mu \;, \\ \overline{S^{\mu\nu}} &= S^{\mu\nu} \;, \\ \overline{i\gamma_5} &= i\gamma_5 \;, \\ \overline{\gamma^\mu\gamma_5} &= \gamma^\mu\gamma_5 \;, \\ \overline{i\gamma_5 S^{\mu\nu}} &= i\gamma_5 S^{\mu\nu} \;. \end{aligned} \tag{221}$$

The barred spinors satisfy the equations

$$\begin{aligned} \overline{u}_s(\mathbf{p})(\not{p} + m) &= 0 \;, \\ \overline{v}_s(\mathbf{p})(-\not{p} + m) &= 0 \;. \end{aligned} \tag{222}$$

It is not very hard to work out $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ from eq. (218), but it is even easier to use various tricks that will sidestep any need for the explicit formulae. Consider, for example, $\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p})$; from eqs. (218) and (219), we see that $\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = \overline{u}_{s'}(\mathbf{0})u_s(\mathbf{0})$, and this is easy to compute from eqs. (212) and (216). We find

$$\begin{aligned} \overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= +2m\,\delta_{s's} \;, \\ \overline{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= -2m\,\delta_{s's} \;, \\ \overline{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= 0 \;, \\ \overline{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= 0 \;. \end{aligned} \tag{223}$$

Also useful are the *Gordon identities*,

$$\begin{aligned} 2m\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) &= \overline{u}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]u_s(\mathbf{p}) \;, \\ -2m\,\overline{v}_{s'}(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) &= \overline{v}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]v_s(\mathbf{p}) \;. \end{aligned} \tag{224}$$

To derive them, start with

$$\begin{aligned} \gamma^\mu \not{p} &= \tfrac{1}{2}\{\gamma^\mu, \not{p}\} + \tfrac{1}{2}[\gamma^\mu, \not{p}] \\ &= -p^\mu - 2iS^{\mu\nu}p_\nu \;. \end{aligned} \tag{225}$$

Similarly,

$$\begin{aligned} \not{p}'\gamma^\mu &= \tfrac{1}{2}\{\gamma^\mu, \not{p}'\} - \tfrac{1}{2}[\gamma^\mu, \not{p}'] \\ &= -p'^\mu + 2iS^{\mu\nu}p'_\nu \;. \end{aligned} \tag{226}$$

Add eqs. (225) and (226), sandwich them between $\overline{u}'$ and u spinors (or $\overline{v}'$ and v spinors), and use eqs. (207) and (222). An important special case is $p' = p$; then, using eq. (223), we find

$$\begin{aligned} \overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) &= 2p^\mu \delta_{s's} \;, \\ \overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) &= 2p^\mu \delta_{s's} \;. \end{aligned} \tag{227}$$

With a little more effort, we can also show

$$\begin{aligned} \overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) &= 0 \;, \\ \overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) &= 0 \;. \end{aligned} \tag{228}$$

We will need eqs. (227) and (228) in the next section.

Consider now the spin sums $\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p})$ and $\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p})$, each of which is a 4×4 matrix. The sum over eigenstates of S_z should remove any memory of the spin-quantization axis, and so the result should be expressible in terms of the four-vector p^μ and various gamma matrices, with all vector indices contracted. In the rest frame, $\not{p} = -m\gamma^0$, and it is easy to check that $\sum_{s=\pm} u_s(\mathbf{0})\overline{u}_s(\mathbf{0}) = m\gamma^0 + m$ and $\sum_{s=\pm} v_s(\mathbf{0})\overline{v}_s(\mathbf{0}) = m\gamma^0 - m$. We therefore conclude that

$$\begin{aligned} \sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= -\not{p} + m \;, \\ \sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= -\not{p} - m \;. \end{aligned} \tag{229}$$

We will make extensive use of eq. (229) when we calculate scattering cross sections for spin-one-half particles.

From eq. (229), we can get $u_+(\mathbf{p})\overline{u}_+(\mathbf{p})$, etc, by applying appropriate spin projection matrices. In the rest frame, we have

$$\begin{aligned} \tfrac{1}{2}(1+2sS_z)u_{s'}(\mathbf{0}) &= \delta_{ss'}\,u_{s'}(\mathbf{0})\;, \\ \tfrac{1}{2}(1-2sS_z)v_{s'}(\mathbf{0}) &= \delta_{ss'}\,v_{s'}(\mathbf{0})\;. \end{aligned} \tag{230}$$

In order to boost these projection matrices to a more general frame, we first recall that

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{231}$$

This allows us to write $S_z = \frac{i}{2}\gamma^1\gamma^2$ as $S_z = -\frac{1}{2}\gamma_5\gamma^3\gamma^0$. In the rest frame, we can write γ^0 as $-\not{p}/m$, and γ^3 as $\not{z}$, where $z^\mu = (0,\hat{\mathbf{z}})$; thus we have

$$S_z = \tfrac{1}{2m}\gamma_5\not{z}\not{p}\;. \tag{232}$$

Now we can boost S_z to any other frame simply by replacing $\not{z}$ and $\not{p}$ with their values in that frame. (Note that, in any frame, z^μ satisfies $z^2 = 1$ and $z\cdot p = 0$.) Boosting eq. (230) then yields

$$\begin{aligned} \tfrac{1}{2}(1-s\gamma_5\not{z})u_{s'}(\mathbf{p}) &= \delta_{ss'}\,u_{s'}(\mathbf{p})\;, \\ \tfrac{1}{2}(1-s\gamma_5\not{z})v_{s'}(\mathbf{p}) &= \delta_{ss'}\,v_{s'}(\mathbf{p})\;, \end{aligned} \tag{233}$$

where we have used eq. (207) to eliminate $\not{p}$. Combining eqs. (229) and (233) we get

$$\begin{aligned} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= \tfrac{1}{2}(1-s\gamma_5\not{z})(-\not{p}+m)\;, \\ v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= \tfrac{1}{2}(1-s\gamma_5\not{z})(-\not{p}-m)\;. \end{aligned} \tag{234}$$

It is interesting to consider the extreme relativistic limit of this formula. Let us take the three-momentum to be in the z direction, so that it is parallel to the spin-quantization axis. The component of the spin in the direction of the three-momentum is called the *helicity*. A fermion with helicity $+1/2$ is said to be *right-handed*, and a fermion with helicity $-1/2$ is said to be *left-handed*. For rapidity η, we have

$$\begin{aligned} \tfrac{1}{m}p^\mu &= (\cosh\eta, 0, 0, \sinh\eta)\;, \\ z^\mu &= (\sinh\eta, 0, 0, \cosh\eta)\;. \end{aligned} \tag{235}$$

The first equation is simply the definition of η, and the second follows from $z^2 = 1$ and $p{\cdot}z = 0$ (along with the knowledge that a boost of a four-vector in the z direction does not change its x and y components). In the limit of large η, we see that

$$z^\mu = \tfrac{1}{m}p^\mu + O(e^{-\eta}) \;. \tag{236}$$

Hence, in eq. (234), we can replace $\not{z}$ with $\not{p}/m$, and then use $(\not{p}/m)(-\not{p}\pm m) = \mp(-\not{p}\pm m)$, which holds for $p^2 = -m^2$. For consistency, we should then also drop the m relative to $\not{p}$, since it is down by a factor of $O(e^{-\eta})$. We get

$$\begin{aligned} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &\to \tfrac{1}{2}(1+s\gamma_5)(-\not{p}) \;, \\ v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &\to \tfrac{1}{2}(1-s\gamma_5)(-\not{p}) \;. \end{aligned} \tag{237}$$

The spinor corresponding to a right-handed fermion (helicity $+1/2$) is $u_+(\mathbf{p})$ for a b-type particle and $v_-(\mathbf{p})$ for a d-type particle. According to eq. (237), either of these is projected by $\frac{1}{2}(1+\gamma_5) = \mathrm{diag}(0,0,1,1)$ onto the lower two components only. In terms of the Dirac field $\Psi(x)$, this is the part that corresponds to the right-handed Weyl field. Similarly, left-handed fermions are projected (in the extreme relativistic limit) onto the upper two spinor components only, corresponding to the left-handed Weyl field.

The case of a massless particle follows from the extreme relativistic limit of a massive particle. In particular, eqs. (207), (222), (223), (227), (228), and (229) are all valid with $m = 0$, and eq. (237) becomes exact.

Finally, for our discussion of parity, time reversal, and charge conjugation in section 40, we will need a number of relationships among the u and v spinors. First, note that $\beta u_s(\mathbf{0}) = +u_s(\mathbf{0})$ and $\beta v_s(\mathbf{0}) = -v_s(\mathbf{0})$. Also, $\beta K^j = -K^j\beta$. We then have

$$\begin{aligned} u_s(-\mathbf{p}) &= +\beta u_s(\mathbf{p}) \;, \\ v_s(-\mathbf{p}) &= -\beta v_s(\mathbf{p}) \;. \end{aligned} \tag{238}$$

Next, we need the charge conjugation matrix

$$\mathcal{C} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \tag{239}$$

which obeys

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C} \;, \tag{240}$$

$$\beta\mathcal{C} = -\mathcal{C}\beta \;, \tag{241}$$

$$\mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = -(\gamma^{\mu})^{\mathrm{T}} \;. \tag{242}$$

Using eqs. (212), (216), and (239), we get $\mathcal{C}\overline{u}_s(\mathbf{0})^{\mathrm{T}} = v_s(\mathbf{0})$ and $\mathcal{C}\overline{v}_s(\mathbf{0})^{\mathrm{T}} = u_s(\mathbf{0})$. Also, eq. (242) implies $\mathcal{C}^{-1}K^j\mathcal{C} = -(K^j)^{\mathrm{T}}$. From this we can conclude that

$$\begin{aligned} \mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} &= v_s(\mathbf{p}) \;, \\ \mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} &= u_s(\mathbf{p}) \;. \end{aligned} \tag{243}$$

Taking the complex conjugate of eq. (243), and using $\overline{u}^{\mathrm{T}*} = \overline{u}^{\dagger} = \beta u$, we get

$$\begin{aligned} u_s^*(\mathbf{p}) &= \mathcal{C}\beta v_s(\mathbf{p}) \;, \\ v_s^*(\mathbf{p}) &= \mathcal{C}\beta u_s(\mathbf{p}) \;. \end{aligned} \tag{244}$$

Next, note that $\gamma_5 u_s(\mathbf{0}) = +s\, v_{-s}(\mathbf{0})$ and $\gamma_5 v_s(\mathbf{0}) = -s\, u_{-s}(\mathbf{0})$, and that $\gamma_5 K^j = K^j\gamma_5$. Therefore

$$\begin{aligned} \gamma_5 u_s(\mathbf{p}) &= +s\, v_{-s}(\mathbf{p}) \;, \\ \gamma_5 v_s(\mathbf{p}) &= -s\, u_{-s}(\mathbf{p}) \;. \end{aligned} \tag{245}$$

Combining eqs. (238), (244), and (245) results in

$$\begin{aligned} u_{-s}^*(-\mathbf{p}) &= -s\,\mathcal{C}\gamma_5 u_s(\mathbf{p}) \;, \\ v_{-s}^*(-\mathbf{p}) &= -s\,\mathcal{C}\gamma_5 v_s(\mathbf{p}) \;. \end{aligned} \tag{246}$$

We will need eq. (238) in our discussion of parity, eq. (243) in our discussion of charge conjugation, and eq. (246) in our discussion of time reversal.

Problems

38.1) Use eq. (218) to compute $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ explicity. Hint: show that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1, and that, for any matrix A with eigenvalues ± 1, $e^{cA} = (\cosh c) + (\sinh c)A$, where c is an arbitrary complex number.

38.2) Verify eq. (221).

38.3) Verify eq. (228).

39: Canonical Quantization of Spinor Fields II

Prerequisite: 38

A Dirac field Ψ with lagrangian

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi \tag{247}$$

obeys the canonical anticommutation relations

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0\ , \tag{248}$$

$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{249}$$

and has the Dirac equation

$$(-i\partial\!\!\!/ + m)\Psi = 0 \tag{250}$$

as its equation of motion. The general solution is

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\Big[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\Big]\ , \tag{251}$$

where $b_s(\mathbf{p})$ and $d_s^\dagger(\mathbf{p})$ are operators; the properties of the four-component spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ were belabored in the previous section.

Let us express $b_s(\mathbf{p})$ and $d_s^\dagger(\mathbf{p})$ in terms of $\Psi(x)$ and $\overline{\Psi}(x)$. We begin with

$$\int d^3x\ e^{-ipx}\Psi(x) = \sum_{s'=\pm}\Big[\tfrac{1}{2\omega}b_{s'}(\mathbf{p})u_{s'}(\mathbf{p}) + \tfrac{1}{2\omega}e^{2i\omega t}d_{s'}^\dagger(-\mathbf{p})v_{s'}(-\mathbf{p})\Big]\,. \tag{252}$$

Next, multiply on the left by $\overline{u}_s(\mathbf{p})\gamma^0$, and use $\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$ and $\overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) = 0$ from section 38. The result is

$$b_s(\mathbf{p}) = \int d^3x\ e^{-ipx}\,\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)\ . \tag{253}$$

Note that $b_s(\mathbf{p})$ is time independent.

To get $b_s^\dagger(\mathbf{p})$, take the hermitian conjugate of eq. (253), using

$$\begin{aligned}
\Big[\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)\Big]^\dagger &= \overline{\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)} \\
&= \overline{\Psi}(x)\overline{\gamma^0}u_s(\mathbf{p}) \\
&= \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p}) \;,
\end{aligned} \tag{254}$$

where, for any general combination of gamma matrices A,

$$\overline{A} \equiv \beta A^\dagger \beta \;. \tag{255}$$

Thus we find

$$b_s^\dagger(\mathbf{p}) = \int d^3x\; e^{ipx}\,\overline{\Psi}(x)\gamma^0 u_s(\mathbf{p}) \;. \tag{256}$$

To extract $d_s^\dagger(\mathbf{p})$ from $\Psi(x)$, we start with

$$\int d^3x\; e^{ipx}\Psi(x) = \sum_{s'=\pm}\Big[\tfrac{1}{2\omega}e^{-2i\omega t}b_{s'}(-\mathbf{p})u_{s'}(-\mathbf{p}) + \tfrac{1}{2\omega}d_{s'}^\dagger(\mathbf{p})v_{s'}(\mathbf{p})\Big] . \tag{257}$$

Next, multiply on the left by $\overline{v}_s(\mathbf{p})\gamma^0$, and use $\overline{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$ and $\overline{v}_s(\mathbf{p})\gamma^0 u_{s'}(-\mathbf{p}) = 0$ from section 38. The result is

$$d_s^\dagger(\mathbf{p}) = \int d^3x\; e^{ipx}\,\overline{v}_s(\mathbf{p})\gamma^0\Psi(x) \;. \tag{258}$$

To get $d_s(\mathbf{p})$, take the hermitian conjugate of eq. (258), which yields

$$d_s(\mathbf{p}) = \int d^3x\; e^{-ipx}\,\overline{\Psi}(x)\gamma^0 v_s(\mathbf{p}) \;. \tag{259}$$

Next, let us work out the anticommutation relations of the b and d operators (and their hermitian conjugates). From eq. (248), it is immediately clear that

$$\begin{aligned}
\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} &= 0 \;, \\
\{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= 0 \;, \\
\{b_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} &= 0 \;,
\end{aligned} \tag{260}$$

because these involve only the anticommutator of Ψ with itself, and this vanishes. Of course, hermitian conjugation also yields

$$\begin{aligned}
\{b_s^\dagger(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= 0 \;, \\
\{d_s^\dagger(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} &= 0 \;, \\
\{b_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= 0 \;.
\end{aligned} \tag{261}$$

Now consider

$$\begin{aligned}
\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{-ipx+ip'y}\, \overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x),\overline{\Psi}(y)\}\gamma^0 u_{s'}(\mathbf{p}') \\
&= \int d^3x\, e^{-i(p-p')x}\, \overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 u_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, \overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'} \;.
\end{aligned} \tag{262}$$

In the first line, we are free to set $x^0 = y^0$ because $b_s(\mathbf{p})$ and $b_{s'}^\dagger(\mathbf{p}')$ are actually time independent. In the third, we used $(\gamma^0)^2 = 1$, and in the fourth, $\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$.

Similarly,

$$\begin{aligned}
\{d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{ipx-ip'y}\, \overline{v}_s(\mathbf{p})\gamma^0\{\Psi(x),\overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}') \\
&= \int d^3x\, e^{i(p-p')x}\, \overline{v}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, \overline{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'} \;.
\end{aligned} \tag{263}$$

And finally,

$$\begin{aligned}
\{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{-ipx-ip'y}\, \overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x),\overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}') \\
&= \int d^3x\, e^{-i(p+p')x}\, \overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}')\, \overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) \\
&= 0 \;.
\end{aligned} \tag{264}$$

According to the discussion in section 3, eqs. (260–264) are exactly what we need to describe the creation and annihilation of fermions. In this case, we have two different kinds: b-type and d-type, each with two possible spin states, $s=+$ and $s=-$.

Next, let us evaluate the hamiltonian

$$H = \int d^3x\, \overline{\Psi}(-i\gamma^i\partial_i + m)\Psi \tag{265}$$

in terms of the b and d operators. We have

$$\begin{aligned}
(-i\gamma^i\partial_i + m)\Psi &= \sum_{s=\pm}\int \widetilde{dp}\,\Bigl(-i\gamma^i\partial_i + m\Bigr)\Bigl(\,b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\Bigr) \\
&= \sum_{s=\pm}\int \widetilde{dp}\,\Bigl[\,b_s(\mathbf{p})(+\gamma^i p_i + m)u_s(\mathbf{p})e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^i p_i + m)v_s(\mathbf{p})e^{-ipx}\,\Bigr] \\
&= \sum_{s=\pm}\int \widetilde{dp}\,\Bigl[\,b_s(\mathbf{p})(\gamma^0\omega)u_s(\mathbf{p})e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^0\omega)v_s(\mathbf{p})e^{-ipx}\,\Bigr].
\end{aligned} \tag{266}$$

Therefore

$$\begin{aligned}
H &= \sum_{s,s'}\int \widetilde{dp}\,\widetilde{dp}'\,d^3x\,\Bigl(b_{s'}^\dagger(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'x} + d_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'x}\Bigr) \\
&\qquad\qquad \times\,\omega\,\Bigl(b_s(\mathbf{p})\gamma^0 u_s(\mathbf{p})e^{ipx} - d_s^\dagger(\mathbf{p})\gamma^0 v_s(\mathbf{p})e^{-ipx}\Bigr) \\
&= \sum_{s,s'}\int \widetilde{dp}\,\widetilde{dp}'\,d^3x\,\omega\,\Bigl[\,b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})\,\overline{u}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{-i(p'-p)x} \\
&\qquad\qquad - b_{s'}^\dagger(\mathbf{p}')d_s^\dagger(\mathbf{p})\,\overline{u}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{-i(p'+p)x} \\
&\qquad\qquad + d_{s'}(\mathbf{p}')b_s(\mathbf{p})\,\overline{v}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{+i(p'+p)x} \\
&\qquad\qquad - d_{s'}(\mathbf{p}')d_s^\dagger(\mathbf{p})\,\overline{v}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{+i(p'-p)x}\,\Bigr] \\
&= \sum_{s,s'}\int \widetilde{dp}\;\tfrac{1}{2}\Bigl[\,b_{s'}^\dagger(\mathbf{p})b_s(\mathbf{p})\,\overline{u}_{s'}(\mathbf{p})\gamma^0 u_s(\mathbf{p}) \\
&\qquad\qquad - b_{s'}^\dagger(-\mathbf{p})d_s^\dagger(\mathbf{p})\,\overline{u}_{s'}(-\mathbf{p})\gamma^0 v_s(\mathbf{p})\;e^{+2i\omega t} \\
&\qquad\qquad + d_{s'}(-\mathbf{p})b_s(\mathbf{p})\,\overline{v}_{s'}(-\mathbf{p})\gamma^0 u_s(\mathbf{p})\;e^{-2i\omega t}
\end{aligned}$$

$$
\begin{aligned}
&\qquad - d_{s'}(\mathbf{p}) d_s^\dagger(\mathbf{p})\, \overline{v}_{s'}(\mathbf{p})\gamma^0 v_s(\mathbf{p}) \Big] \\
&= \sum_s \int \widetilde{dp}\;\, \omega \Big[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \Big]. \qquad (267)
\end{aligned}
$$

Using eq. (263), we can rewrite this as

$$
H = \sum_{s=\pm} \int \widetilde{dp}\;\, \omega \Big[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \Big] - 4\mathcal{E}_0 V \;, \qquad (268)
$$

where $\mathcal{E}_0 = \frac{1}{2} \int d^3k\, \omega$ is the zero-point energy per unit volume that we found for a real scalar field in section 3, and $V = (2\pi)^3 \delta^3(\mathbf{0}) = \int d^3x$ is the volume of space. That the zero-point energy is negative rather than positive is characteristic of fermions; that it is larger in magnitude by a factor of four is due to the four types of particles that are associated with a Dirac field. We can cancel off this constant energy by including a constant term $-4\mathcal{E}_0$ in the original lagrangian density; from here on, we will assume that this has been done.

The ground state of the hamiltonian (268) is the *vacuum state* $|0\rangle$ that is annihilated by every $b_s(\mathbf{p})$ and $d_s(\mathbf{p})$,

$$
b_s(\mathbf{p})|0\rangle = d_s(\mathbf{p})|0\rangle = 0 \;. \qquad (269)
$$

Then, we can interpret the $b_s^\dagger(\mathbf{p})$ operator as creating a b-type particle with momentum $\mathbf{p}$, energy $\omega = (\mathbf{p}^2 + m^2)^{1/2}$, and spin $S_z = \frac{1}{2}s$, and the $d_s^\dagger(\mathbf{p})$ operator as creating a d-type particle with the same properties. The b-type and d-type particles are distinguished by the value of the charge $Q = \int d^3x\, j^0$, where $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ is the Noether current associated with the invariance of Ł under the U(1) transformation $\Psi \to e^{-i\alpha}\Psi$, $\overline{\Psi} \to e^{+i\alpha}\overline{\Psi}$. Following the same procedure that we used for the hamiltonian, we can show that

$$
\begin{aligned}
Q &= \int d^3x\; \overline{\Psi}\gamma^0\Psi \\
&= \sum_{s=\pm} \int \widetilde{dp}\; \Big[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \Big] \\
&= \sum_{s=\pm} \int \widetilde{dp}\; \Big[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \Big] + \text{constant} \;; \qquad (270)
\end{aligned}
$$

Thus the conserved charge Q counts the total number of b-type particles minus the total number of d-type particles. (We are free to shift the overall value of Q to remove the constant term, and so we shall.) In quantum electrodynamics, we will identify the b-type particles as electrons and the d-type particles as positrons.

Now consider a Majorana field Ψ with lagrangian

$$\mathcal{L} = \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\partial\!\!\!/\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi \; . \tag{271}$$

The equation of motion for Ψ is once again the Dirac equation, and so the general solution is once again given by eq. (251). However, Ψ must also obey the Majorana condition $\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$. Starting from the barred form of eq. (251),

$$\overline{\Psi}(x) = \sum_{s=\pm}\int \widetilde{dp}\;\Big[b_s^\dagger(\mathbf{p})\overline{u}_s(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})\overline{v}_s(\mathbf{p})e^{ipx}\Big]\,, \tag{272}$$

we have

$$\mathcal{C}\overline{\Psi}^{\mathrm{T}}(x) = \sum_{s=\pm}\int \widetilde{dp}\;\Big[b_s^\dagger(\mathbf{p})\,\mathcal{C}\overline{u}_s^{\mathrm{T}}(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})\,\mathcal{C}\overline{v}_s^{\mathrm{T}}(\mathbf{p})e^{ipx}\Big]\,. \tag{273}$$

From section 38, we have

$$\begin{aligned}\mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} &= v_s(\mathbf{p})\;,\\ \mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} &= u_s(\mathbf{p})\;,\end{aligned} \tag{274}$$

and so

$$\mathcal{C}\overline{\Psi}^{\mathrm{T}}(x) = \sum_{s=\pm}\int \widetilde{dp}\;\Big[b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})u_s(\mathbf{p})e^{ipx}\Big]\;. \tag{275}$$

Comparing eqs. (251) and (275), we see that we will have $\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$ if

$$d_s(\mathbf{p}) = b_s(\mathbf{p})\;. \tag{276}$$

Thus a free Majorana field can be written as

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\;\Big[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\Big]\;. \tag{277}$$

The anticommutation relations for a Majorana field,

$$\{\Psi_\alpha(\mathbf{x},t),\Psi_\beta(\mathbf{y},t)\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\;, \tag{278}$$

$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\;, \tag{279}$$

can be used to show that

$$\begin{aligned}\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} &= 0 \ , \\ \{b_s(\mathbf{p}), b^\dagger_{s'}(\mathbf{p}')\} &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\delta_{ss'} \ , \end{aligned} \tag{280}$$

as we would expect.

The hamiltonian for the Majorana field Ψ is

$$\begin{aligned} H &= \tfrac{1}{2}\int d^3x \ \Psi^{\mathrm{T}}\mathcal{C}(-i\gamma^i\partial_i + m)\Psi \\ &= \tfrac{1}{2}\int d^3x \ \overline{\Psi}(-i\gamma^i\partial_i + m)\Psi \ , \end{aligned} \tag{281}$$

and we can work through the same manipulations that led to eq. (267); the only differences are an extra overall factor of one-half, and $d_s(\mathbf{p}) = b_s(\mathbf{p})$. Thus we get

$$H = \tfrac{1}{2}\sum_{s=\pm}\int \widetilde{dp} \ \omega \left[b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) - b_s(\mathbf{p})b^\dagger_s(\mathbf{p}) \right]. \tag{282}$$

Note that this would reduce to a constant if we tried to use commutators rather than anticommutators in eq. (280), a reflection of the spin-statistics theorem. Using eq. (280) as it is, we find

$$H = \sum_{s=\pm}\int \widetilde{dp}\,\omega\, b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) - 2\mathcal{E}_0 V. \tag{283}$$

Again, we can (and will) cancel off the zero-point energy by including a term $-2\mathcal{E}_0$ in the original lagrangian density.

The Majorana lagrangian has no U(1) symmetry. Thus there is no associated charge, and only one kind of particle (with two possible spin states).

Problems

39.1) Verify eq. (270).

39.2) Show that

$$
\begin{aligned}
U(\Lambda)^{-1}b_s^\dagger(\mathbf{p})U(\Lambda) &= b_s^\dagger(\Lambda^{-1}\mathbf{p}) \;, \\
U(\Lambda)^{-1}d_s^\dagger(\mathbf{p})U(\Lambda) &= d_s^\dagger(\Lambda^{-1}\mathbf{p}) \;,
\end{aligned}
\tag{284}
$$

and hence that

$$
U(\Lambda)|p,s,q\rangle = |\Lambda p,s,q\rangle \;, \tag{285}
$$

where

$$
\begin{aligned}
|p,s,+\rangle &\equiv b_s^\dagger(\mathbf{p})|0\rangle \;, \\
|p,s,-\rangle &\equiv d_s^\dagger(\mathbf{p})|0\rangle
\end{aligned}
\tag{286}
$$

are single-particle states.

40: Parity, Time Reversal, and Charge Conjugation

Prerequisite: 39

Recall that, under a Lorentz transformation Λ implemented by the unitary operator $U(\Lambda)$, a Dirac (or Majorana) field transforms as

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x) \ . \tag{287}$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, the matrix $D(\Lambda)$ is given by

$$D(1{+}\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu} \ , \tag{288}$$

where the Lorentz generator matrices are

$$S^{\mu\nu} = \tfrac{i}{4}[\gamma^\mu, \gamma^\nu] \ . \tag{289}$$

In this section, we will consider the two Lorentz transformations that cannot be reached via a sequence of infinitesimal transformations away from the identity: parity and time reversal. We begin with parity.

Define the parity transformation

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{290}$$

and the corresponding unitary operator

$$P \equiv U(\mathcal{P}) \ . \tag{291}$$

Now we have

$$P^{-1}\Psi(x)P = D(\mathcal{P})\Psi(\mathcal{P}x) \ . \tag{292}$$

The question we wish to answer is, what is the matrix $D(\mathcal{P})$?

First of all, if we make a second parity transformation, we get

$$P^{-2}\Psi(x)P^2 = D(\mathcal{P})^2\Psi(x) \; , \tag{293}$$

and it is tempting to conclude that we should have $D(\mathcal{P})^2 = 1$, so that we return to the original field. This is correct for scalar fields, since they are themselves observable. With fermions, however, it takes an even number of fields to construct an observable. Therefore we need only require the weaker condition $D(\mathcal{P})^2 = \pm 1$.

We will also require the particle creation and annihilation operators to transform in a simple way. Because

$$P^{-1}\mathbf{P}P = -\mathbf{P} \; , \tag{294}$$

$$P^{-1}\mathbf{J}P = +\mathbf{J} \; , \tag{295}$$

where $\mathbf{P}$ is the total three-momentum operator and $\mathbf{J}$ is the total angular momentum operator, a parity transformation should reverse the three-momentum while leaving the spin direction unchanged. We therefore require

$$\begin{aligned} P^{-1}b_s^\dagger(\mathbf{p})P &= \eta\, b_s^\dagger(-\mathbf{p}) \; , \\ P^{-1}d_s^\dagger(\mathbf{p})P &= \eta\, d_s^\dagger(-\mathbf{p}) \; , \end{aligned} \tag{296}$$

where η is a possible phase factor that (by the previous argument about observables) should satisfy $\eta^2 = \pm 1$. We could in principle assign different phase factors to the b and d operators, but we choose them to be the same so that the parity transformation is compatible with the Majorana condition $d_s(\mathbf{p}) = b_s(\mathbf{p})$. Writing the mode expansion of the free field

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx} \right] , \tag{297}$$

the parity transformation reads

$$\begin{aligned} P^{-1}\Psi(x)P &= \sum_{s=\pm} \int \widetilde{dp} \left[\Big(P^{-1}b_s(\mathbf{p})P\Big)u_s(\mathbf{p})e^{ipx} + \Big(P^{-1}d_s^\dagger(\mathbf{p})P\Big)v_s(\mathbf{p})e^{-ipx} \right] \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[\eta^* b_s(-\mathbf{p})u_s(\mathbf{p})e^{ipx} + \eta d_s^\dagger(-\mathbf{p})v_s(\mathbf{p})e^{-ipx} \right] \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[\eta^* b_s(\mathbf{p})u_s(-\mathbf{p})e^{ip\mathcal{P}x} + \eta d_s^\dagger(\mathbf{p})v_s(-\mathbf{p})e^{-ip\mathcal{P}x} \right] . \end{aligned} \tag{298}$$

In the last line, we have changed the integration variable from $\mathbf{p}$ to $-\mathbf{p}$. We now use a result from section 38, namely that

$$\begin{aligned} u_s(-\mathbf{p}) &= +\beta u_s(\mathbf{p}) \;, \\ v_s(-\mathbf{p}) &= -\beta v_s(\mathbf{p}) \;, \end{aligned} \tag{299}$$

where

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} . \tag{300}$$

Then, if we choose $\eta = -i$, eq. (298) becomes

$$\begin{aligned} P^{-1}\Psi(x)P &= \sum_{s=\pm} \int \widetilde{dp} \left[ib_s(\mathbf{p})\beta u_s(\mathbf{p}) e^{ip\mathcal{P}x} + id_s^\dagger(\mathbf{p})\beta v_s(\mathbf{p}) e^{-ip\mathcal{P}x} \right] \\ &= i\beta\,\Psi(\mathcal{P}x) \;. \end{aligned} \tag{301}$$

Thus we see that $D(\mathcal{P}) = i\beta$. (We could also have chosen $\eta = i$, resulting in $D(\mathcal{P}) = -i\beta$; either choice is acceptable.)

The factor of i has an interesting physical consequence. Consider a state of an electron and positron with zero center-of-mass momentum,

$$|\phi\rangle = \int \widetilde{dp}\; \phi(\mathbf{p}) b_s^\dagger(\mathbf{p}) d_{s'}^\dagger(-\mathbf{p})|0\rangle \;; \tag{302}$$

here $\phi(\mathbf{p})$ is the momentum-space wave function. Let us assume that the vacuum is parity invariant: $P|0\rangle = P^{-1}|0\rangle = |0\rangle$. Let us also assume that the wave function has definite parity: $\phi(-\mathbf{p}) = (-)^\ell \phi(\mathbf{p})$. Then, applying the inverse parity operator on $|\phi\rangle$, we get

$$\begin{aligned} P^{-1}|\phi\rangle &= \int \widetilde{dp}\; \phi(\mathbf{p}) \Big(P^{-1} b_s^\dagger(\mathbf{p}) P\Big)\big(P^{-1} d_{s'}^\dagger(-\mathbf{p}) P\Big) P^{-1}|0\rangle \;. \\ &= (-i)^2 \int \widetilde{dp}\; \phi(\mathbf{p}) b_s^\dagger(-\mathbf{p}) d_{s'}^\dagger(\mathbf{p})|0\rangle \\ &= (-i)^2 \int \widetilde{dp}\; \phi(-\mathbf{p}) b_s^\dagger(\mathbf{p}) d_{s'}^\dagger(-\mathbf{p})|0\rangle \\ &= -(-)^\ell |\phi\rangle \;. \end{aligned} \tag{303}$$

Thus, the parity of this state is opposite to that of its wave function; an electron-positron pair has an *intrinsic parity* of -1. This also applies to a pair of Majorana fermions. This influences the selection rules for fermion pair annihilation in theories which conserve parity. (A pair of electrons also has negative intrinsic parity, but this is not interesting because the electrons are prevented from annihilating by charge conservation.)

It is interesting to see what eq. (301) implies for the two Weyl fields that comprise the Dirac field. Recalling that

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix} , \tag{304}$$

we see from eqs. (300) and (301) that

$$\begin{aligned} P^{-1}\chi_a(x)P &= i\xi^{\dagger\dot{a}}(\mathcal{P}x) \; , \\ P^{-1}\xi^{\dagger\dot{a}}(x)P &= i\chi_a(\mathcal{P}x) \; . \end{aligned} \tag{305}$$

Thus a parity transformation exchanges a left-handed field for a right-handed one.

If we take the hermitian conjugate of eq. (305), then raise the index on one side while lowering it on the other (and remember that this introduces a relative minus sign!), we get

$$\begin{aligned} P^{-1}\chi^{\dagger\dot{a}}(x)P &= i\xi_a(\mathcal{P}x) \; , \\ P^{-1}\xi_a(x)P &= i\chi^{\dagger\dot{a}}(\mathcal{P}x) \; . \end{aligned} \tag{306}$$

Comparing eqs. (305) and (306), we see that they are compatible with the Majorana condition $\chi_a(x) = \xi_a(x)$.

Next we take up time reversal. Define the time-reversal transformation

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \tag{307}$$

and the corresponding operator

$$T \equiv U(\mathcal{T}) \; . \tag{308}$$

Now we have

$$T^{-1}\Psi(x)T = D(\mathcal{T})\Psi(\mathcal{T}x)\ . \tag{309}$$

The question we wish to answer is, what is the matrix $D(\mathcal{T})$?

As with parity, we can conclude that $D(\mathcal{T})^2 = \pm 1$, and we will require the particle creation and annihilation operators to transform in a simple way. Because

$$T^{-1}\mathbf{P}T = -\mathbf{P}\ , \tag{310}$$

$$T^{-1}\mathbf{J}T = -\mathbf{J}\ , \tag{311}$$

where $\mathbf{P}$ is the total three-momentum operator and $\mathbf{J}$ is the total angular momentum operator, a time-reversal transformation should reverse the direction of both the three-momentum and the spin. We therefore require

$$\begin{aligned} T^{-1}b^\dagger_s(\mathbf{p})T &= \zeta_s\, b^\dagger_{-s}(-\mathbf{p})\ , \\ T^{-1}d^\dagger_s(\mathbf{p})T &= \zeta_s\, d^\dagger_{-s}(-\mathbf{p})\ . \end{aligned} \tag{312}$$

This time we allow for possible s-dependence of the phase factor, which should satisfy $\zeta^*_s\zeta_{-s} = \pm 1$. Also, we recall from section 23 that T must be an antiunitary operator, so that $T^{-1}iT = -i$. Then we have

$$\begin{aligned} T^{-1}\Psi(x)T &= \sum_{s=\pm}\int \widetilde{dp}\,\Big[\Big(T^{-1}b_s(\mathbf{p})T\Big)u^*_s(\mathbf{p})e^{-ipx} + \Big(T^{-1}d^\dagger_s(\mathbf{p})T\Big)v^*_s(\mathbf{p})e^{ipx}\Big] \\ &= \sum_{s=\pm}\int \widetilde{dp}\,\Big[\zeta^*_s b_{-s}(-\mathbf{p})u^*_s(\mathbf{p})e^{-ipx} + \zeta_s d^\dagger_{-s}(-\mathbf{p})v^*_s(\mathbf{p})e^{ipx}\Big] \qquad (313) \\ &= \sum_{s=\pm}\int \widetilde{dp}\,\Big[\zeta^*_{-s} b_{s}(\mathbf{p})u^*_{-s}(-\mathbf{p})e^{ip\mathcal{T}x} + \zeta_{-s} d^\dagger_{s}(\mathbf{p})v^*_{-s}(-\mathbf{p})e^{-ip\mathcal{T}x}\Big]\ . \end{aligned}$$

In the last line, we have changed the integration variable from $\mathbf{p}$ to $-\mathbf{p}$, and the summation variable from s to $-s$. We now use a result from section 38, namely that

$$\begin{aligned} u^*_{-s}(-\mathbf{p}) &= -s\,\mathcal{C}\gamma_5 u_s(\mathbf{p})\ , \\ v^*_{-s}(-\mathbf{p}) &= -s\,\mathcal{C}\gamma_5 v_s(\mathbf{p})\ . \end{aligned} \tag{314}$$

Then, if we choose $\zeta_s = s$, eq. (313) becomes

$$T^{-1}\Psi(x)T = \mathcal{C}\gamma_5\Psi(\mathcal{T}x)\ . \tag{315}$$

Thus we see that $D(\mathcal{T}) = \mathcal{C}\gamma_5$. (We could also have chosen $\zeta_s = -s$, resulting in $D(\mathcal{T}) = -\mathcal{C}\gamma_5$; either choice is acceptable.)

As with parity, we can consider the effect of time reversal on the Weyl fields. Using eqs. (304), (315),

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ab} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} , \tag{316}$$

and

$$\gamma_5 = \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} , \tag{317}$$

we see that

$$\begin{aligned} T^{-1}\chi_a(x)T &= +\chi^a(\mathcal{T}x) , \\ T^{-1}\xi^{\dagger\dot{a}}(x)T &= -\xi^\dagger_{\dot{a}}(\mathcal{T}x) . \end{aligned} \tag{318}$$

Thus left-handed Weyl fields transform into left-handed Weyl fields (and right-handed into right-handed) under time reversal.

If we take the hermitian conjugate of eq. (318), then raise the index on one side while lowering it on the other (and remember that this introduces a relative minus sign!), we get

$$\begin{aligned} T^{-1}\chi^{\dagger\dot{a}}(x)T &= -\chi^\dagger_{\dot{a}}(\mathcal{T}x) , \\ T^{-1}\xi_a(x)T &= +\xi^a(\mathcal{T}x) . \end{aligned} \tag{319}$$

Comparing eqs. (318) and (319), we see that they are compatible with the Majorana condition $\chi_a(x) = \xi_a(x)$.

It is interesting and important to evaluate the transformation properties of fermion bilinears of the form $\overline{\Psi}A\Psi$, where A is some combination of gamma matrices. We will consider A's that satisfy $\overline{A} = A$, where $\overline{A} \equiv \beta A^\dagger \beta$; in this case, $\overline{\Psi}A\Psi$ is hermitian.

Let us begin with parity transformations. From $\overline{\Psi} = \Psi^\dagger\beta$ and eq. (301) we get

$$P^{-1}\overline{\Psi}(x)P = -i\overline{\Psi}(\mathcal{P}x)\beta , \tag{320}$$

Combining eqs. (301) and (320) we find

$$P^{-1}\Big(\overline{\Psi}A\Psi\Big)P = \overline{\Psi}\Big(\beta A\beta\Big)\Psi \ , \tag{321}$$

where we have suppressed the spacetime arguments (which transform in the obvious way). For various particular choices of A we have

$$\begin{aligned}
\beta 1\beta &= +1 \ , \\
\beta i\gamma_5\beta &= -i\gamma_5 \ , \\
\beta\gamma^0\beta &= +\gamma^0 \ , \\
\beta\gamma^i\beta &= -\gamma^i \ , \\
\beta\gamma^0\gamma_5\beta &= -\gamma^0\gamma_5 \ , \\
\beta\gamma^i\gamma_5\beta &= +\gamma^i\gamma_5 \ .
\end{aligned} \tag{322}$$

Therefore, the corresponding hermitian bilinears transform as

$$\begin{aligned}
P^{-1}\Big(\overline{\Psi}\Psi\Big)P &= +\,\overline{\Psi}\Psi \ , \\
P^{-1}\Big(\overline{\Psi}i\gamma_5\Psi\Big)P &= -\,\overline{\Psi}i\gamma_5\Psi \ , \\
P^{-1}\Big(\overline{\Psi}\gamma^\mu\Psi\Big)P &= +\,\mathcal{P}^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi \ , \\
P^{-1}\Big(\overline{\Psi}\gamma^\mu\gamma_5\Psi\Big)P &= -\,\mathcal{P}^\mu{}_\nu\overline{\Psi}\gamma^\nu\gamma_5\Psi \ ,
\end{aligned} \tag{323}$$

Thus we see that $\overline{\Psi}\Psi$ and $\overline{\Psi}\gamma^\mu\Psi$ are even under a parity transformation, while $\overline{\Psi}i\gamma_5\Psi$ and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ are odd. We say that $\overline{\Psi}\Psi$ is a scalar, $\overline{\Psi}\gamma^\mu\Psi$ is a vector or *polar vector*, $\overline{\Psi}i\gamma_5\Psi$ is a pseudoscalar, and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ is a *pseudovector* or *axial vector*.

Turning to time reversal, from eq. (315) we get

$$T^{-1}\overline{\Psi}(x)T = \overline{\Psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1} \ . \tag{324}$$

Combining eqs. (315) and (324), along with $T^{-1}AT = A^*$, we find

$$T^{-1}\Big(\overline{\Psi}A\Psi\Big)T = \overline{\Psi}\Big(\gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5\Big)\Psi \ , \tag{325}$$

where we have suppressed the spacetime arguments (which transform in the obvious way). Recall that $\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}$ and that $\mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5$. Also,

γ^0 and γ_5 are real, hermitian, and square to one, while γ^i is antihermitian. Finally, γ_5 anticommutes with γ^μ. Using all of this info, we find

$$\begin{aligned}
\gamma_5 \mathcal{C}^{-1} 1^* \mathcal{C} \gamma_5 &= +1 \ , \\
\gamma_5 \mathcal{C}^{-1} (i\gamma_5)^* \mathcal{C} \gamma_5 &= -i\gamma_5 \ , \\
\gamma_5 \mathcal{C}^{-1} (\gamma^0)^* \mathcal{C} \gamma_5 &= +\gamma^0 \ , \\
\gamma_5 \mathcal{C}^{-1} (\gamma^i)^* \mathcal{C} \gamma_5 &= -\gamma^i \ , \\
\gamma_5 \mathcal{C}^{-1} (\gamma^0\gamma_5)^* \mathcal{C} \gamma_5 &= +\gamma^0\gamma_5 \ , \\
\gamma_5 \mathcal{C}^{-1} (\gamma^i\gamma_5)^* \mathcal{C} \gamma_5 &= -\gamma^i\gamma_5 \ .
\end{aligned} \tag{326}$$

Therefore,

$$\begin{aligned}
T^{-1}\Big(\overline{\Psi}\Psi\Big)T &= +\,\overline{\Psi}\Psi \ , \\
T^{-1}\Big(\overline{\Psi}i\gamma_5\Psi\Big)T &= -\,\overline{\Psi}i\gamma_5\Psi \ , \\
T^{-1}\Big(\overline{\Psi}\gamma^\mu\Psi\Big)T &= -\,\mathcal{T}^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi \ , \\
T^{-1}\Big(\overline{\Psi}\gamma^\mu\gamma_5\Psi\Big)T &= -\,\mathcal{T}^\mu{}_\nu\overline{\Psi}\gamma^\nu\gamma_5\Psi \ .
\end{aligned} \tag{327}$$

Thus we see that $\overline{\Psi}\Psi$ is even under time reversal, while $\overline{\Psi}i\gamma_5\Psi$, $\overline{\Psi}\gamma^\mu\Psi$, and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ are odd.

For completeness we will also consider the transformation properties of bilinears under charge conjugation. Recall that

$$\begin{aligned}
C^{-1}\Psi(x)C &= \mathcal{C}\overline{\Psi}^{\mathrm{T}}(x) \ , \\
C^{-1}\overline{\Psi}(x)C &= \Psi^{\mathrm{T}}(x)\mathcal{C} \ .
\end{aligned} \tag{328}$$

The bilinear $\overline{\Psi}A\Psi$ therefore transforms as

$$C^{-1}\Big(\overline{\Psi}A\Psi\Big)C = \Psi^{\mathrm{T}}\mathcal{C}A\mathcal{C}\overline{\Psi}^{\mathrm{T}} \ . \tag{329}$$

Since all indices are contracted, we can rewrite the right-hand side as its transpose, with an extra minus sign for exchanging the order of the two fermion fields. We get

$$C^{-1}\Big(\overline{\Psi}A\Psi\Big)C = -\overline{\Psi}\mathcal{C}^{\mathrm{T}}A^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\Psi \ . \tag{330}$$

Recalling that $\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{-1} = -\mathcal{C}$, we have

$$C^{-1}\Big(\overline{\Psi}A\Psi\Big)C = \overline{\Psi}\Big(\mathcal{C}^{-1}A^{\mathrm{T}}\mathcal{C}\Big)\Psi \; . \tag{331}$$

Once again we can go through the list:

$$\begin{aligned}
\mathcal{C}^{-1}1^{\mathrm{T}}\mathcal{C} &= +1 \; , \\
\mathcal{C}^{-1}(i\gamma_5)^{\mathrm{T}}\mathcal{C} &= +i\gamma_5 \; , \\
\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} &= -\gamma^\mu \; , \\
\mathcal{C}^{-1}(\gamma^\mu\gamma_5)^{\mathrm{T}}\mathcal{C} &= +\gamma^\mu\gamma_5 \; .
\end{aligned} \tag{332}$$

Therefore,

$$\begin{aligned}
C^{-1}\Big(\overline{\Psi}\Psi\Big)C &= +\,\overline{\Psi}\Psi \; , \\
C^{-1}\Big(\overline{\Psi}i\gamma_5\Psi\Big)C &= +\,\overline{\Psi}i\gamma_5\Psi \; , \\
C^{-1}\Big(\overline{\Psi}\gamma^\mu\Psi\Big)C &= -\,\overline{\Psi}\gamma^\mu\Psi \; , \\
C^{-1}\Big(\overline{\Psi}\gamma^\mu\gamma_5\Psi\Big)C &= +\,\overline{\Psi}\gamma^\mu\gamma_5\Psi \; .
\end{aligned} \tag{333}$$

Thus we see that $\overline{\Psi}\Psi$, $\overline{\Psi}i\gamma_5\Psi$, and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ are even under charge conjugation, while $\overline{\Psi}\gamma^\mu\Psi$ is odd. For a Majorana field, this implies $\overline{\Psi}\gamma^\mu\Psi = 0$.

Let us consider the combined effects of the three transformations (C, P, and T) on the bilinears. From eqs. (323), (327), and (333), we have

$$\begin{aligned}
(CPT)^{-1}\Big(\overline{\Psi}\Psi\Big)CPT &= +\,\overline{\Psi}\Psi \; , \\
(CPT)^{-1}\Big(\overline{\Psi}i\gamma_5\Psi\Big)CPT &= +\,\overline{\Psi}i\gamma_5\Psi \; , \\
(CPT)^{-1}\Big(\overline{\Psi}\gamma^\mu\Psi\Big)CPT &= -\,\overline{\Psi}\gamma^\mu\Psi \; , \\
(CPT)^{-1}\Big(\overline{\Psi}\gamma^\mu\gamma_5\Psi\Big)CPT &= -\,\overline{\Psi}\gamma^\mu\gamma_5\Psi \; ,
\end{aligned} \tag{334}$$

where we have used $\mathcal{P}^\mu{}_\nu\mathcal{T}^\nu{}_\rho = -\delta^\mu{}_\rho$. We see that $\overline{\Psi}\Psi$ and $\overline{\Psi}i\gamma_5\Psi$ are both even under CPT, while $\overline{\Psi}\gamma^\mu\Psi$ and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ are both odd. These are (it turns out) examples of a more general rule: a fermion bilinear with n vector indices (and no uncontracted spinor indices) is even (odd) under CPT if n is even (odd). This also applies if we allow derivatives acting on the fields, since each factor of ∂_μ is odd under the combination PT and even under C.

For scalar and vector fields, it is always possible to choose the phase factors in the C, P, and T transformations so that, overall, they obey the same rule: a hermitian combination of fields and derivatives is even or odd depending on the total number of uncontracted vector indices. Putting this together with our result for fermion bilinears, we see that any hermitian combination of any set of fields (scalar, vector, Dirac, Majorana) and their derivatives that is a Lorentz scalar (and so carries no indices) is even under CPT. Since the lagrangian must be formed out of such combinations, we have Ł$(x) \to$ Ł$(-x)$ under CPT, and so the action $S = \int d^4x\,$Ł is invariant. This is the CPT theorem.

Problems

40.1) Find the transformation properties of $\overline{\Psi} S^{\mu\nu} \Psi$ and $\overline{\Psi} i S^{\mu\nu} \gamma_5 \Psi$ under P, T, and C. Verify that they are both even under CPT, as claimed.

41: LSZ Reduction for Spin-One-Half Particles

Prerequisite: 39

Let us now consider how to construct appropriate initial and final states for scattering experiments. We will first consider the case of a Dirac field Ψ, and assume that its interactions respect the U(1) symmetry that gives rise to the conserved current $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ and its associated charge Q.

In the free theory, we can create a state of one particle by acting on the vacuum state with a creation operator:

$$|p, s, +\rangle = b^\dagger_s(\mathbf{p})|0\rangle \ , \tag{335}$$

$$|p, s, -\rangle = d^\dagger_s(\mathbf{p})|0\rangle \ , \tag{336}$$

where the label $\pm$ on the ket indicates the value of the U(1) charge Q, and

$$b^\dagger_s(\mathbf{p}) = \int d^3x \; e^{ipx}\, \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p}) \ , \tag{337}$$

$$d^\dagger_s(\mathbf{p}) = \int d^3x \; e^{ipx}\, \overline{v}_s(\mathbf{p})\gamma^0 \Psi(x) \ . \tag{338}$$

Recall that $b^\dagger_s(\mathbf{p})$ and $d^\dagger_s(\mathbf{p})$ are time independent in the free theory. The states $|p, s, \pm\rangle$ have the Lorentz-invariant normalization

$$\langle p, s, q|p', s', q'\rangle = (2\pi)^3\, 2\omega\, \delta^3(\mathbf{p} - \mathbf{p}')\, \delta_{ss'}\, \delta_{qq'} \ , \tag{339}$$

where $\omega = (\mathbf{p}^2 + m^2)^{1/2}$.

Let us consider an operator that (in the free theory) creates a particle with definite spin and charge, localized in momentum space near $\mathbf{p}_1$, and localized in position space near the origin:

$$b^\dagger_1 \equiv \int d^3p \; f_1(\mathbf{p}) b^\dagger_{s_1}(\mathbf{p}) \ , \tag{340}$$

where

$$f_1(\mathbf{p}) \propto \exp[-(\mathbf{p}-\mathbf{p}_1)^2/4\sigma^2] \tag{341}$$

is an appropriate wave packet, and σ is its width in momentum space. If we time evolve (in the Schrödinger picture) the state created by this time-independent operator, then the wave packet will propagate (and spread out). The particle will thus be localized far from the origin as $t \to \pm\infty$. If we consider instead an initial state of the form $|i\rangle = b_1^\dagger b_2^\dagger |0\rangle$, where $\mathbf{p}_1 \neq \mathbf{p}_2$, then we have two particles that are widely separated in the far past.

Let us guess that this still works in the interacting theory. One complication is that $b_s^\dagger(\mathbf{p})$ will no longer be time independent, and so $b_1^\dagger$, eq. (340), becomes time dependent as well. Our guess for a suitable initial state for a scattering experiment is then

$$|i\rangle = \lim_{t\to-\infty} b_1^\dagger(t) b_2^\dagger(t)|0\rangle \ . \tag{342}$$

By appropriately normalizing the wave packets, we can make $\langle i|i\rangle = 1$, and we will assume that this is the case. Similarly, we can consider a final state

$$|f\rangle = \lim_{t\to+\infty} b_{1'}^\dagger(t) b_{2'}^\dagger(t)|0\rangle \ , \tag{343}$$

where $\mathbf{p}_1' \neq \mathbf{p}_2'$, and $\langle f|f\rangle = 1$. This describes two widely separated particles in the far future. (We could also consider acting with more creation operators, if we are interested in the production of some extra particles in the collision of two, or using $d^\dagger$ operators instead of $b^\dagger$ operators for some or all of the initial and final particles.) Now the scattering amplitude is simply given by $\langle f|i\rangle$.

We need to find a more useful expression for $\langle f|i\rangle$. To this end, let us note that

$$\begin{aligned}
b_1^\dagger(-\infty) &- b_1^\dagger(+\infty) \\
&= -\int_{-\infty}^{+\infty} dt\ \partial_0 b_1^\dagger(t) \\
&= -\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \partial_0\Big(e^{ipx}\,\overline{\Psi}(x)\gamma^0 u_{s_1}(\mathbf{p})\Big)\ . \\
&= -\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \overline{\Psi}(x)\Big(\gamma^0\overleftarrow{\partial}_0 - i\gamma^0 p^0\Big)u_{s_1}(\mathbf{p})e^{ipx}
\end{aligned}$$

$$= -\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \overline{\Psi}(x)\Big(\gamma^0\overleftarrow{\partial}_0 - i\gamma^i p_i - im\Big)u_{s_1}(\mathbf{p})e^{ipx}$$
$$= -\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \overline{\Psi}(x)\Big(\gamma^0\overleftarrow{\partial}_0 - \gamma^i\overrightarrow{\partial}_i - im\Big)u_{s_1}(\mathbf{p})e^{ipx}$$
$$= -\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \overline{\Psi}(x)\Big(\gamma^0\overleftarrow{\partial}_0 + \gamma^i\overleftarrow{\partial}_i - im\Big)u_{s_1}(\mathbf{p})e^{ipx}$$
$$= i\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \overline{\Psi}(x)(+i\overleftarrow{\not\partial} + m)u_s(\mathbf{p})e^{ipx}u_{s_1}(\mathbf{p})e^{ipx}\ . \tag{344}$$

The first equality is just the fundamental theorem of calculus. To get the second, we substituted the definition of $b_1^\dagger(t)$, and combined the d^3x from this definition with the dt to get d^4x. The third comes from straightforward evaluation of the time derivatives. The fourth uses $(\not p + m)u_s(\mathbf{p}) = 0$. The fifth writes ip_i as ∂_i acting on e^{ipx}. The sixth uses integration by parts to move the ∂_i onto the field $\overline{\Psi}(x)$; here the wave packet is needed to avoid a surface term. The seventh simply identifies $\gamma^0\partial_0 + \gamma^i\partial_i$ as $\not\partial$.

In free-field theory, the right-hand side of eq. (344) is zero, since $\Psi(x)$ obeys the Dirac equation, which, after barring it, reads

$$\overline{\Psi}(x)(+i\overleftarrow{\not\partial} + m) = 0\ . \tag{345}$$

In an interacting theory, however, the right-hand side of eq. (344) will not be zero.

We will also need the hermitian conjugate of eq. (344), which (after some slight rearranging) reads

$$b_1(+\infty) - b_1(-\infty)$$
$$= i\int d^3p\ f_1(\mathbf{p}) \int d^4x\ e^{-ipx}\ \overline{u}_{s_1}(\mathbf{p})(-i\not\partial + m)\Psi(x)\ , \tag{346}$$

and the analogous formulae for the d operators,

$$d_1^\dagger(-\infty) - d_1^\dagger(+\infty)$$
$$= -i\int d^3p\ f_1(\mathbf{p}) \int d^4x\ e^{ipx}\ \overline{v}_{s_1}(\mathbf{p})(-i\not\partial + m)\Psi(x)\ , \tag{347}$$

$$d_1(+\infty) - d_1(-\infty)$$
$$= -i\int d^3p\ f_1(\mathbf{p}) \int d^4x\ \overline{\Psi}(x)(+i\overleftarrow{\not\partial} + m)v_{s_1}(\mathbf{p})e^{-ipx}\ . \tag{348}$$

Let us now return to the scattering amplitude we were considering,

$$\langle f|i\rangle = \langle 0|b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle \ . \tag{349}$$

Note that the operators are in time order. Thus, if we feel like it, we can put in a *time-ordering symbol* without changing anything:

$$\langle f|i\rangle = \langle 0|\,\mathrm{T}\, b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle \ . \tag{350}$$

The symbol T means the product of operators to its right is to be ordered, not as written, but with operators at later times to the left of those at earlier times. However, *there is an extra minus sign if this rearrangement involves an odd number of exchanges of these anticommuting operators.*

Now let us use eqs. (344) and (346) in eq. (350). The time-ordering symbol automatically moves all $b_{i'}(-\infty)$'s to the right, where they annihilate $|0\rangle$. Similarly, all $b_i^\dagger(+\infty)$'s move to the left, where they annihilate $\langle 0|$.

The wave packets no longer play a key role, and we can take the $\sigma \to 0$ limit in eq. (341), so that $f_1(\mathbf{p}) = \delta^3(\mathbf{p}-\mathbf{p}_1)$. The initial and final states now have a delta-function normalization, the multiparticle generalization of eq. (339). We are left with the *Lehmann-Symanzik-Zimmerman reduction formula* for spin-one-half particles,

$$\begin{aligned}
\langle f|i\rangle = i^4 \int & d^4x_1\, d^4x_2\, d^4x_{1'}\, d^4x_{2'} \\
& \times e^{-ip_1'x_1'}\, [\overline{u}_{s_{1'}}(\mathbf{p}_{1'})(-i\partial\!\!\!/_{1'} + m)]_{\alpha_{1'}} \\
& \times e^{-ip_2'x_2'}\, [\overline{u}_{s_{2'}}(\mathbf{p}_{2'})(-i\partial\!\!\!/_{2'} + m)]_{\alpha_{2'}} \\
& \times \langle 0|\,\mathrm{T}\,\Psi_{\alpha_{2'}}(x_{2'})\Psi_{\alpha_{1'}}(x_{1'})\overline{\Psi}_{\alpha_1}(x_1)\overline{\Psi}_{\alpha_2}(x_2)|0\rangle \\
& \times [(+i\overleftarrow{\partial\!\!\!/}_1 + m)u_{s_1}(\mathbf{p}_1)]_{\alpha_1}\, e^{ip_1x_1} \\
& \times [(+i\overleftarrow{\partial\!\!\!/}_2 + m)u_{s_2}(\mathbf{p}_2)]_{\alpha_2}\, e^{ip_2x_2} \ .
\end{aligned} \tag{351}$$

The generalization of the LSZ formula to other processes should be clear; insert a time-ordering symbol, and make the following replacements:

$$b_s^\dagger(\mathbf{p})_{\mathrm{in}} \to +i\int d^4x\, \overline{\Psi}(x)(+i\overleftarrow{\partial\!\!\!/} + m)u_s(\mathbf{p})\, e^{+ipx} \ , \tag{352}$$

$$b_s(\mathbf{p})_{\rm out} \to +i \int d^4x \; e^{-ipx} \, \overline{u}_s(\mathbf{p})(-i\partial\!\!\!/ + m)\Psi(x) \; , \tag{353}$$

$$d_s^\dagger(\mathbf{p})_{\rm in} \to -i \int d^4x \; e^{+ipx} \, \overline{v}_s(\mathbf{p})(-i\partial\!\!\!/ + m)\Psi(x) \; , \tag{354}$$

$$d_s(\mathbf{p})_{\rm out} \to -i \int d^4x \; \overline{\Psi}(x)(+i\overleftarrow{\partial\!\!\!/} + m)v_s(\mathbf{p}) \, e^{-ipx} \; , \tag{355}$$

where we have used the subscripts "in" and "out" to denote $t \to -\infty$ and $t \to +\infty$, respectively.

All of this holds for a Majorana field as well. In that case, $d_s(\mathbf{p}) = b_s(\mathbf{p})$, and we can use *either* eq. (352) *or* eq. (354) for the incoming particles, and *either* eq. (353) *or* eq. (355) for the outgoing particles, whichever is more convenient. The Majorana condition $\overline{\Psi} = \Psi^{\rm T}\mathcal{C}$ guarantees that the results will be equivalent.

As in the case of a scalar field, we cheated a little in our derivation of the LSZ formula, because we assumed that the creation operators of *free* field theory would work comparably in the *interacting* theory. After performing an analysis that is entirely analogous to what we did for the scalar in section 5, we come to the same conclusion: the LSZ formula holds provided the field is properly normalized. For a Dirac field, we must require

$$\langle 0|\Psi(x)|0\rangle = 0 \; , \tag{356}$$

$$\langle p,s,+|\Psi(x)|0\rangle = 0 \; , \tag{357}$$

$$\langle p,s,-|\Psi(x)|0\rangle = v_s(p)e^{-ipx} \; , \tag{358}$$

$$\langle p,s,+|\overline{\Psi}(x)|0\rangle = \overline{u}_s(p)e^{-ipx} \; , \tag{359}$$

$$\langle p,s,-|\overline{\Psi}(x)|0\rangle = 0 \; , \tag{360}$$

where $\langle 0|0\rangle = 1$, and the one-particle states are normalized according to eq. (339).

The zeros on the right-hand sides of eqs. (357) and (360) are required by charge conservation. To see this, start with $[Q, \Psi(x)] = -\Psi(x)$, take the matrix elements indicated, and use $Q|0\rangle = 0$ and $Q|p,s,\pm\rangle = \pm|p,s,\pm\rangle$.

The zero on the right-hand side of eq. (356) is required by Lorentz invariance. To see this, start with $[M^{\mu\nu}, \Psi(0)] = S^{\mu\nu}\Psi(0)$, and take the expectation value in the vacuum state $|0\rangle$. If $|0\rangle$ is Lorentz invariant (as we will assume), then it is annihilated by the Lorentz generators $M^{\mu\nu}$, which means that we must have $S^{\mu\nu}\langle 0|\Psi(0)|0\rangle = 0$; this is possible for all μ and ν only if $\langle 0|\Psi(0)|0\rangle = 0$, which (by translation invariance) is possible only if $\langle 0|\Psi(x)|0\rangle = 0$.

The right-hand sides of eqs. (358) and (359) are similarly fixed by Lorentz invariance: only the overall scale might be different in an interacting theory. However, the LSZ formula is correctly normalized if and only if eqs. (358) and (359) hold as written. We will enforce this by rescaling (or, one might say, *renormalizing*) $\Psi(x)$ by an overall constant. This is just a change of the name of the operator of interest, and does not affect the physics. However, the rescaled $\Psi(x)$ will obey eqs. (358) and (359). (These two equations are related by charge conjugation, and so actually constitute only one condition on Ψ.)

For a Majorana field, there is no conserved charge, and we have

$$\langle 0|\Psi(x)|0\rangle = 0 \ , \tag{361}$$

$$\langle p,s|\Psi(x)|0\rangle = v_s(p)e^{-ipx} \ , \tag{362}$$

$$\langle p,s|\overline{\Psi}(x)|0\rangle = \overline{u}_s(p)e^{-ipx} \ , \tag{363}$$

instead of eqs. (356–360).

The renormalization of Ψ necessitates including appropriate Z factors in the lagrangian. Consider, for example,

$$Ł = iZ\overline{\Psi}\partial\!\!\!/\Psi - Z_m m\overline{\Psi}\Psi - \tfrac{1}{4}Z_g g(\overline{\Psi}\Psi)^2 \ , \tag{364}$$

where Ψ is a Dirac field, and g is a coupling constant. We must choose the three constants Z, Z_m, and Z_g so that the following three conditions are satisfied: m is the mass of a single particle; g is fixed by some appropriate scattering cross section; and eq. (358) and is obeyed. [Eq. (359) then follows by charge conjugation.]

Next, we must develop the tools needed to compute the correlation functions $\langle 0|\mathrm{T}\Psi_{\alpha_{1'}}(x_{1'})\ldots\overline{\Psi}_{\alpha_1}(x_1)\ldots|0\rangle$ in an interacting quantum field theory.

42: The Free Fermion Propagator

Prerequisite: 39

Consider a free Dirac field

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp}\, \Big[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \Big] \;, \tag{365}$$

$$\overline{\Psi}(y) = \sum_{s'=\pm} \int \widetilde{dp}'\, \Big[b_{s'}^\dagger(\mathbf{p}') \overline{u}_{s'}(\mathbf{p}') e^{-ip'y} + d_{s'}(\mathbf{p}') \overline{v}_{s'}(\mathbf{p}') e^{ip'y} \Big] \;, \tag{366}$$

where

$$b_s(\mathbf{p})|0\rangle = d_s(\mathbf{p})|0\rangle = 0 \;, \tag{367}$$

and

$$\{ b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}') \} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')\, 2\omega \delta_{ss'} \;, \tag{368}$$

$$\{ d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}') \} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')\, 2\omega \delta_{ss'} \;, \tag{369}$$

and all the other possible anticommutators between b and d operators (and their hermitian conjugates) vanish.

We wish to compute the Feynman propagator

$$S(x-y)_{\alpha\beta} \equiv i\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle \;, \tag{370}$$

where T denotes the time-ordered product,

$$\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y) \equiv \theta(x^0-y^0)\Psi_\alpha(x)\overline{\Psi}_\beta(y) - \theta(y^0-x^0)\overline{\Psi}_\beta(y)\Psi_\alpha(x) \;, \tag{371}$$

and $\theta(t)$ is the unit step function. Note the minus sign in the second term; this is needed because $\Psi_\alpha(x)\overline{\Psi}_\beta(y) = -\overline{\Psi}_\beta(y)\Psi_\alpha(x)$ when $x^0 \neq y^0$.

We can now compute $\langle 0|\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle$ and $\langle 0|\overline{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle$ by inserting eqs. (365) and (366), and then using eqs. (367–369). We get

$$
\begin{aligned}
&\langle 0|\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle \\
&\qquad = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}{}'\, e^{ipx}\, e^{-ip'y}\, u_s(\mathbf{p})_\alpha \overline{u}_{s'}(\mathbf{p}')_\beta\, \langle 0|b_s(\mathbf{p})b^\dagger_{s'}(\mathbf{p}')|0\rangle \\
&\qquad = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}{}'\, e^{ipx}\, e^{-ip'y}\, u_s(\mathbf{p})_\alpha \overline{u}_{s'}(\mathbf{p}')_\beta\, (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'} \\
&\qquad = \sum_{s} \int \widetilde{dp}\, e^{ip(x-y)}\, u_s(\mathbf{p})_\alpha \overline{u}_s(\mathbf{p})_\beta \\
&\qquad = \int \widetilde{dp}\, e^{ip(x-y)}\, (-\not{p}+m)_{\alpha\beta}\ .
\end{aligned}
\tag{372}
$$

To get the last line, we used a result from section 38. Similarly,

$$
\begin{aligned}
&\langle 0|\overline{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle \\
&\qquad = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}{}'\, e^{-ipx}\, e^{ip'y}\, v_s(\mathbf{p})_\alpha \overline{v}_{s'}(\mathbf{p}')_\beta\, \langle 0|d_{s'}(\mathbf{p}')d^\dagger_{s}(\mathbf{p})|0\rangle \\
&\qquad = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}{}'\, e^{-ipx}\, e^{ip'y}\, v_s(\mathbf{p})_\alpha \overline{v}_{s'}(\mathbf{p}')_\beta\, (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'} \\
&\qquad = \sum_{s} \int \widetilde{dp}\, e^{-ip(x-y)}\, v_s(\mathbf{p})_\alpha \overline{v}_s(\mathbf{p})_\beta \\
&\qquad = \int \widetilde{dp}\, e^{-ip(x-y)}\, (-\not{p}-m)_{\alpha\beta}\ .
\end{aligned}
\tag{373}
$$

We can combine eqs. (372) and (373) into a compact formula for the time-ordered product by means of the identity

$$
\begin{aligned}
\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}f(p)}{p^2+m^2-i\epsilon} = {}& i\theta(x^0-y^0)\int \widetilde{dp}\, e^{ip(x-y)}\, f(p) \\
&+ i\theta(y^0-x^0)\int \widetilde{dp}\, e^{-ip(x-y)}\, f(-p)\ ,
\end{aligned}
\tag{374}
$$

where $f(p)$ is a polynomial in p; the derivation of eq. (374) was sketched in section 8. We get

$$
\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not{p}+m)_{\alpha\beta}}{p^2+m^2-i\epsilon}\ , \tag{375}
$$

and so

$$S(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not{p}+m)_{\alpha\beta}}{p^2+m^2-i\epsilon}\ . \tag{376}$$

Note that $S(x-y)$ is a Green's function for the Dirac wave operator:

$$\begin{aligned}(-i\not{\partial}_x+m)_{\alpha\beta}S(x-y)_{\beta\gamma} &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(\not{p}+m)_{\alpha\beta}(-\not{p}+m)_{\beta\gamma}}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(p^2+m^2)\delta_{\alpha\gamma}}{p^2+m^2-i\epsilon} \\ &= \delta^4(x-y)\delta_{\alpha\gamma}\ . \end{aligned} \tag{377}$$

Similarly,

$$\begin{aligned}S(x-y)_{\alpha\beta}(+i\overleftarrow{\not{\partial}}_y+m)_{\beta\gamma} &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not{p}+m)_{\alpha\beta}(\not{p}+m)_{\beta\gamma}}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(p^2+m^2)\delta_{\alpha\gamma}}{p^2+m^2-i\epsilon} \\ &= \delta^4(x-y)\delta_{\alpha\gamma}\ . \end{aligned} \tag{378}$$

We can also consider $\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)|0\rangle$ and $\langle 0|\mathrm{T}\overline{\Psi}_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle$, but it is easy to see that now there is no way to pair up a b with a $b^\dagger$ or a d with a $d^\dagger$, and so

$$\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)|0\rangle = 0\ , \tag{379}$$

$$\langle 0|\mathrm{T}\overline{\Psi}_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = 0\ . \tag{380}$$

Next, consider a Majorana field

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\, \Big[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\Big]\ , \tag{381}$$

$$\overline{\Psi}(y) = \sum_{s'=\pm}\int \widetilde{dp}'\, \Big[b_{s'}^\dagger(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'y} + b_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'y}\Big]\ . \tag{382}$$

It is easy to see that $\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle$ is the same as it is in the Dirac case; the only difference in the calculation is that we would have b and $b^\dagger$ in place of d and $d^\dagger$ in the second line of eq. (373), and this does not change the final result. Thus,

$$i\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = S(x-y)_{\alpha\beta}\ , \tag{383}$$

where $S(x-y)$ is given by eq. (376).

However, eqs. (379) and (380) no longer hold for a Majorana field. Instead, the Majorana condition $\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$, which can be rewritten as $\Psi^{\mathrm{T}} = \overline{\Psi}\mathcal{C}^{-1}$, implies

$$\begin{aligned} i\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)|0\rangle &= i\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\gamma(y)|0\rangle(\mathcal{C}^{-1})_{\gamma\beta} \\ &= [S(x-y)\mathcal{C}^{-1}]_{\alpha\beta}\ . \end{aligned} \tag{384}$$

Similarly, using $\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{-1}$, we can write the Majorana condition as $\overline{\Psi}^{\mathrm{T}} = \mathcal{C}^{-1}\Psi$, and so

$$\begin{aligned} i\langle 0|\mathrm{T}\overline{\Psi}_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle &= i(\mathcal{C}^{-1})_{\alpha\gamma}\langle 0|\mathrm{T}\Psi_\gamma(x)\overline{\Psi}_\beta(y)|0\rangle \\ &= [\mathcal{C}^{-1}S(x-y)]_{\alpha\beta}\ . \end{aligned} \tag{385}$$

Of course, $\mathcal{C}^{-1} = -\mathcal{C}$, but it will prove more convenient to leave eqs. (384) and (385) as they are.

We can also consider the vacuum expectation value of a time-ordered product of more than two fields. In the Dirac case, we must have an equal number of Ψ's and $\overline{\Psi}$'s to get a nonzero result; and then, the Ψ's and $\overline{\Psi}$'s must pair up to form propagators. There is an extra minus sign if the ordering of the fields in their pairs is an odd permutation of the original ordering. For example,

$$\begin{aligned} i^2\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)\Psi_\gamma(z)\overline{\Psi}_\delta(w)|0\rangle = &+ S(x-y)_{\alpha\beta}\,S(z-w)_{\gamma\delta} \\ &- S(x-w)_{\alpha\delta}\,S(z-y)_{\gamma\beta}\ . \end{aligned} \tag{386}$$

In the Majorana case, we may as well let all the fields be Ψ's (since we can always replace a $\overline{\Psi}$ with $\Psi^{\mathrm{T}}\mathcal{C}$). Then we must pair them up in all possible ways. There is an extra minus sign if the ordering of the fields in their pairs is an odd permutation of the original ordering. For example,

$$i^2\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)\Psi_\gamma(z)\Psi_\delta(w)|0\rangle = +[S(x-y)\mathcal{C}^{-1}]_{\alpha\beta}\,[S(z-w)\mathcal{C}^{-1}]_{\gamma\delta}$$

$$
\begin{aligned}
&- [S(x-z)\mathcal{C}^{-1}]_{\alpha\gamma}\, [S(y-w)\mathcal{C}^{-1}]_{\beta\delta} \\
&+ [S(x-w)\mathcal{C}^{-1}]_{\alpha\delta}\, [S(y-z)\mathcal{C}^{-1}]_{\beta\gamma}\, .
\end{aligned}
\tag{387}
$$

Note that the ordering within a pair does not matter, since

$$
[S(x-y)\mathcal{C}^{-1}]_{\alpha\beta} = -[S(y-x)\mathcal{C}^{-1}]_{\beta\alpha}\, . \tag{388}
$$

This follows from anticommutation of the fields and eq. (384); it can also be proven directly using $\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -(\gamma^\mu)^{\mathrm{T}}$ and $\mathcal{C}^{-1} = \mathcal{C}^{\mathrm{T}} = -\mathcal{C}$.

43: The Path Integral for Fermion Fields

Prerequisite: 9, 42

We would like to write down a path integral formula for the vacuum-expectation value of a time-ordered product of free Dirac or Majorana fields. Recall that for a real scalar field with

$$
\begin{aligned}
\mathcal{L}_0 &= -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 \\
&= -\tfrac{1}{2}\varphi(-\partial^2+m^2)\varphi - \tfrac{1}{2}\partial_\mu(\varphi\partial^\mu\varphi) \,,
\end{aligned}
\tag{389}
$$

we have

$$
\langle 0|\mathrm{T}\varphi(x_1)\ldots|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta J(x_1)}\ldots Z_0(J)\Big|_{J=0} \,,
\tag{390}
$$

where

$$
Z_0(J) = \int \mathcal{D}\varphi \;\exp\Big[i\int d^4x\,(\mathcal{L}_0 + J\varphi)\Big] .
\tag{391}
$$

In this formula, we use the epsilon trick (see section 6) of replacing m^2 with $m^2 - i\epsilon$ to construct the vacuum as the initial and final state. Then we get

$$
Z_0(J) = \exp\Big[\frac{i}{2}\int d^4x\,d^4y\,J(x)\Delta(x-y)J(y)\Big] ,
\tag{392}
$$

where the Feynman propagator

$$
\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}
\tag{393}
$$

is the inverse of the Klein-Gordon wave operator:

$$
(-\partial_x^2 + m^2)\Delta(x-y) = \delta^4(x-y) \,.
\tag{394}
$$

For a complex scalar field with

$$\mathcal{L}_0 = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi$$
$$= -\varphi^\dagger(-\partial^2 + m^2)\varphi - \partial_\mu(\varphi^\dagger\partial^\mu\varphi) \;, \tag{395}$$

we have instead

$$\langle 0|\mathrm{T}\varphi(x_1)\ldots\varphi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta J^\dagger(x_1)}\ldots\frac{1}{i}\,\frac{\delta}{\delta J(y_1)}\ldots Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} \;, \tag{396}$$

where

$$Z_0(J^\dagger,J) = \int\mathcal{D}\varphi^\dagger\mathcal{D}\varphi\;\exp\Big[i\int d^4x\,(\mathcal{L}_0 + J^\dagger\varphi + \varphi^\dagger J)\Big]$$
$$= \exp\Big[i\int d^4x\,d^4y\;J^\dagger(x)\Delta(x-y)J(y)\Big] . \tag{397}$$

We treat J and $J^\dagger$ as independent variables when evaluating eq. (396).

In the case of a fermion field, we should have something similar, except that we need to account for the extra minus signs from anticommutation. For this to work out, a functional derivative with respect to an anticommuting variable must itself be treated as anticommuting. Thus if we define an anticommuting source $\eta(x)$ for a Dirac field, we can write

$$\frac{\delta}{\delta\eta(x)}\int d^4y\,\Big[\overline{\eta}(y)\Psi(y) + \overline{\Psi}(y)\eta(y)\Big] = -\overline{\Psi}(x) \;, \tag{398}$$
$$\frac{\delta}{\delta\overline{\eta}(x)}\int d^4y\,\Big[\overline{\eta}(y)\Psi(y) + \overline{\Psi}(y)\eta(y)\Big] = +\Psi(x) \;. \tag{399}$$

The minus sign in eq. (398) arises because the $\delta/\delta\eta$ must pass through $\overline{\Psi}$ before reaching η.

Thus, consider a free Dirac field with

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi$$
$$= -\overline{\Psi}(-i\partial\!\!\!/ + m)\Psi \;. \tag{400}$$

A natural guess for the appropriate path-integral formula, based on analogy with eq. (397), is

$$\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\ldots\overline{\Psi}_{\beta_1}(y_1)\ldots|0\rangle$$
$$= \frac{1}{i}\,\frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)}\;\ldots\;i\,\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\;\ldots\;Z_0(\overline{\eta},\eta)\Big|_{\eta=\overline{\eta}=0} \;, \tag{401}$$

where

$$Z_0(\overline{\eta},\eta) = \int \mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\;\exp\Bigl[i\int d^4x\,(Ł_0 + \overline{\eta}\Psi + \overline{\Psi}\eta)\Bigr]$$
$$= \exp\Bigl[i\int d^4x\,d^4y\;\overline{\eta}(x)S(x-y)\eta(y)\Bigr]\,, \tag{402}$$

and the Feynman propagator

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{(-\not{p}+m)e^{ip(x-y)}}{p^2+m^2-i\epsilon} \tag{403}$$

is the inverse of the Dirac wave operator:

$$(-i\not{\partial}_x + m)S(x-y) = \delta^4(x-y)\;. \tag{404}$$

Note that each $\delta/\delta\eta$ in eq. (401) comes with a factor of i rather than the usual $1/i$; this reflects the extra minus sign of eq. (398). We treat η and $\overline{\eta}$ as independent variables when evaluating eq. (401). It is straightforward to check (by working out a few examples) that eqs. (401–404) do indeed reproduce the result of section 42 for the vacuum expectation value of a time-ordered product of Dirac fields.

This is really all we need to know. Recall that, for a complex scalar field with interactions specified by $Ł_1(\varphi^\dagger,\varphi)$, we have

$$Z(J^\dagger,J)\;\propto\;\exp\Bigl[\,i\int d^4x\;Ł_1\Bigl(\frac{1}{i}\frac{\delta}{\delta J(x)},\frac{1}{i}\frac{\delta}{\delta J^\dagger(x)}\Bigr)\Bigr]\,Z_0(J^\dagger,J)\;, \tag{405}$$

where the overall normalization is fixed by $Z(0,0)=1$. Thus, for a Dirac field with interactions specified by $Ł_1(\overline{\Psi},\Psi)$, we have

$$Z(\overline{\eta},\eta)\;\propto\;\exp\Bigl[\,i\int d^4x\;Ł_1\Bigl(i\frac{\delta}{\delta\eta(x)},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\Bigr)\Bigr]\,Z_0(\overline{\eta},\eta)\;, \tag{406}$$

where again the overall normalization is fixed by $Z(0,0)=1$. Vacuum expectation values of time-ordered products of Dirac fields in an interacting theory will now be given by eq. (401), but with $Z_0(\overline{\eta},\eta)$ replaced by $Z(\overline{\eta},\eta)$. Then, just as for a scalar field, this will lead to a Feynman-diagram expansion for $Z(\overline{\eta},\eta)$. There are two extra complications: we must keep track of the spinor indices, and we must keep track of the extra minus signs from anticommutation. Both tasks are straightforward; we will take them up in section 45.

Next, let us consider a Majorana field with

$$\begin{aligned} Ł_0 &= \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\partial\!\!\!/\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi \\ &= -\tfrac{1}{2}\Psi^{\mathrm{T}}\mathcal{C}(-i\partial\!\!\!/ + m)\Psi\ . \end{aligned} \tag{407}$$

A natural guess for the appropriate path-integral formula, based on analogy with eq. (390), is

$$\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\ldots|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta\eta_{\alpha_1}(x_1)}\ \ldots\ Z_0(\eta)\Big|_{\eta=0}\ , \tag{408}$$

where

$$\begin{aligned} Z_0(\eta) &= \int\mathcal{D}\Psi\ \exp\Bigl[i\int d^4x\,(Ł_0 + \eta^{\mathrm{T}}\Psi)\Bigr] \\ &= \exp\Bigl[-\frac{i}{2}\int d^4x\,d^4y\ \eta^{\mathrm{T}}(x)S(x-y)\mathcal{C}^{-1}\eta(y)\Bigr]\,. \end{aligned} \tag{409}$$

The Feynman propagator $S(x-y)\mathcal{C}^{-1}$ is the inverse of the Majorana wave operator $\mathcal{C}(-i\partial\!\!\!/ + m)$:

$$\mathcal{C}(-i\partial\!\!\!/_x + m)S(x-y)\mathcal{C}^{-1} = \delta^4(x-y)\ . \tag{410}$$

It is straightforward to check (by working out a few examples) that eqs. (408–410) do indeed reproduce the result of section 42 for the vacuum expectation value of a time-ordered product of Majorana fields. The extra minus sign in eq. (409), as compared with eq. (402), arises because all functional derivative in eq. (408) are accompanied by $1/i$, rather than half by $1/i$ and half by i, as in eq. (401).

44: Formal Development of Fermionic Path Integrals

Prerequisite: 43

In section 43, we formally defined the fermionic path integral for a free Dirac field Ψ via

$$\begin{aligned} Z_0(\overline{\eta},\eta) &= \int \mathcal{D}\Psi\, \mathcal{D}\overline{\Psi}\ \exp\Big[i \int d^4x\, \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \overline{\eta}\Psi + \overline{\Psi}\eta \Big] \\ &= \exp\Big[i \int d^4x\, d^4y\ \overline{\eta}(x) S(x-y) \eta(y) \Big] , \end{aligned} \tag{411}$$

where the Feynman propagator $S(x-y)$ is the inverse of the Dirac wave operator:

$$(-i\partial\!\!\!/_x + m) S(x-y) = \delta^4(x-y) \ . \tag{412}$$

We would like to find a mathematical framework that allows us to derive this formula, rather than postulating it by analogy.

Consider a set of *anticommuting numbers* or *Grassmann variables* ψ_i that obey

$$\{\psi_i, \psi_j\} = 0 \ , \tag{413}$$

where $i = 1, \ldots, n$. Let us begin with the very simplest case of $n = 1$, and thus a single anticommuting number ψ that obeys $\psi^2 = 0$. We can define a function $f(\psi)$ of such an object via a Taylor expansion; because $\psi^2 = 0$, this expansion ends with the second term:

$$f(\psi) = a + \psi b \ . \tag{414}$$

The reason for writing the coefficient b to the right of the variable ψ will become clear in a moment.

Next we would like to define the derivative of $f(\psi)$ with respect to ψ. Before we can do so, we must decide if $f(\psi)$ itself is to be commuting or anticommuting; generally we will be interested in functions that are themselves commuting. In this case, a in eq. (414) should be treated as an ordinary commuting number, but b should be treated as an anticommuting number: $\{b,b\} = \{b,\psi\} = 0$. In this case, $f(\psi) = a + \psi b = a - b\psi$.

Now we can define two kinds of derivatives. The *left derivative* of $f(\psi)$ with respect to ψ is given by the coefficient of ψ when $f(\psi)$ is written with the ψ always on the far left:

$$\partial_\psi f(\psi) = +b \; . \tag{415}$$

Similarly, the *right derivative* of $f(\psi)$ with respect to ψ is given by the coefficient of ψ when $f(\psi)$ is written with the ψ always on the far right:

$$f(\psi)\overleftarrow{\partial}_\psi = -b \; . \tag{416}$$

Generally, when we write a derivative with respect to a Grassmann variable, we mean the left derivative. However, in section 37, when we wrote the canonical momentum for a fermionic field ψ as $\pi = \partial Ł/\partial(\partial_0\psi)$, we actually meant the right derivative. (This is a standard, though rarely stated, convention.) Correspondingly, we wrote the hamiltonian density as $\mathcal{H} = \pi\partial_0\psi - Ł$, with the $\partial_0\psi$ to the right of π.

Finally, we would like to define a definite integral, analogous to integrating a real variable x from minus to plus infinity. The key features of such an integral over x (when it converges) are linearity,

$$\int_{-\infty}^{+\infty} dx\, cf(x) = c\int_{-\infty}^{+\infty} dx\, f(x) \; , \tag{417}$$

and invariance under shifts of the dependent variable x by a constant:

$$\int_{-\infty}^{+\infty} dx\, f(x+a) = \int_{-\infty}^{+\infty} dx\, f(x) \; . \tag{418}$$

Up to an overall numerical factor that is the same for every $f(\psi)$, the only possible nontrivial definition of $\int d\psi\, f(\psi)$ that is both linear and shift invariant is

$$\int d\psi\, f(\psi) = b \; . \tag{419}$$

Now let us generalize this to $n > 1$. We have

$$f(\psi) = a + \psi_i b_i + \tfrac{1}{2}\psi_{i_1}\psi_{i_2}c_{i_1 i_2} + \ldots + \tfrac{1}{n!}\psi_{i_1}\ldots\psi_{i_n}d_{i_1\ldots i_n} \ , \tag{420}$$

where the indices are implicitly summed. Here we have written the coefficients to the right of the variables to facilitate left-differentiation. These coefficients are completely antisymmetric on exchange of any two indices. The left derivative of $f(\psi)$ with respect to ψ_j is

$$\tfrac{\partial}{\partial\psi_j}f(\psi) = b_j + \psi_i c_{ji} + \ldots + \tfrac{1}{(n-1)!}\psi_{i_2}\ldots\psi_{i_n}d_{ji_2\ldots i_n} \ . \tag{421}$$

Next we would like to find a linear, shift-invariant definition of the integral of $f(\psi)$. Note that the antisymmetry of the coefficients implies that

$$d_{i_1\ldots i_n} = d\,\varepsilon_{i_1\ldots i_n} \ . \tag{422}$$

where d is a just a number (ordinary if f is commuting and n is even, Grassmann if f is commuting and n is odd, etc.), and $\varepsilon_{i_1\ldots i_n}$ is the completely antisymmetric Levi-Civita symbol with $\varepsilon_{1\ldots n} = +1$. This number d is a candidate (in fact, up to an overall numerical factor, the only candidate!) for the integral of $f(\psi)$:

$$\int d^n\psi\, f(\psi) = d \ . \tag{423}$$

Although eq. (423) really tells us everything we need to know about $\int d^n\psi$, we can, if we like, write $d^n\psi = d\psi_n \ldots d\psi_1$ (note the backwards ordering), and treat the individual differentials as anticommuting: $\{d\psi_i, d\psi_j\} = 0$, $\{d\psi_i, \psi_j\} = 0$. Then we take $\int d\psi_i = 0$ and $\int d\psi_i\,\psi_j = \delta_{ij}$ as our basic formulae, and use them to derive eq. (423).

Let us work out some consequences of eq. (423). Consider what happens if we make a linear change of variable,

$$\psi_i = J_{ij}\psi'_j \ , \tag{424}$$

where J_{ji} is a matrix of commuting numbers (and therefore can be written on either the left or right of ψ'_j). We now have

$$f(\psi) = a + \ldots + \tfrac{1}{n!}(J_{i_1 j_1}\psi'_{j_1})\ldots(J_{i_n j_n}\psi'_{j_n})\varepsilon_{i_1\ldots i_n}d \ . \tag{425}$$

Next we use

$$\varepsilon_{i_1 \ldots i_n} J_{i_1 j_1} \ldots J_{i_n j_n} = (\det J)\varepsilon_{j_1 \ldots j_n} \ , \tag{426}$$

which holds for any $n \times n$ matrix J, to get

$$f(\psi) = a + \ldots + \tfrac{1}{n!}\psi'_{i_1} \ldots \psi'_{i_n} \varepsilon_{i_1 \ldots i_n} (\det J) d \ . \tag{427}$$

If we now integrate $f(\psi)$ over $d^n\psi'$, eq. (423) tells us that the result is $(\det J)d$. Thus,

$$\int d^n\psi \, f(\psi) = (\det J)^{-1} \int d^n\psi' \, f(\psi) \ . \tag{428}$$

Recall that, for integrals over commuting real numbers x_i with $x_i = J_{ij}x'_j$, we have instead

$$\int d^n x \, f(x) = (\det J)^{+1} \int d^n x' \, f(x) \ . \tag{429}$$

Note the opposite sign on the power of the determinant.

Now consider a quadratic form $\psi^{\mathrm{T}} M \psi = \psi_i M_{ij} \psi_j$, where M is an antisymmetric matrix of commuting numbers (possibly complex). Let's evaluate the gaussian integral $\int d^n\psi \ \exp(\frac{1}{2}\psi^{\mathrm{T}} M \psi)$. For example, for $n = 2$, we have

$$M = \begin{pmatrix} 0 & +m \\ -m & 0 \end{pmatrix} , \tag{430}$$

and $\psi^{\mathrm{T}} M \psi = 2m\psi_1\psi_2$. Thus $\exp(\frac{1}{2}\psi^{\mathrm{T}} M \psi) = 1 + m\psi_1\psi_2$, and so

$$\int d^n\psi \ \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi) = m \ . \tag{431}$$

For larger n, we use the fact that a complex antisymmetric matrix can be brought to a block-diagonal form via

$$U^{\mathrm{T}} M U = \begin{pmatrix} 0 & +m_1 & \\ -m_1 & 0 & \\ & & \ddots \end{pmatrix} , \tag{432}$$

where U is a unitary matrix, and each m_I is real and positive. (If n is odd there is a final row and column of all zeroes; from here on, we assume n is even.) We can now let $\psi_i = U_{ij}\psi'_j$; then, we have

$$\int d^n\psi \ \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi) = (\det U)^{-1} \prod_{I=1}^{n/2} \int d^2\psi_I \ \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M_I \psi) \ , \tag{433}$$

where M_I represents one of the 2×2 blocks in eq. (432). Each of these two-dimensional integrals can be evaluated using eq. (431), and so

$$\int d^n\psi \, \exp(\tfrac{1}{2}\psi^{\mathrm{T}}M\psi) = (\det U)^{-1} \prod_{I=1}^{n/2} m_I \, . \tag{434}$$

Taking the determinant of eq. (432), we get

$$(\det U)^2 (\det M) = \prod_{I=1}^{n/2} m_I^2 \, . \tag{435}$$

We can therefore rewrite the right-hand side of eq. (434) as

$$\int d^n\psi \, \exp(\tfrac{1}{2}\psi^{\mathrm{T}}M\psi) = (\det M)^{1/2} \, . \tag{436}$$

In this form, there is a sign ambiguity associated with the square root; it is resolved by eq. (434). However, the overall sign (more generally, any overall numerical factor) will never be of concern to us, so we can use eq. (436) without worrying about the correct branch of the square root.

It is instructive to compare eq. (436) with the corresponding gaussian integral for commuting real numbers,

$$\int d^n x \, \exp(-\tfrac{1}{2}x^{\mathrm{T}}Mx) = (2\pi)^{n/2}(\det M)^{-1/2} \, . \tag{437}$$

Here M is a complex symmetric matrix. Again, note the opposite sign on the power of the determinant.

Now let us introduce the notion of *complex* Grassmann variables via

$$\begin{aligned} \chi &\equiv \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \, , \\ \bar\chi &\equiv \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \, . \end{aligned} \tag{438}$$

We can invert this to get

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \bar\chi \\ \chi \end{pmatrix} . \tag{439}$$

The determinant of this transformation matrix is $-i$, and so

$$d^2\psi = d\psi_2 d\psi_1 = (-i)^{-1} d\chi \, d\bar\chi \, . \tag{440}$$

Also, $\psi_1\psi_2 = -i\bar{\chi}\chi$. Thus we have

$$\int d\chi\, d\bar{\chi}\, \bar{\chi}\chi = (-i)(-i)^{-1} \int d\psi_2 d\psi_1\, \psi_1\psi_2 = 1\ . \tag{441}$$

Thus, if we have a function

$$f(\chi,\bar{\chi}) = a + \chi b + \bar{\chi} c + \bar{\chi}\chi d\ , \tag{442}$$

its integral is

$$\int d\chi\, d\bar{\chi}\, f(\chi,\bar{\chi}) = d\ . \tag{443}$$

In particular,

$$\int d\chi\, d\bar{\chi}\ \exp(m\bar{\chi}\chi) = m\ . \tag{444}$$

Let us now consider n complex Grassmann variables χ_i and their complex conjugates, $\bar{\chi}_i$. We define

$$d^n\chi\, d^n\bar{\chi} \equiv d\chi_n d\bar{\chi}_n \ldots d\chi_1 d\bar{\chi}_1\ . \tag{445}$$

Then under a change of variable, $\chi_i = J_{ij}\chi'_j$ and $\bar{\chi}_i = K_{ij}\bar{\chi}'_j$, we have

$$d^n\chi\, d^n\bar{\chi} = (\det J)^{-1}(\det K)^{-1}\, d^n\chi'\, d^n\bar{\chi}'\ . \tag{446}$$

Note that we need not require $K_{ij} = J^*_{ij}$, because, as far as the integral is concerned, it is does not matter whether or not $\bar{\chi}_i$ is the complex conjugate of χ_i.

We now have enough information to evaluate $\int d^n\chi\, d^n\bar{\chi}\ \exp(\chi^\dagger M\chi)$, where M is a general complex matrix. We make the change of variable $\chi = U\chi'$ and $\chi^\dagger = \chi'^\dagger V$, where U and V are unitary matrices with the property that VMU is diagonal with positive real entries m_i. Then we get

$$\begin{aligned}\int d^n\chi\, d^n\bar{\chi}\ \exp(\chi^\dagger M\chi) &= (\det U)^{-1}(\det V)^{-1}\prod_{i=1}^{n}\int d\chi_i d\bar{\chi}_i\ \exp(m_i\bar{\chi}_i\chi_i)\\ &= (\det U)^{-1}(\det V)^{-1}\prod_{i=1}^{n} m_i\\ &= \det M\ . \end{aligned}\tag{447}$$

This can be compared to the analogous integral for commuting complex variables $z_i = (x_i + iy_i)/\sqrt{2}$ and $\bar{z} = (x_i - iy_i)/\sqrt{2}$, with $d^n z\, d^n \bar{z} = d^n x\, d^n y$, namely

$$\int d^n z\, d^n \bar{z}\, \exp(-z^\dagger M z) = (2\pi)^n (\det M)^{-1} . \tag{448}$$

We can now generalize eqs. (436) and (447) by shifting the integration variables, and using shift invariance of the integrals. Thus, by making the replacement $\psi \to \psi - M^{-1}\eta$ in eq. (436), we get

$$\int d^n \psi\, \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi + \eta^{\mathrm{T}}\psi) = (\det M)^{1/2} \exp(\tfrac{1}{2}\eta^{\mathrm{T}} M^{-1}\eta) . \tag{449}$$

(In verifying this, remember that M and its inverse are both antisymmetric.) Similarly, by making the replacements $\chi \to \chi - M^{-1}\eta$ and $\chi^\dagger \to \chi^\dagger - \eta^\dagger M^{-1}$ in eq. (447), we get

$$\int d^n \chi\, d^n \bar{\chi}\, \exp(\chi^\dagger M \chi + \eta^\dagger \chi + \chi^\dagger \eta) = (\det M) \exp(-\eta^\dagger M^{-1} \eta) . \tag{450}$$

We can now see that eq. (411) is simply a particular case of eq. (450), with the index on the complex Grassmann variable generalized to include both the ordinary spin index α and the continuous spacetime argument x of the field $\Psi_\alpha(x)$. Similarly, eq. (409) for the path integral for a free Majorana field is simply a particular case of eq. (449). In both cases, the determinant factors are constants (that is, independent of the fields and sources) that we simply absorb into the overall normalization of the path integral. We will meet determinants that cannot be so neatly absorbed in sections 53 and 70.

45: The Feynman Rules for Dirac Fields and Yukawa Theory

Prerequisite: 10, 13, 41, 43

In this section we will derive the Feynman rules for *Yukawa theory*, a theory with a Dirac field Ψ (with mass m) and a real scalar field φ (with mass M), interacting via

$$Ł_1 = g\varphi\overline{\Psi}\Psi \ , \tag{451}$$

where g is a coupling constant. In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

In four spacetime dimensions, φ has mass dimension $[\varphi] = 1$ and Ψ has mass dimension $[\Psi] = \frac{3}{2}$; thus the coupling constant g is dimensionless: $[g] = 0$. As discussed in section 13, this is generally the most interesting situation.

Note that $Ł_1$ is invariant under the U(1) transformation $\Psi \to e^{-i\alpha}\Psi$, as is the free Dirac lagrangian. Thus, the corresponding Noether current $\overline{\Psi}\gamma^\mu\Psi$ is still conserved, and the associated charge Q (which counts the number of b-type particles minus the number of d-type particles) is constant in time.

We can think of Q as electric charge, and identify the b-type particle as the electron e^-, and the d-type particle as the positron e^+. The scalar particle is electrically neutral (and could, for example, be thought of as the Higgs boson; see Part III).

We now use the general result of sections 9 and 43 to write

$$Z(\overline{\eta},\eta,J) \propto \exp\left[ig\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\right)\right]Z_0(\overline{\eta},\eta)Z_0(J) \ , \tag{452}$$

where

$$Z_0(\overline{\eta},\eta) = \exp\left[i\int d^4x\, d^4y\ \overline{\eta}(x)S(x-y)\eta(y)\right], \tag{453}$$

$$Z_0(J) = \exp\Big[\frac{i}{2}\int d^4x\,d^4y\;J(x)\Delta(x-y)J(y)\Big]\,, \tag{454}$$

and

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{(-\not{p}+m)e^{ip(x-y)}}{p^2+m^2-i\epsilon}\,, \tag{455}$$

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+M^2-i\epsilon} \tag{456}$$

are the appropriate Feynman propagators for the corresponding free fields. We impose the normalization $Z(0,0,0)=1$, and write

$$Z(\overline{\eta},\eta,J) = \exp[iW(\overline{\eta},\eta,J)]\,. \tag{457}$$

Then $iW(\overline{\eta},\eta,J)$ can be expressed as a series of connected Feynman diagrams with sources.

We use a dashed line to stand for the scalar propagator $\frac{1}{i}\Delta(x-y)$, and a solid line to stand for the fermion propagator $\frac{1}{i}S(x-y)$. The only allowed vertex joins two solid lines and one dashed line; the associated vertex factor is ig. The blob at the end of a dashed line stands for the φ source $i\int d^4x\,J(x)$, and the blob at the end of a solid line for either the Ψ source $i\int d^4x\,\overline{\eta}(x)$, or the $\overline{\Psi}$ source $i\int d^4x\,\eta(x)$. To tell which is which, we adopt the "arrow rule" of problem 9.3: the blob stands for $i\int d^4x\,\eta(x)$ if the arrow on the attached line points *away* from the blob, and the blob stands for $i\int d^4x\,\overline{\eta}(x)$ if the arrow on the attached line points *towards* the blob. Because $Ł_1$ involves one Ψ and one $\overline{\Psi}$, we also have the rule that, at each vertex, *one arrow must point towards the vertex, and one away.* The first few tree diagrams that contribute to $iW(\overline{\eta},\eta,J)$ are shown in fig. (1). We omit tadpole diagrams; as in φ^3 theory, these can be cancelled by shifting the φ field, or, equivalently, adding a term linear in φ to Ł. The LSZ formula is valid only after all tadpole diagrams have been cancelled in this way.

The spin indices on the fermionic sources and propagators are all contracted in the obvious way. For example, the complete expression corresponding to fig. (1)(b) is

$$\text{Fig. (1)(b)} = i^3\Big(\tfrac{1}{i}\Big)^3(ig)\int d^4x\,d^4y\,d^4z\,d^4w$$

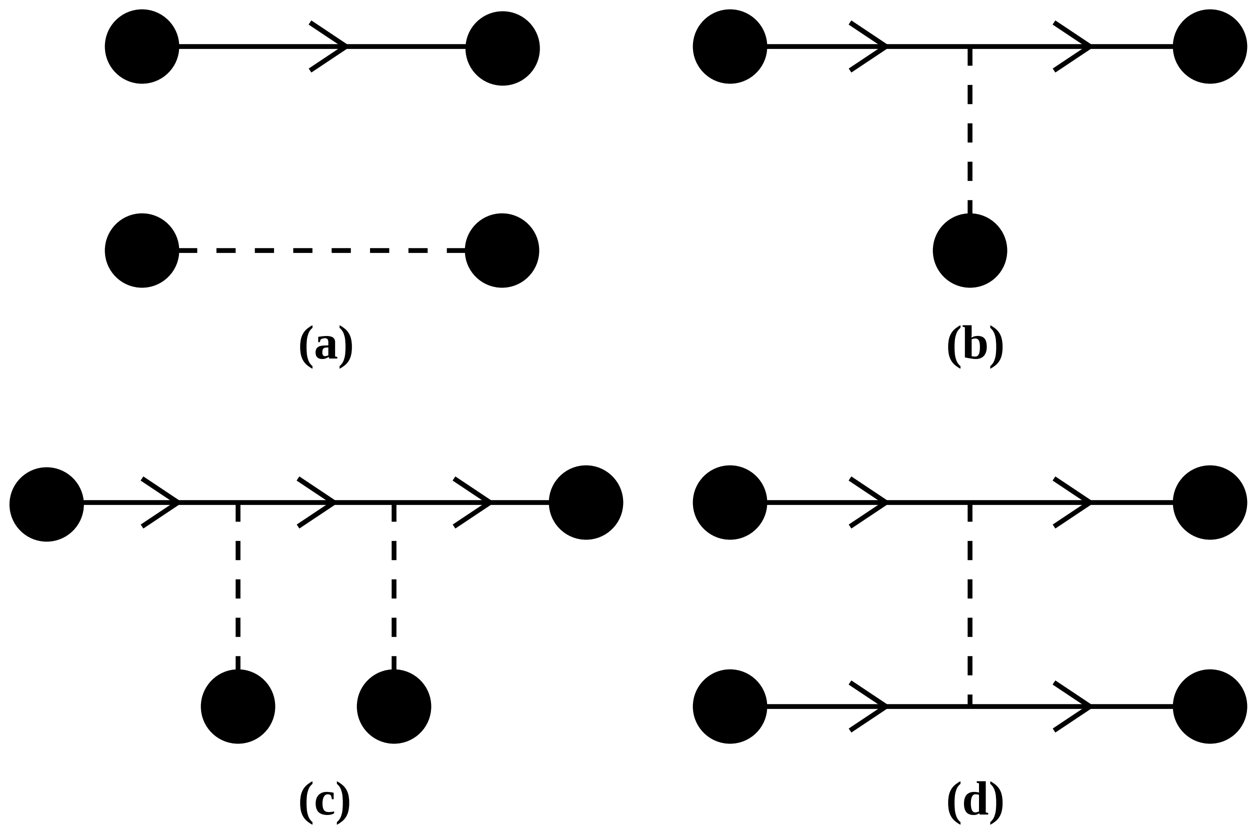

Figure 1: Tree contributions to $iW(\overline{\eta}, \eta, J)$ with four or fewer sources.

$$\times \Big[\overline{\eta}(x)S(x-y)S(y-z)\eta(z)\Big]$$
$$\times \Delta(y-w)J(w)\ . \tag{458}$$

Our main purpose in this section is to compute the tree-level amplitudes for various two-body elastic scattering processes, such as $e^-\varphi \to e^-\varphi$ and $e^+e^- \to \varphi\varphi$; for these, we will need to evaluate the tree-level contributions to connected correlation functions of the form $\langle 0|\mathrm{T}\Psi\overline{\Psi}\varphi\varphi|0\rangle_{\mathrm{C}}$. Other processes of interest include $e^-e^- \to e^-e^-$ and $e^+e^- \to e^+e^-$; for these, we will need to evaluate the tree-level contributions to connected correlation functions of the form $\langle 0|\mathrm{T}\Psi\overline{\Psi}\Psi\overline{\Psi}|0\rangle_{\mathrm{C}}$.

For $\langle 0|\mathrm{T}\Psi\overline{\Psi}\varphi\varphi|0\rangle_{\mathrm{C}}$, the relevant tree-level contribution to $iW(\overline{\eta},\eta,J)$ is given by fig. (1)(d). We have

$$\begin{aligned}
&\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)\varphi(z_1)\varphi(z_2)|0\rangle_{\mathrm{C}} \\
&\quad= \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\,i\frac{\delta}{\delta\eta_\beta(y)}\,\frac{1}{i}\frac{\delta}{\delta J(z_1)}\,\frac{1}{i}\frac{\delta}{\delta J(z_2)}iW(\overline{\eta},\eta,J)\Big|_{\overline{\eta}=\eta=J=0} \\
&\quad= \left(\tfrac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2 \\
&\qquad\qquad \times [S(x-w_1)S(w_1-w_2)S(w_2-y)]_{\alpha\beta} \\
&\qquad\qquad \times \Delta(z_1-w_1)\Delta(z_2-w_2) \\
&\qquad + \Big(z_1 \leftrightarrow z_2\Big) + O(g^4)\ .
\end{aligned} \tag{459}$$

The corresponding diagrams, with sources removed, are shown in fig. (2).

For $\langle 0|\mathrm{T}\Psi\overline{\Psi}\Psi\overline{\Psi}|0\rangle_{\mathrm{C}}$, the relevant tree-level contribution to $iW(\overline{\eta},\eta,J)$ is given by fig. (1)(c), which has a symmetry factor $S=2$. We have

$$\begin{aligned}
&\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\overline{\Psi}_{\beta_1}(y_1)\Psi_{\alpha_2}(x_2)\overline{\Psi}_{\beta_2}(y_2)|0\rangle_{\mathrm{C}} \\
&\quad= \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)}\,i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\,\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_2}(x_2)}\,i\frac{\delta}{\delta\eta_{\beta_2}(y_2)}iW(\overline{\eta},\eta,J)\Big|_{\overline{\eta}=\eta=J=0}
\end{aligned} \tag{460}$$

The two η derivatives can act on the two η's in the diagram in two different ways; ditto for the two $\overline{\eta}$ derivatives. This results in four different terms, but two of them are algebraic duplicates of the other two; this duplication

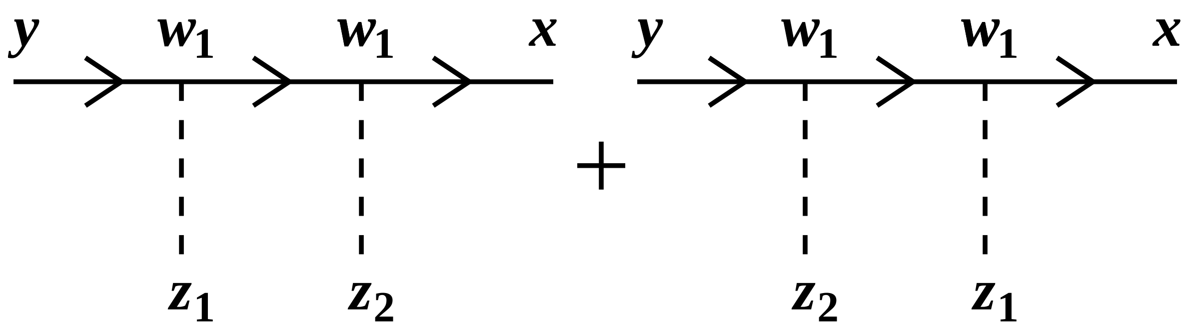

Figure 2: Diagrams corresponding to eq. (459).

cancels the symmetry factor (which is a general result). We get

$$\begin{aligned}
&\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\overline{\Psi}_{\beta_1}(y_1)\Psi_{\alpha_2}(x_2)\overline{\Psi}_{\beta_2}(y_2)|0\rangle_{\mathrm{C}} \\
&\quad = \left(\tfrac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2 \\
&\qquad\qquad\qquad \times [S(x_1{-}w_1)S(w_1{-}y_1)]_{\alpha_1\beta_1} \\
&\qquad\qquad\qquad \times \Delta(w_1{-}w_2) \\
&\qquad\qquad\qquad \times [S(x_2{-}w_2)S(w_2{-}y_2)]_{\alpha_2\beta_2} \\
&\qquad\qquad - \Big((y_1,\beta_1) \leftrightarrow (y_2,\beta_2)\Big) + O(g^4)\;. \qquad (461)
\end{aligned}$$

The corresponding diagrams, with sources removed, are shown in fig. (3). Note, however, that we now have a *relative minus sign* between the two diagrams, due to the anticommutation of the derivatives with respect to $\overline{\eta}$. The general rule is this: there is a relative minus sign between any two diagrams that are identical *except for a swap of the position and spin labels between two external fermion lines.*

Let us now consider a particular scattering process: $e^-\varphi \to e^-\varphi$. The scattering amplitude is

$$\langle f|i\rangle = \langle 0|\,\mathrm{T}\,a(\mathbf{k}')_{\mathrm{out}} b_{s'}(\mathbf{p}')_{\mathrm{out}} b_s^\dagger(\mathbf{p})_{\mathrm{in}} a^\dagger(\mathbf{k})_{\mathrm{in}}\,|0\rangle\;. \qquad (462)$$

Next we use the replacements

$$b_s^\dagger(\mathbf{p})_{\mathrm{in}} \to i\int d^4y\; \overline{\Psi}(y)(+i\overleftarrow{\partial\!\!\!/} + m)u_s(\mathbf{p})\, e^{+ipy}\;, \qquad (463)$$

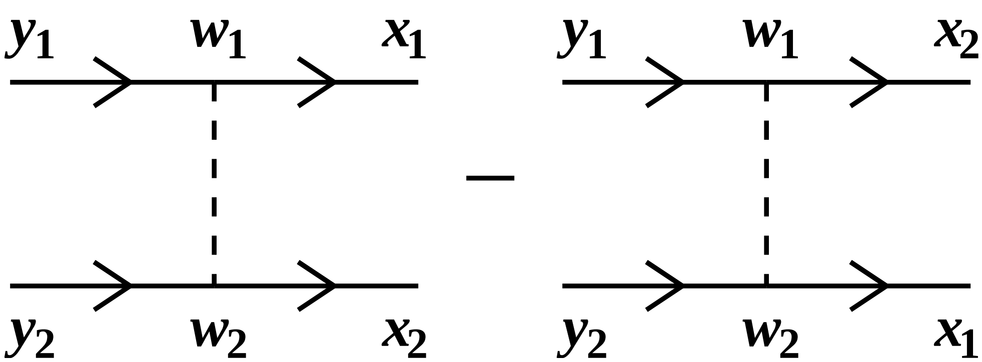

Figure 3: Diagrams corresponding to eq. (461).

$$b_{s'}(\mathbf{p}')_{\rm out} \to i\int d^4x\; e^{-ipx}\;\overline{u}_{s'}(\mathbf{p}')(-i\partial\!\!\!/ + m)\Psi(x)\;, \tag{464}$$

$$a^\dagger(\mathbf{k})_{\rm in} \to i\int d^4z_1\; e^{+ikz_1}(-\partial^2 + m^2)\varphi(z_1)\;, \tag{465}$$

$$a(\mathbf{k}')_{\rm out} \to i\int d^4z_2\; e^{-ik'z_2}(-\partial^2 + m^2)\varphi(z_2)\;. \tag{466}$$

We substitute these into eq. (462), and then use eq. (459). The wave operators (either Klein-Gordon or Dirac) act on the external propagators, and convert them to delta functions. After using eqs. (455) and (456) for the internal propagators, all dependence on the various spacetime coordinates is in the form of plane-wave factors, as in section 10. Integrating over the internal coordinates then generates delta functions that conserve four-momentum at each vertex. The only new feature arises from the spinor factors $u_s(\mathbf{p})$ and $\overline{u}_{s'}(\mathbf{p}')$. We find that $u_s(\mathbf{p})$ is associated with the external fermion line whose arrow points *towards* the vertex, and that $\overline{u}_{s'}(\mathbf{p}')$ is associated with the external fermion line whose arrow points *away* from the vertex. We can therefore draw the momentum-space diagrams of fig. (4), and write down the associated tree-level expression for the $e^-\varphi \to e^-\varphi$ scattering amplitude,

$$i\mathcal{T}_{e^-\varphi\to e^-\varphi} = \tfrac{1}{i}(ig)^2\,\overline{u}_{s'}(\mathbf{p}')\left[\frac{-p\!\!\!/ - k\!\!\!/ + m}{-s+m^2} + \frac{-p\!\!\!/ + k\!\!\!/' + m}{-u+m^2}\right]u_s(\mathbf{p})\;, \tag{467}$$

where $s = -(p+k)^2$ and $u = -(p-k')^2$. (We can safely ignore the $i\epsilon$'s in the propagators, because their denominators cannot vanish for any physically

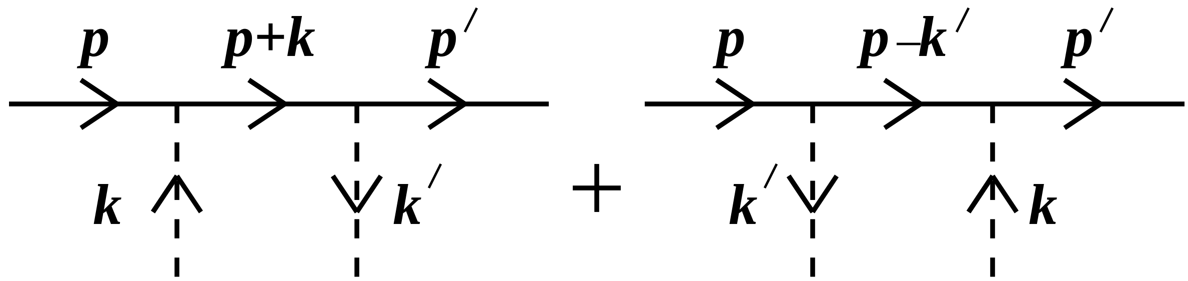

Figure 4: Diagrams for $e^-\varphi \to e^-\varphi$, corresponding to eq. (467).

allowed values of s and u.) We will see how to turn this into a more useful expression in section 46.

Next consider the process $e^+\varphi \to e^+\varphi$. We now have

$$\langle f|i\rangle = \langle 0|\,\mathrm{T}\, a(\mathbf{k}')_{\rm out} d_{s'}(\mathbf{p}')_{\rm out} d^\dagger_s(\mathbf{p})_{\rm in} a^\dagger(\mathbf{k})_{\rm in}\,|0\rangle\ . \tag{468}$$

The relevant replacements are

$$d^\dagger_s(\mathbf{p})_{\rm in} \to -i\int d^4x\ e^{+ipx}\ \overline{v}_s(\mathbf{p})(-i\partial\!\!\!/ + m)\Psi(x)\ , \tag{469}$$

$$d_{s'}(\mathbf{p}')_{\rm out} \to -i\int d^4y\ \overline{\Psi}(y)(+i\overleftarrow{\partial\!\!\!/} + m)v_{s'}(\mathbf{p}')\, e^{-ipy}\ , \tag{470}$$

$$a^\dagger(\mathbf{k})_{\rm in} \to i\int d^4z_1\ e^{+ikz_1}(-\partial^2 + m^2)\varphi(z_1)\ , \tag{471}$$

$$a(\mathbf{k}')_{\rm out} \to i\int d^4z_2\ e^{-ikz_2}(-\partial^2 + m^2)\varphi(z_2)\ . \tag{472}$$

We substitute these into eq. (468), and then use eq. (459). This ultimately leads to the momentum-space Feynman diagrams of fig. (5). Note that we must now label the external fermion lines with *minus* their four-momenta; this is characteristic of d-type particles. (The same phenomenon occurs for a complex scalar field; see problem 10.1.) Regarding the spinor factors, we find that $-\overline{v}_s(\mathbf{p})$ is associated with the external fermion line whose arrow points *away* from the vertex, and $-v_{s'}(\mathbf{p}')$ with the external fermion line whose arrow points *towards* the vertex. The minus signs attached to each v and $\overline{v}$ can be consistently dropped, however, as they only affect the overall sign

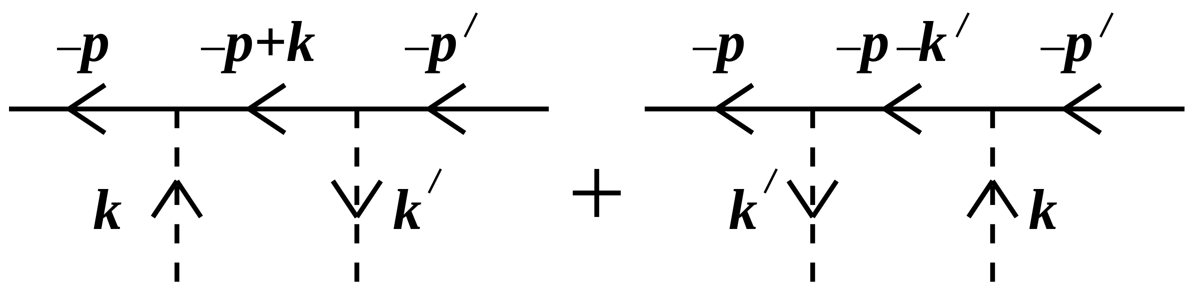

Figure 5: Diagrams for $e^+\varphi \to e^+\varphi$, corresponding to eq. (473).

of the amplitude (and not the relative signs among contributing diagrams). The tree-level expression for the $e^+\varphi \to e^+\varphi$ amplitude is then

$$i\mathcal{T}_{e^+\varphi\to e^+\varphi} = \tfrac{1}{i}(ig)^2\,\overline{v}_s(\mathbf{p})\left[\frac{\not{p}-\not{k}'+m}{-u+m^2}+\frac{\not{p}+\not{k}+m}{-s+m^2}\right]v_{s'}(\mathbf{p}')\;, \tag{473}$$

where again $s = -(p+k)^2$ and $u = -(p-k')^2$.

After working out a few more of these (you might try your hand at some of them before reading ahead), we can abstract the following set of Feynman rules.

1) For each *incoming electron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with the electron's four-momentum, p_i.

2) For each *outgoing electron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with the electron's four-momentum, p_i'.

3) For each *incoming positron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with *minus* the positron's four-momentum, $-p_i$.

4) For each *outgoing positron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with *minus* the positron's four-momentum, $-p_i'$.

5) For each *incoming scalar*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with the scalar's four-momentum, k_i.

6) For each *outgoing scalar*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with the scalar's four-momentum, k_i'.

7) The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one dashed line (whose arrow can point in either direction). Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent*.

8) Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9) The value of a diagram consists of the following factors:

for each incoming or outgoing scalar, 1;

for each incoming electron, $u_{s_i}(\mathbf{p}_i)$;

for each outgoing electron, $\overline{u}_{s_i'}(\mathbf{p}_i')$;

for each incoming positron, $\overline{v}_{s_i}(\mathbf{p}_i)$;

for each outgoing positron, $v_{s_i'}(\mathbf{p}_i')$;

for each vertex, ig;

for each internal scalar line, $-i/(k^2+M^2-i\epsilon)$,

where k is the four-momentum of that line;

for each internal fermion line, $-i(-\not{p}+m)/(p^2+m^2-i\epsilon)$,

where p is the four-momentum of that line.

10) Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$. Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any.

11) Two diagrams that are identical *except for the momentum and spin labels on two external fermion lines* that *have their arrows pointing in the same direction* (either both towards or both away from the vertex) have a relative minus sign.

12) The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of the contributing diagrams.

There are additional rules for counterterms and loops, but we will postpone those to section 51.

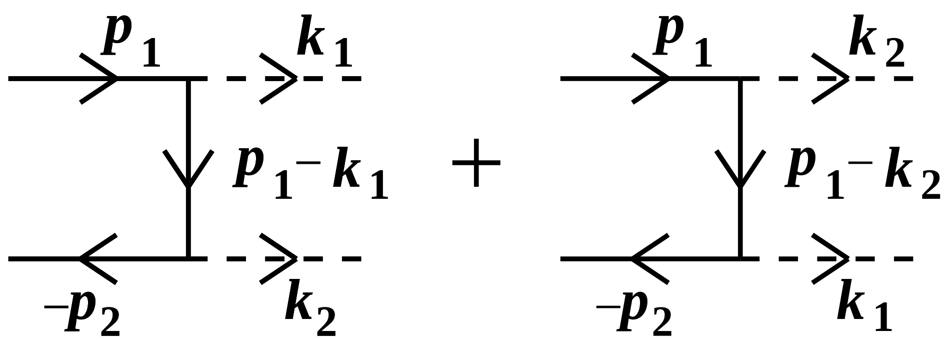

Figure 6: Diagrams for $e^+e^- \to \varphi\varphi$, corresponding to eq. (474).

Let us apply these rules to $e^+e^- \to \varphi\varphi$. Let the initial electron and positron have four-momenta p_1 and p_2, respectively, and the two final scalars have four-momenta k_1' and k_2'. The relevant diagrams are shown in fig. (6), and the result is

$$i\mathcal{T}_{e^+e^-\to\varphi\varphi} = \tfrac{1}{i}(ig)^2\,\overline{v}_{s_2}(\mathbf{p}_2)\left[\frac{-\not{p}_1 + \not{k}_1' + m}{-t + m^2} + \frac{-\not{p}_1 + \not{k}_2' + m}{-u + m^2}\right]u_{s_1}(\mathbf{p}_1)\;, \tag{474}$$

where $t = -(p_1 - k_1')^2$ and $u = -(p_1 - k_2')^2$.

Next, consider $e^-e^- \to e^-e^-$. Let the initial electrons have four-momenta p_1 and p_2, and the final electrons have four-momenta p_1' and p_2'. The relevant diagrams are shown in fig. (7). It is clear that they are identical except for the labels on the two external fermion lines that have arrows pointing away from their vertices. Thus, according to rule #11, these diagrams have a relative minus sign. (Which diagram gets the extra minus sign is a matter of convention, and is physically irrelevant.) Thus the result is

$$i\mathcal{T}_{e^-e^-\to e^-e^-} = \tfrac{1}{i}(ig)^2\left[\frac{(\overline{u}_1'u_1)(\overline{u}_2'u_2)}{-t + M^2} - \frac{(\overline{u}_2'u_1)(\overline{u}_1'u_2)}{-u + M^2}\right], \tag{475}$$

where u_1 is short for $u_{s_1}(\mathbf{p}_1)$, etc., and $t = -(p_1 - p_1')^2$, $u = -(p_1 - p_2')^2$.

One more: $e^+e^- \to e^+e^-$. Let the initial electron and positron have four-momenta p_1 and p_2, respectively, and the final electron and positron have four-momenta p_1' and p_2', respectively. The relevant diagrams are shown in fig. (8). If we redraw them in the topologically equivalent manner shown in

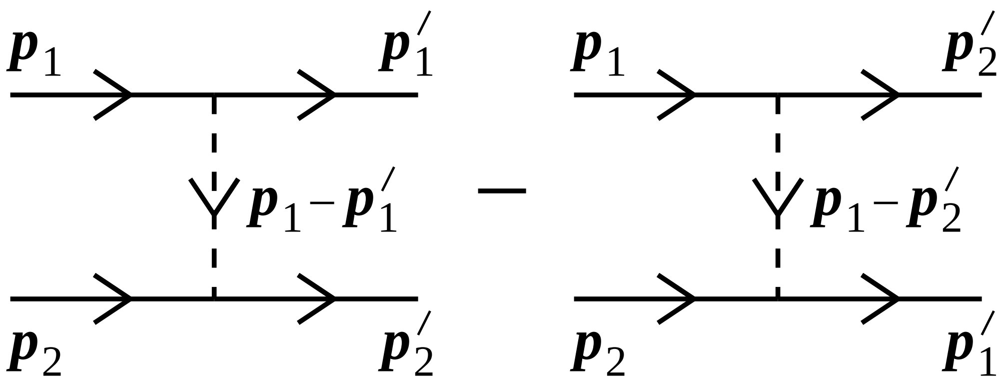

Figure 7: Diagrams for $e^-e^- \to e^-e^-$, corresponding to eq. (475).

fig. (9), then it becomes clear that they are identical except for the labels on the two external fermion lines that have arrows pointing away from their vertices. Thus, according to rule #11, these diagrams have a relative minus sign. (Which diagram gets the extra minus sign is a matter of convention, and is physically irrelevant.) Thus the result is

$$i\mathcal{T}_{e^+e^-\to e^+e^-} = \tfrac{1}{i}(ig)^2 \left[\frac{(\overline{u}_1' u_1)(\overline{v}_2 v_2')}{-t + M^2} - \frac{(\overline{v}_2 u_1)(\overline{u}_1' v_2')}{-u + M^2} \right], \tag{476}$$

where $s = -(p_1 + p_2)^2$ and $t = -(p_1 - p_1')^2$.

Problems

45.1a) Determine how $\varphi(x)$ must transform under parity, time reversal, and charge conjugation in order for these to all be symmetries of the theory. (Prerequisite: 39)

b) Same question, but with the interaction given by $\text{Ł}_1 = ig\varphi\overline{\Psi}\gamma_5\Psi$ instead of eq. (451).

45.2) Use the Feynman rules to write down (at tree level) $i\mathcal{T}$ for the processes $e^+e^+ \to e^+e^+$ and $\varphi\varphi \to e^+e^-$.

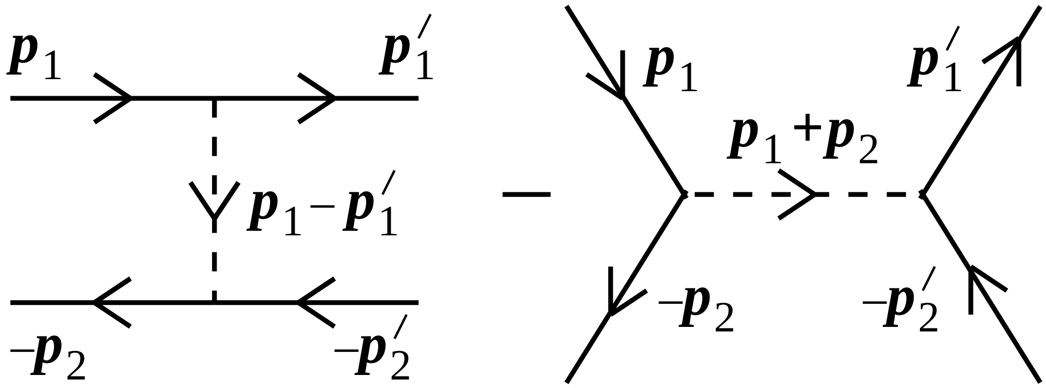

Figure 8: Diagrams for $e^+e^- \to e^+e^-$, corresponding to eq. (476).

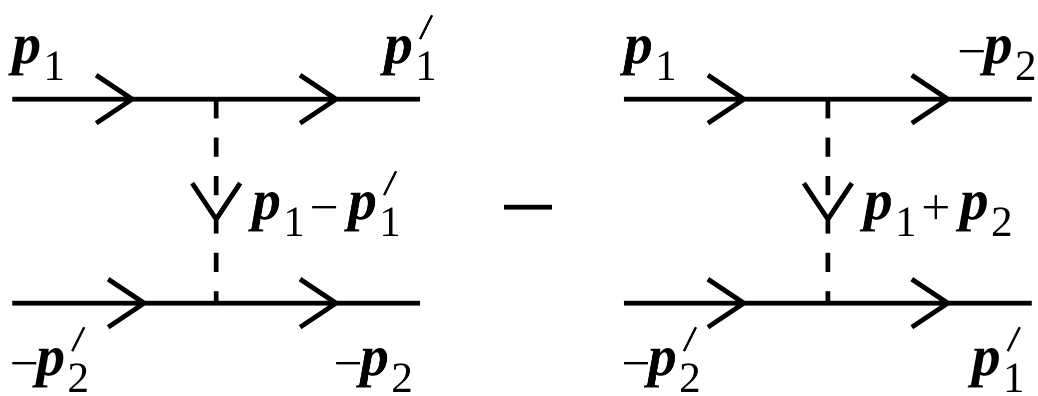

Figure 9: Same as fig. (8), but with the diagrams redrawn to show more clearly that, according to rule #11, they have a relative minus sign.

46: Spin Sums

Prerequisite: 45

In the last section, we calculated various tree-level scattering amplitudes in Yukawa theory. For example, for $e^-\varphi \to e^-\varphi$ we found

$$\mathcal{T} = g^2\,\overline{u}_{s'}(\mathbf{p}')\left[\frac{-\not{p}-\not{k}+m}{-s+m^2}+\frac{-\not{p}+\not{k}'+m}{-u+m^2}\right]u_s(\mathbf{p})\;, \tag{477}$$

where $s = -(p+k)^2$ and $u = -(p-k')^2$. In order to compute the corresponding cross section, we must evaluate $|\mathcal{T}|^2 = \mathcal{T}\mathcal{T}^*$. We begin by simplifying eq. (477) a little; we use $(\not{p}+m)u_s(\mathbf{p}) = 0$ to replace the $-\not{p}$ in each numerator with m. We then abbreviate eq. (477) as

$$\mathcal{T} = \overline{u}'Au\;, \tag{478}$$

where

$$A \equiv g^2\left[\frac{-\not{k}+2m}{m^2-s}+\frac{\not{k}'+2m}{m^2-u}\right]. \tag{479}$$

Then we have

$$\mathcal{T}^* = \overline{\mathcal{T}} = \overline{\overline{u}'Au} = \overline{u}\overline{A}u'\;, \tag{480}$$

where in general $\overline{A} \equiv \beta A^\dagger\beta$, and, for the particular A of eq. (479), $\overline{A} = A$. Thus we have

$$\begin{aligned}
|\mathcal{T}|^2 &= (\overline{u}'Au)(\overline{u}Au') \\
&= \sum_{\alpha\beta\gamma\delta}\overline{u}'_\alpha A_{\alpha\beta}u_\beta\overline{u}_\gamma A_{\gamma\delta}u'_\delta \\
&= \sum_{\alpha\beta\gamma\delta}u'_\delta\overline{u}'_\alpha A_{\alpha\beta}u_\beta\overline{u}_\gamma A_{\gamma\delta} \\
&= \mathrm{Tr}\Big[(u'\overline{u}')A(u\overline{u})A\Big]\;.
\end{aligned} \tag{481}$$

Next, we use a result from section 38:

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} + m) \; , \tag{482}$$

where $s = \pm$ tells us whether the spin is up or down along the spin quantization axis z. We then have

$$|\mathcal{T}|^2 = \tfrac{1}{4}\mathrm{Tr}\Big[(1 - s'\gamma_5 \not{z}')(-\not{p}' + m)A(1 - s\gamma_5 \not{z})(-\not{p} + m)A\Big] \; . \tag{483}$$

We now simply need to take traces of products of gamma matrices; we will work out the technology for this in the next section.

However, in practice, we are often not interested in (or are unable to easily measure or prepare) the spin states of the scattering particles. Thus, if we know that an electron with momentum p' landed in our detector, but know nothing about its spin, we should *sum* $|\mathcal{T}|^2$ over the two possible spin states of this outgoing electron. Similarly, if the spin state of the initial electron is not specially prepared for each scattering event, then we should *average* $|\mathcal{T}|^2$ over the two possible spin states of this initial electron. Then we can use

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m \tag{484}$$

in place of eq. (482).

Let us, then, take $|\mathcal{T}|^2$, sum over all final spins, and average over all initial spins, and call the result $\langle|\mathcal{T}|^2\rangle$. In the present case, we have

$$\begin{aligned} \langle|\mathcal{T}|^2\rangle &\equiv \tfrac{1}{2}\sum_{s,s'} |\mathcal{T}|^2 \\ &= \tfrac{1}{2}\mathrm{Tr}\Big[(-\not{p}' + m)A(-\not{p} + m)A\Big] \; , \end{aligned} \tag{485}$$

which is much less cumbersome than eq. (483).

Next let's try something a little harder, namely $e^+e^- \to e^+e^-$. We found in section 45 that

$$\mathcal{T} = g^2 \left[\frac{(\overline{u}_1' u_1)(\overline{v}_2 v_2')}{M^2 - t} - \frac{(\overline{v}_2 u_1)(\overline{u}_1' v_2')}{M^2 - s} \right] . \tag{486}$$

We then have

$$\overline{\mathcal{T}} = g^2 \left[\frac{(\overline{u}_1 u_1')(\overline{v}_2' v_2)}{M^2 - t} - \frac{(\overline{u}_1 v_2)(\overline{v}_2' u_1')}{M^2 - s} \right] . \tag{487}$$

When we multiply $\mathcal{T}$ by $\overline{\mathcal{T}}$, we will get four terms. We want to arrange the factors in each of them so that every u and every v stands just to the left of the corresponding $\overline{u}$ and $\overline{v}$. In this way, we get

$$
\begin{aligned}
|\mathcal{T}|^2 = \; & + \frac{g^4}{(M^2{-}t)^2} \, \mathrm{Tr}\Big[u_1\overline{u}_1 u_1'\overline{u}_1'\Big] \, \mathrm{Tr}\Big[v_2'\overline{v}_2' v_2\overline{v}_2\Big] \\
& + \frac{g^4}{(M^2{-}s)^2} \, \mathrm{Tr}\Big[u_1\overline{u}_1 v_2\overline{v}_2\Big] \, \mathrm{Tr}\Big[v_2'\overline{v}_2' u_1'\overline{u}_1'\Big] \\
& - \frac{g^4}{(M^2{-}t)(M^2{-}s)} \, \mathrm{Tr}\Big[u_1\overline{u}_1 v_2\overline{v}_2 v_2'\overline{v}_2' u_1'\overline{u}_1'\Big] \\
& - \frac{g^4}{(M^2{-}s)(M^2{-}t)} \, \mathrm{Tr}\Big[u_1\overline{u}_1 u_1'\overline{u}_1' v_2'\overline{v}_2' v_2\overline{v}_2\Big] \; . \qquad (488)
\end{aligned}
$$

Then we average over initial spins and sum over final spins, and use eq. (484) and

$$
\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m \; . \qquad (489)
$$

We then must evaluate traces of products of up to four gamma matrices.

47: Gamma Matrix Technology

Prerequisite: 36

In this section, we will learn some tricks for handling gamma matrices. We need the following information as a starting point:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \ , \tag{490}$$

$$\gamma_5^2 = 1 \ , \tag{491}$$

$$\{\gamma^\mu, \gamma_5\} = 0 \ , \tag{492}$$

$$\mathrm{Tr}\, 1 = 4 \ . \tag{493}$$

Now consider the trace of the product of n gamma matrices. We have

$$\begin{aligned} \mathrm{Tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] &= \mathrm{Tr}[\gamma_5^2 \gamma^{\mu_1} \gamma_5^2 \ldots \gamma_5^2 \gamma^{\mu_n}] \\ &= \mathrm{Tr}[(\gamma_5 \gamma^{\mu_1} \gamma_5) \ldots (\gamma_5 \gamma^{\mu_n} \gamma_5)] \\ &= \mathrm{Tr}[(-\gamma_5^2 \gamma^{\mu_1}) \ldots (-\gamma_5^2 \gamma^{\mu_n})] \\ &= (-1)^n\, \mathrm{Tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] \ . \end{aligned} \tag{494}$$

We used eq. (491) to get the first equality, the cyclic property of the trace for the second, eq. (492) for the third, and eq. (491) again for the fourth. If n is odd, eq. (494) tells us that this trace is equal to minus itself, and must therefore be zero:

$$\mathrm{Tr}[\,\text{odd \# of } \gamma^\mu\text{'s}\,] = 0 \ . \tag{495}$$

Similarly,

$$\mathrm{Tr}[\,\gamma_5\,(\,\text{odd \# of } \gamma^\mu\text{'s}\,)\,] = 0 \ . \tag{496}$$

Next, consider $\mathrm{Tr}[\gamma^\mu\gamma^\nu]$. We have

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\gamma^\nu\gamma^\mu] \\
&= \tfrac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \\
&= -g^{\mu\nu}\,\mathrm{Tr}\,1 \\
&= -4g^{\mu\nu}\ .
\end{aligned} \tag{497}$$

The first equality follows from the cyclic property of the trace, the second averages the left- and right-hand sides of the first, the third uses eq. (490), and the fourth uses eq. (493).

A slightly nicer way of expressing eq. (497) is to introduce two arbitrary four-vectors a^μ and b^μ, and write

$$\mathrm{Tr}[\not{a}\not{b}] = -4(ab)\ , \tag{498}$$

where $\not{a} = a_\mu\gamma^\mu$, $\not{b} = b_\mu\gamma^\mu$, and $(ab) = a^\mu b_\mu$.

Next consider $\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}]$. We evaluate this by moving $\not{a}$ to the right, using eq. (490), which is now more usefully written as

$$\not{a}\not{b} = -\not{b}\not{a} - 2(ab)\ . \tag{499}$$

Using this repeatedly, we have

$$\begin{aligned}
\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] &= -\mathrm{Tr}[\not{b}\not{a}\not{c}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}] \\
&= +\mathrm{Tr}[\not{b}\not{c}\not{a}\not{d}] + 2(ac)\mathrm{Tr}[\not{b}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}] \\
&= -\mathrm{Tr}[\not{b}\not{c}\not{d}\not{a}] - 2(ad)\mathrm{Tr}[\not{b}\not{c}] + 2(ac)\mathrm{Tr}[\not{b}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}]\ .
\end{aligned} \tag{500}$$

Now we note that the first term on the right-hand side of the last line is, by the cyclic property of the trace, actually equal to minus the left-hand side. We can then move this term to the left-hand side to get

$$2\,\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = \ -2(ad)\mathrm{Tr}[\not{b}\not{c}] + 2(ac)\mathrm{Tr}[\not{b}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}]\ . \tag{501}$$

Finally, we evaluate each $\mathrm{Tr}[\not{a}\not{b}]$ with eq. (498), and divide by two:

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4\Big[(ad)(bc) - (ac)(bd) + (ab)(cd)\Big]\ . \tag{502}$$

This is our final result for this trace.

Clearly, we can use the same technique to evaluate the trace of the product of any even number of gamma matrices.

Next, let's consider traces that involve γ_5's and γ^μ's. Since $\{\gamma_5, \gamma^\mu\} = 0$, we can always bring all the γ_5's together by moving them through the γ^μ's (generating minus signs as we go). Then, since $\gamma_5^2 = 1$, we end up with either one γ_5 or none. So we need only consider $\mathrm{Tr}[\gamma_5 \gamma^{\mu_1} \ldots \gamma^{\mu_n}]$. And, according to eq. (496), we need only be concerned with even n.

Recall that an explicit formula for γ_5 is

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{503}$$

Eq. (502) then implies

$$\mathrm{Tr}\,\gamma_5 = 0 \ . \tag{504}$$

Similarly, the six-matrix generalization of eq. (502) yields

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0 \ . \tag{505}$$

Finally, consider $\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]$. The only way to get a nonzero result is to have the four vector indices take on four different values. If we consider the special case $\mathrm{Tr}[\gamma_5\gamma^3\gamma^2\gamma^1\gamma^0]$, plug in eq. (503), and then use $(\gamma^i)^2 = -1$ and $(\gamma^0)^2 = 1$, we get $i(-1)^3\,\mathrm{Tr}\,1 = -4i$, or equivalently

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\varepsilon^{\mu\nu\rho\sigma} \ , \tag{506}$$

where $\varepsilon^{0123} = \varepsilon^{3210} = +1$.

Another category of gamma matrix combinations that we will eventually encounter is $\gamma^\mu \not{a} \ldots \gamma_\mu$. The simplest of these is

$$\begin{aligned} \gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu \\ &= \tfrac{1}{2} g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} \\ &= -g_{\mu\nu}g^{\mu\nu} \\ &= -d \ . \end{aligned} \tag{507}$$

To get the second equality, we used the fact that $g_{\mu\nu}$ is symmetric, and so only the symmetric part of $\gamma^\mu\gamma^\nu$ contributes. In the last line, d is the number of spacetime dimensions. Of course, our entire spinor formalism has been built around $d = 4$, but we will need formal results for $d = 4{-}\varepsilon$ when we dimensionally regulate loop diagrams involving fermions.

We move on to evaluate

$$\begin{aligned}
\gamma^\mu \not{a} \gamma_\mu &= \gamma^\mu(-\gamma_\mu \not{a} - 2a_\mu) \\
&= -\gamma^\mu \gamma_\mu \not{a} - 2\not{a} \\
&= (d{-}2)\not{a} \, .
\end{aligned} \tag{508}$$

We continue with

$$\begin{aligned}
\gamma^\mu \not{a} \not{b} \gamma_\mu &= (-\not{a}\gamma^\mu - 2a^\mu)(-\gamma_\mu \not{b} - 2b_\mu) \\
&= \not{a}\gamma^\mu\gamma_\mu \not{b} + 2\not{a}\not{b} + 2\not{a}\not{b} + 4(ab) \\
&= 4(ab) - (d{-}4)\not{a}\not{b} \, .
\end{aligned} \tag{509}$$

And finally,

$$\begin{aligned}
\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu &= (-\not{a}\gamma^\mu - 2a^\mu)\not{b}(-\gamma_\mu \not{c} - 2c_\mu) \\
&= \not{a}\gamma^\mu \not{b}\gamma_\mu \not{c} + 2\not{b}\not{a}\not{c} + 2\not{a}\not{c}\not{b} + 4(ac)\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2\not{b}\not{a}\not{c} + 2[\not{a}\not{c} + 2(ac)]\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2\not{b}\not{a}\not{c} - 2\not{c}\not{a}\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2[-\not{a}\not{b} - 2(ab)]\not{c} - 2\not{c}\not{a}\not{b} \\
&= (d{-}4)\not{a}\not{b}\not{c} - 4(ab)\not{c} - 2\not{c}\not{a}\not{b} \\
&= (d{-}4)\not{a}\not{b}\not{c} + 2\not{c}\,[-2(ab) - \not{a}\not{b}\,] \\
&= 2\not{c}\not{b}\not{a} + (d{-}4)\not{a}\not{b}\not{c} \, .
\end{aligned} \tag{510}$$

48: Spin-Averaged Cross Sections in Yukawa Theory

Prerequisite: 46, 47

In section 46, we computed $|\mathcal{T}|^2$ for (among other processes) $e^+e^- \to e^+e^-$. We take the incoming and outgoing electrons to have momenta p_1 and p_1', respectively, and the incoming and outgoing positrons to have momenta p_2 and p_2', respectively. We have $p_i^2 = p_i'^2 = -m^2$, where m is the electron (and positron) mass. The Mandelstam variables are

$$\begin{aligned} s &= -(p_1+p_2)^2 = -(p_1'+p_2')^2 \; , \\ t &= -(p_1-p_1')^2 = -(p_2-p_2')^2 \; , \\ u &= -(p_1-p_2')^2 = -(p_2-p_1')^2 \; , \end{aligned} \tag{511}$$

and they obey $s+t+u = 4m^2$. Our result was

$$|\mathcal{T}|^2 = g^4\left[\frac{\Phi_{ss}}{(M^2-s)^2} - \frac{\Phi_{st}+\Phi_{ts}}{(M^2-s)(M^2-t)} + \frac{\Phi_{tt}}{(M^2-t)^2}\right], \tag{512}$$

where M is the scalar mass, and

$$\begin{aligned} \Phi_{ss} &= \mathrm{Tr}\Big[u_1\overline{u}_1 v_2\overline{v}_2\Big]\,\mathrm{Tr}\Big[v_2'\overline{v}_2' u_1'\overline{u}_1'\Big] \; , \\ \Phi_{tt} &= \mathrm{Tr}\Big[u_1\overline{u}_1 u_1'\overline{u}_1'\Big]\,\mathrm{Tr}\Big[v_2'\overline{v}_2' v_2\overline{v}_2\Big] \; , \\ \Phi_{st} &= \mathrm{Tr}\Big[u_1\overline{u}_1 u_1'\overline{u}_1' v_2'\overline{v}_2' v_2\overline{v}_2\Big] \; , \\ \Phi_{ts} &= \mathrm{Tr}\Big[u_1\overline{u}_1 v_2\overline{v}_2 v_2'\overline{v}_2' u_1'\overline{u}_1'\Big] \; . \end{aligned} \tag{513}$$

Next, we average over the two initial spins and sum over the two final spins to get

$$\langle|\mathcal{T}|^2\rangle = \tfrac{1}{4}\textstyle\sum_{s_1,s_2,s_1',s_2'} |\mathcal{T}|^2 \; . \tag{514}$$

Then we use

$$\begin{aligned}\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= -\not{p} + m \;, \\ \sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= -\not{p} - m \;,\end{aligned} \tag{515}$$

to get

$$\langle\Phi_{ss}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1{+}m)(-\not{p}_2{-}m)\Big]\,\mathrm{Tr}\Big[(-\not{p}_2'{-}m)(-\not{p}_1'{+}m)\Big]\,, \tag{516}$$

$$\langle\Phi_{tt}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1{+}m)(-\not{p}_1'{+}m)\Big]\,\mathrm{Tr}\Big[(-\not{p}_2'{-}m)(-\not{p}_2{-}m)\Big]\,, \tag{517}$$

$$\langle\Phi_{st}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1{+}m)(-\not{p}_1'{+}m)(-\not{p}_2'{-}m)(-\not{p}_2{-}m)\Big]\,, \tag{518}$$

$$\langle\Phi_{ts}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1{+}m)(-\not{p}_2{-}m)(-\not{p}_2'{-}m)(-\not{p}_1'{+}m)\Big]\,. \tag{519}$$

It is now merely tedious to evaluate these traces with the technology of section 47.

For example,

$$\begin{aligned}\mathrm{Tr}\Big[(-\not{p}_1{+}m)(-\not{p}_2{-}m)\Big] &= \mathrm{Tr}[\not{p}_1\not{p}_2] - m^2\,\mathrm{Tr}\,1 \\ &= -4(p_1p_2) - 4m^2 \;,\end{aligned} \tag{520}$$

It is convenient to write four-vector products in terms of the Mandelstam variables. We have

$$\begin{aligned}p_1p_2 = p_1'p_2' &= -\tfrac{1}{2}(s-2m^2) \;, \\ p_1p_1' = p_2p_2' &= +\tfrac{1}{2}(t-2m^2) \;, \\ p_1p_2' = p_1'p_2 &= +\tfrac{1}{2}(u-2m^2) \;,\end{aligned} \tag{521}$$

and so

$$\mathrm{Tr}\Big[(-\not{p}_1{+}m)(-\not{p}_2{-}m)\Big] = 2s - 8m^2 \;. \tag{522}$$

Thus, we can easily work out eqs. (516) and (517):

$$\langle\Phi_{ss}\rangle = (s-4m^2)^2 \;, \tag{523}$$

$$\langle\Phi_{tt}\rangle = (t-4m^2)^2 \;. \tag{524}$$

Obviously, if we start with $\langle\Phi_{ss}\rangle$ and make the swap $s \leftrightarrow t$, we get $\langle\Phi_{tt}\rangle$. We could have anticipated this from eqs. (516) and (517): if we start with the right-hand side of eq. (516) and make the swap $p_2 \leftrightarrow -p_1'$, we get the right-hand side of eq. (517). But from eq. (521), we see that this momentum swap is equivalent to $s \leftrightarrow t$.

Let's move on to $\langle\Phi_{st}\rangle$ and $\langle\Phi_{ts}\rangle$. These two are also related by $p_2 \leftrightarrow -p_1'$, and so we only need to compute one of them. We have

$$\begin{aligned}
\langle\Phi_{st}\rangle &= \tfrac{1}{4}\mathrm{Tr}[\not{p}_1\not{p}_1'\not{p}_2'\not{p}_2] + \tfrac{1}{4}m^2\,\mathrm{Tr}[\not{p}_1\not{p}_1' - \not{p}_1\not{p}_2' - \not{p}_1\not{p}_2 - \not{p}_1'\not{p}_2' - \not{p}_1'\not{p}_2 + \not{p}_2'\not{p}_2] \\
&\quad + \tfrac{1}{4}m^4\,\mathrm{Tr}\,1 \\
&= (p_1p_1')(p_2p_2') - (p_1p_2')(p_2p_1') + (p_1p_2)(p_1'p_2') \\
&\quad - m^2[p_1p_1' - p_1p_2' - p_1p_2 - p_1'p_2' + p_1'p_2 + p_2p_2'] + m^4 \\
&= -\tfrac{1}{2}st + 2m^2u\;. \qquad (525)
\end{aligned}$$

To get the last line, we used eq. (521), and then simplified it as much as possible via $s+t+u = 4m^2$. Since our result is symmetric on $s \leftrightarrow t$, we have $\langle\Phi_{ts}\rangle = \langle\Phi_{st}\rangle$.

Putting all of this together, we get

$$\langle|\mathcal{T}|^2\rangle = g^4\left[\frac{(s-4m^2)^2}{(M^2-s)^2} + \frac{st-4m^2u}{(M^2-s)(M^2-t)} + \frac{(t-4m^2)^2}{(M^2-t)^2}\right]. \qquad (526)$$

This can then be converted to a differential cross section (in any frame) via the formulae of section 11.

Let's do one more: $e^-\varphi \to e^-\varphi$. We take the incoming and outgoing electrons to have momenta p and p', respectively, and the incoming and outgoing scalars to have momenta k and k', respectively. We then have $p^2 = p'^2 = -m^2$ and $k^2 = k'^2 = -M^2$. The Mandelstam variables are

$$\begin{aligned}
s &= -(p+k)^2 = -(p'+k')^2\;, \\
t &= -(p-p')^2 = -(k-k')^2\;, \\
u &= -(p-k')^2 = -(k-p')^2\;, \qquad (527)
\end{aligned}$$

and they obey $s+t+u = 2m^2+2M^2$. Our result in section 46 was

$$\langle|\mathcal{T}|^2\rangle = \tfrac{1}{2}\mathrm{Tr}\Big[A(-\not{p}+m)A(-\not{p}'+m)\Big]\;, \qquad (528)$$

where

$$A = g^2 \left[\frac{-\not{k} + 2m}{m^2 - s} + \frac{\not{k}' + 2m}{m^2 - u} \right] . \tag{529}$$

Thus we have

$$\langle |\mathcal{T}|^2 \rangle = g^4 \Biggl[\frac{\langle \Phi_{ss} \rangle}{(m^2 - s)^2} + \frac{\langle \Phi_{su} \rangle + \langle \Phi_{us} \rangle}{(m^2 - s)(m^2 - u)} + \frac{\langle \Phi_{uu} \rangle}{(m^2 - u)^2} \Biggr] , \tag{530}$$

where now

$$\langle \Phi_{ss} \rangle = \tfrac{1}{2} \mathrm{Tr} \Bigl[(-\not{p}'+m)(-\not{k}+2m)(-\not{p}+m)(-\not{k}+2m) \Bigr] , \tag{531}$$

$$\langle \Phi_{uu} \rangle = \tfrac{1}{2} \mathrm{Tr} \Bigl[(-\not{p}'+m)(+\not{k}'+2m)(-\not{p}+m)(+\not{k}'+2m) \Bigr] , \tag{532}$$

$$\langle \Phi_{su} \rangle = \tfrac{1}{2} \mathrm{Tr} \Bigl[(-\not{p}'+m)(-\not{k}+2m)(-\not{p}+m)(+\not{k}'+2m) \Bigr] , \tag{533}$$

$$\langle \Phi_{us} \rangle = \tfrac{1}{2} \mathrm{Tr} \Bigl[(-\not{p}'+m)(+\not{k}'+2m)(-\not{p}+m)(-\not{k}+2m) \Bigr] . \tag{534}$$

We can evaluate these in terms of the Mandelstam variables by using our trace technology, along with

$$\begin{aligned} pk = p'k' &= -\tfrac{1}{2}(s - m^2 - M^2) , \\ pp' &= +\tfrac{1}{2}(t - 2m^2) , \\ kk' &= +\tfrac{1}{2}(t - 2M^2) , \\ pk' = p'k &= +\tfrac{1}{2}(u - m^2 - M^2) . \end{aligned} \tag{535}$$

Examining eqs. (531) and (532), we see that $\langle \Phi_{ss} \rangle$ and $\langle \Phi_{uu} \rangle$ are transformed into each other by $k \leftrightarrow -k'$. Examining eqs. (533) and (534), we see that $\langle \Phi_{su} \rangle$ and $\langle \Phi_{us} \rangle$ are also transformed into each other by $k \leftrightarrow -k'$. From eq. (535), we see that this is equivalent to $s \leftrightarrow u$. Thus we need only compute $\langle \Phi_{ss} \rangle$ and $\langle \Phi_{su} \rangle$, and then take $s \leftrightarrow u$ to get $\langle \Phi_{uu} \rangle$ and $\langle \Phi_{us} \rangle$. This is, again, merely tedious, and the results are

$$\langle \Phi_{ss} \rangle = -su + m^2(9s + u) + 7m^4 - 8m^2M^2 + M^4 , \tag{536}$$

$$\langle \Phi_{uu} \rangle = -su + m^2(9u + s) + 7m^4 - 8m^2M^2 + M^4 , \tag{537}$$

$$\langle \Phi_{su} \rangle = +su + 3m^2(s + u) + 9m^4 - 8m^2M^2 - M^4 . \tag{538}$$

$$\langle \Phi_{us} \rangle = +su + 3m^2(s + u) + 9m^4 - 8m^2M^2 - M^4 . \tag{539}$$

48.1) The tedium of these calculations is greatly alleviated by making use of a symbolic manipulation program like Mathematica or Maple. One approach is brute force: compute 4×4 matrices like $\not{p}$ in the CM frame, and take their products and traces. If you are familiar with a symbolic-manipulation program, write one that does this. See if you can verify eqs. (536–539).

48.2) Compute $\langle|\mathcal{T}|^2\rangle$ for $e^-e^- \to e^-e^-$. You should find that your result is the same as that for $e^+e^- \to e^+e^-$, but with $s \leftrightarrow u$, and an extra overall minus sign. This relationship is known as *crossing symmetry*.

48.3) Compute $\langle|\mathcal{T}|^2\rangle$ for $e^+e^- \to \varphi\varphi$. You should find that your result is the same as that for $e^-\varphi \to e^-\varphi$, but with $s \leftrightarrow t$, and an extra overall minus sign. This is another example of crossing symmetry.

48.4) Suppose that $M > 2m$, so that the scalar can decay to an electron-positron pair.

a) Compute the decay rate, summed over final spins.

b) Compute $|\mathcal{T}|^2$ for decay into an electron with spin s_1 and a positron with spin s_2. Take the fermion three-momenta to be along the z axis, and let the x-axis be the spin-quantization axis. You should find that $|\mathcal{T}|^2 = 0$ if $s_1 = -s_2$, or if $M = 2m$ (so that the outgoing three-momentum of each fermion is zero). Discuss this in light of conservation of angular momentum and of parity. (Prerequisite: 40.)

c) Compute the rate for decay into an electron with helicity s_1 and a positron with helicity s_2. (See section 38 for the definition of helicity.) You should find that the decay rate is zero if $s_1 = -s_2$. Discuss this in light of conservation of angular momentum and of parity.

d) Now consider changing the interaction to $\mathcal{L}_1 = ig\varphi\overline{\Psi}\gamma_5\Psi$, and compute the spin-summed decay rate. Explain (in light of conservation of angular momentum and of parity) why the decay rate is larger than it was without the $i\gamma_5$ in the interaction.

e) Repeat parts (b) and (c) for the new form of the interaction, and explain any differences in the results.

49: The Feynman Rules for Majorana Fields

Prerequisite: 45

In this section we will deduce the Feynman rules for Yukawa theory, but with a Majorana field instead of a Dirac field. We can think of the particles associated with the Majorana field as massive neutrinos.

We have

$$\begin{aligned} Ł_1 &= \tfrac{1}{2}g\varphi\overline{\Psi}\Psi \\ &= \tfrac{1}{2}g\varphi\Psi^{\mathrm{T}}\mathcal{C}\Psi \ , \end{aligned} \tag{540}$$

where Ψ be a Majorana field (with mass m) and φ is a real scalar field (with mass M), and g is a coupling constant. In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

From section 41, we have the LSZ rules appropriate for a Majorana field,

$$b_s^\dagger(\mathbf{p})_{\mathrm{in}} \to -i\int d^4x\; e^{+ipx}\;\overline{v}_s(\mathbf{p})(-i\partial\!\!\!/ + m)\Psi(x) \tag{541}$$

$$= +i\int d^4x\; \Psi^{\mathrm{T}}(x)\mathcal{C}(+i\overleftarrow{\partial\!\!\!/} + m)u_s(\mathbf{p})e^{+ipx} \ , \tag{542}$$

$$b_{s'}(\mathbf{p}')_{\mathrm{out}} \to +i\int d^4x\; e^{-ip'x}\;\overline{u}_{s'}(\mathbf{p}')(-i\partial\!\!\!/ + m)\Psi(x) \ , \tag{543}$$

$$= -i\int d^4x\; e^{-ip'x}\;\Psi^{\mathrm{T}}(x)\mathcal{C}(+i\overleftarrow{\partial\!\!\!/} + m)v_{s'}(\mathbf{p}')e^{-ip'x} \ . \tag{544}$$

Eq. (542) follows from eq. (541) by taking the transpose of the right-hand side, and using $\overline{v}_{s'}(\mathbf{p}')^{\mathrm{T}} = -\mathcal{C}u_{s'}(\mathbf{p}')$ and $(-i\partial\!\!\!/ + m)^{\mathrm{T}} = \mathcal{C}(+i\partial\!\!\!/ + m)\mathcal{C}^{-1}$; similarly, eq. (544) follows from eq. (543). Which form we use depends on convenience, and is best chosen on a diagram-by-diagram basis, as we will see shortly.

Eqs. (541–544) lead us to compute correlation functions containing Ψ's, but not $\overline{\Psi}$'s. In position space, this leads to Feynman rules where the fermion propagator is $\frac{1}{i}S(x-y)\mathcal{C}^{-1}$, and the $\varphi\Psi\Psi$ vertex is $ig\mathcal{C}$; the factor of $\frac{1}{2}$ in $\mathcal{L}_1$ is killed by a symmetry factor of 2! that arises from having two identical Ψ fields in $\mathcal{L}_1$. In a particular diagram, as we move along a fermion line, the $\mathcal{C}^{-1}$ in the propagator will cancel against the $\mathcal{C}$ in the vertex, leaving over a final $\mathcal{C}^{-1}$ at one end. This $\mathcal{C}^{-1}$ can be canceled by a $\mathcal{C}$ from eq. (542) (for an incoming particle) or (544) (for an outgoing particle). On the other hand, for the other end of the same line, we should use either eq. (541) (for an incoming particle) or eq. (543) (for an outgoing particle) to avoid introducing an extra $\mathcal{C}$ at *that* end. In this way, we can avoid ever having explicit factors of $\mathcal{C}$ in our Feynman rules.

Using this approach, the Feynman rules for this theory are as follows.

1) The total number of incoming and outgoing neutrinos is always even; call this number $2n$. Draw n solid lines. Connect them with internal dashed lines, using a vertex that joins one dashed and two solid lines. Also, attach an external dashed line for each incoming or outgoing scalar. In this way, draw all possible diagrams that are *topologically inequivalent*.

2) Draw arrows on each segment of each solid line; keep the arrow direction continuous along each line.

3) Label each external dashed line with the momentum of an incoming or outgoing scalar. If the particle is incoming, draw an arrow on the dashed line that points *towards* the vertex; If the particle is outgoing, draw an arrow on the dashed line that points *away* from the vertex.

4) Label each external solid line with the momentum of an incoming or outgoing neutrino, but include a minus sign with the momentum if (a) the particle is incoming and the arrow points *away* from the vertex, or (b) the particle is outgoing and the arrow points *towards* the vertex.

5) Do this labeling of external lines in all possible *inequivalent* ways. Two diagrams are considered *equivalent* if they can be transformed into each other by reversing all the arrows on one or more fermion lines, and correspondingly changing the signs of the external momenta on each reversed-arrow line. The process of arrow reversal contributes a minus sign for each reversed-arrow line.

6) Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9) The value of a diagram consists of the following factors:

for each incoming or outgoing scalar, 1;

for each incoming neutrino labeled with $+p_i$, $u_{s_i}(\mathbf{p}_i)$;

for each incoming neutrino labeled with $-p_i$, $\overline{v}_{s_i}(\mathbf{p}_i)$;

for each outgoing neutrino labeled with $+p'_i$, $\overline{u}_{s'_i}(\mathbf{p}'_i)$;

for each outgoing neutrino labeled with $-p'_i$, $v_{s'_i}(\mathbf{p}'_i)$;

for each vertex, ig;

for each internal scalar line, $-i/(k^2 + M^2 - i\epsilon)$,

where k is the four-momentum of that line;

for each internal fermion line, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$,

where p is the four-momentum of that line.

10) Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$. Go along the complete fermion line, following the arrows backwards, and writing down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any.

11) Two diagrams that are identical *except for the momentum and spin labels on two external fermion lines* that *have their arrows pointing in the same direction* (either both towards or both away from the vertex) have a relative minus sign.

12) The value of $i\mathcal{T}$ is given by a sum over the values of all these diagrams.

There are additional rules for counterterms and loops, but we will postpone those to section 51.

Let's look at the simplest process, $\varphi \to \nu\nu$. There are two possible diagrams for this, shown in fig. (10). However, according to rule #5, these two diagrams are equivalent. The first one evaluates to

$$i\mathcal{T} = ig\,\overline{v}'_2 u'_1 \,, \tag{545}$$

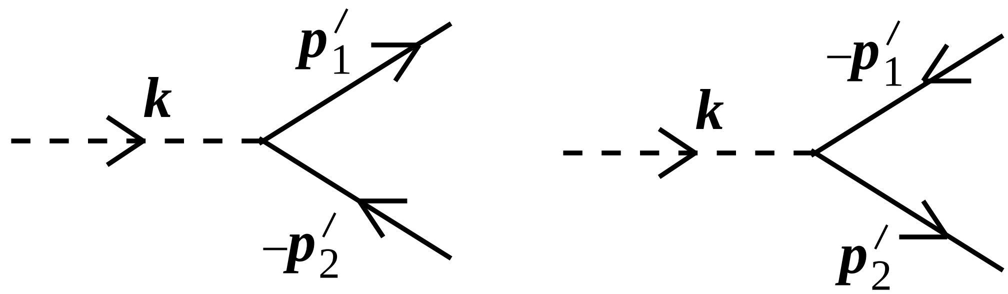

Figure 10: Two equivalent diagrams for $\varphi \to \nu\nu$, corresponding to eqs. (545) and (546), respectively.

while the second gives

$$i\mathcal{T} = -ig\,\overline{v}_1' u_2' \; . \tag{546}$$

The minus sign comes from the last part of rule #5: reversing the arrows on one fermion line gives an extra minus sign. These two versions of $\mathcal{T}$ should of course yield the same result; to check this, note that

$$\begin{aligned} \overline{v}_1 u_2 &= [\overline{v}_1 u_2]^{\mathrm{T}} \\ &= u_2^{\mathrm{T}}\, \overline{v}_1^{\mathrm{T}} \\ &= \overline{v}_2 \mathcal{C}^{-1} \mathcal{C}^{-1} u_1 \\ &= -\overline{v}_2 u_1 \; , \end{aligned} \tag{547}$$

as required.

In general, for processes with a total of just two incoming and outgoing neutrinos, such as $\nu\varphi \to \nu\varphi$ or $\nu\nu \to \varphi\varphi$, these rules give (up to an irrelevant overall sign) the same result for $i\mathcal{T}$ as we would get for the corresponding process in the Dirac case, $e^-\varphi \to e^-\varphi$ or $e^+e^- \to \varphi\varphi$. (Note, however, that in the Dirac case, we have $Ł_1 = g\varphi\overline{\Psi}\Psi$, as compared with $Ł_1 = \frac{1}{2} g\varphi\overline{\Psi}\Psi$ in the Majorana case.)

The differences between Dirac and Majorana fermions become more pronounced for $\nu\nu \to \nu\nu$. Now there are *three* inequivalent contributing diagrams, shown in fig. (11). The corresponding amplitude can be written as

$$i\mathcal{T} = \tfrac{1}{i}(ig)^2 \left[\frac{(\overline{u}_1' u_1)(\overline{u}_2' u_2)}{-t + M^2} - \frac{(\overline{u}_2' u_1)(\overline{u}_1' u_2)}{-u + M^2} + \frac{(\overline{v}_2 u_1)(\overline{u}_1' v_2')}{-s + M^2} \right] , \tag{548}$$

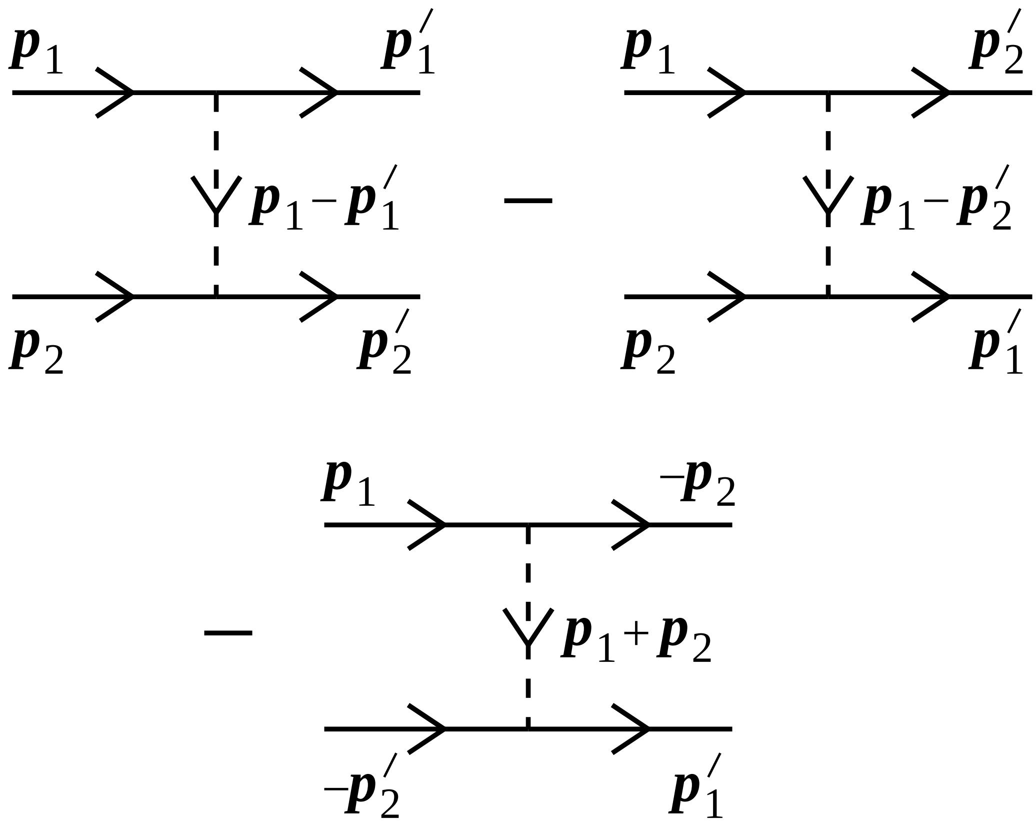

Figure 11: Diagrams for $\nu\nu \to \nu\nu$, corresponding to eq. (548).

where $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$ and $u = -(p_1 - p_2')^2$. Note the relative signs. After taking the absolute square of this expression, we can use relations like eq. (547) on a term-by-term basis to put everything into a form that allows the spin sums to be performed in the standard way.

In fact, we have already done all the necessary work in the Dirac case. The s-s, s-t, and t-t terms in $\langle|\mathcal{T}|^2\rangle$ for $\nu\nu \to \nu\nu$ are the same as those for $e^+e^- \to e^+e^-$, while the t-t, t-u, and u-u terms are the same as those for the crossing-related process $e^-e^- \to e^-e^-$. Finally, the s-u terms can be obtained from the s-t terms via $t \leftrightarrow u$, or equivalently from the t-u terms via $t \leftrightarrow s$. Thus the result is

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle = g^4\biggl[&\frac{(s-4m^2)^2}{(M^2-s)^2} + \frac{st-4m^2u}{(M^2-s)(M^2-t)} \\ &+ \frac{(t-4m^2)^2}{(M^2-t)^2} + \frac{tu-4m^2s}{(M^2-t)(M^2-u)} \\ &+ \frac{(u-4m^2)^2}{(M^2-u)^2} + \frac{us-4m^2t}{(M^2-u)(M^2-s)}\biggr], \end{aligned} \tag{549}$$

which is neatly symmetric on permutations of s, t, and u.

50: Massless Particles and Spinor Helicity

Prerequisite: 48

Scattering amplitudes often simplify greatly if the particles are massless (or can be approximated as massless because the Mandelstam variables all have magnitudes much larger than the particle masses squared). In this section we will explore this phenomenon for spin-one-half (and spin-zero) particles. We will begin developing the technology of *spinor helicity*, which will prove to be of indispensible utility in Part III.

Recall from section 38 that the u spinors for a massless spin-one-half particle obey

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 + s\gamma_5)(-\not{p}) \; , \tag{550}$$

where $s = \pm$ specifies the *helicity*, the component of the particle's spin measured along the axis specified by its three-momentum; in this notation the helicity is $\frac{1}{2}s$. The v spinors obey a similar relation,

$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5)(-\not{p}) \; . \tag{551}$$

In fact, in the massless case, with the phase conventions of section 38, we have $v_s(\mathbf{p}) = u_{-s}(\mathbf{p})$. Thus we can confine our discussion to u-type spinors only, since we need merely change the sign of s to accomodate v-type spinors.

Let us consider a u spinor for a particle of negative helicity. We have

$$u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = \tfrac{1}{2}(1 - \gamma_5)(-\not{p}) \; . \tag{552}$$

Let us define

$$p_{a\dot{a}} \equiv p_\mu \sigma^\mu_{a\dot{a}} \; . \tag{553}$$

Then we also have

$$p^{\dot{a}a} = \varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}} p_{c\dot{c}} = p_\mu \bar{\sigma}^{\mu\dot{a}a} \; . \tag{554}$$

Then, using

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{555}$$

in eq. (552), we find

$$u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = \begin{pmatrix} 0 & -p_{a\dot{a}} \\ 0 & 0 \end{pmatrix}. \tag{556}$$

On the other hand, we know that the lower two components of $u_-(\mathbf{p})$ vanish, and so we can write

$$u_-(\mathbf{p}) = \begin{pmatrix} \phi_a \\ 0 \end{pmatrix}. \tag{557}$$

Here ϕ_a is a two-component numerical spinor; it is not an anticommuting object. Such a commuting spinor is sometimes called a *twistor*. An explicit numerical formula for it (verified in problem 50.1) is

$$\phi_a = \sqrt{2\omega} \begin{pmatrix} -\sin(\tfrac{1}{2}\theta)e^{-i\phi} \\ +\cos(\tfrac{1}{2}\theta) \end{pmatrix}, \tag{558}$$

where θ and ϕ are the polar and azimuthal angles that specify the direction of the three-momentum $\mathbf{p}$, and $\omega = |\mathbf{p}|$. Barring eq. (557) yields

$$\overline{u}_-(\mathbf{p}) = (\,0, \quad \phi^*_{\dot{a}}\,)\,, \tag{559}$$

where $\phi^*_{\dot{a}} = (\phi_a)^*$. Now, combining eqs. (557) and (559), we get

$$u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = \begin{pmatrix} 0 & \phi_a\phi^*_{\dot{a}} \\ 0 & 0 \end{pmatrix}. \tag{560}$$

Comparing with eq. (556), we see that

$$p_{a\dot{a}} = -\phi_a\phi^*_{\dot{a}}\,. \tag{561}$$

This expresses the four-momentum of the particle neatly in terms of the twistor that describes its spin state. The essence of the spinor helicity method is to treat ϕ_a as the fundamental object, and to express the particle's four-momentum in terms of it, via eq. (561).

Given eq. (557), and the phase conventions of section 38, the positive-helicity spinor is

$$u_+(\mathbf{p}) = \begin{pmatrix} 0 \\ \phi^{*\dot a} \end{pmatrix} , \tag{562}$$

where $\phi^{*\dot a} = \varepsilon^{\dot a\dot c}\phi^*_{\dot c}$. Barring eq. (562) yields

$$\overline{u}_+(\mathbf{p}) = (\, \phi^a, \quad 0\,) . \tag{563}$$

Computation of $u_+(\mathbf{p})\overline{u}_+(\mathbf{p})$ via eqs. (562) and (563), followed by comparison with eq. (550) with $s = +$, then reproduces eq. (561), but with the indices raised.

In fact, the decomposition of $p_{a\dot a}$ into the direct product of a twistor and its complex conjugate is unique (up to an overall phase for the twistor). To see this, use $\sigma^\mu = (I, \vec\sigma)$ to write

$$p_{a\dot a} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} . \tag{564}$$

The determinant of this matrix is $-(p^0)^2 + \mathbf{p}^2$, and this vanishes because the particle is (by assumption) massless. Thus $p_{a\dot a}$ has a zero eigenvalue. Therefore, it can be written as a projection onto the eigenvector corresponding to the nonzero eigenvalue. That is what eq. (561) represents, with the nonzero eigenvalue absorbed into the normalization of the eigenvector ϕ_a.

Let us now introduce some useful notation. Let p and k be two four-momenta, and ϕ_a and κ_a the corresponding twistors. We define the twistor product

$$[p\,k] \equiv \phi^a \kappa_a \; . \tag{565}$$

Because $\phi^a\kappa_a = \varepsilon^{ac}\phi_c\kappa_a$, and the twistors commute, we have

$$[k\,p] = -[p\,k] \; . \tag{566}$$

From eqs. (557) and (563), we can see that

$$\overline{u}_+(\mathbf{p})u_-(\mathbf{k}) = [p\,k] \; . \tag{567}$$

Similarly, let us define

$$\langle p\,k\rangle \equiv \phi^*_{\dot a}\kappa^{*\dot a} \; . \tag{568}$$

Comparing with eq. (565) we see that

$$\langle p\,k\rangle = [k\,p]^* \;, \tag{569}$$

which implies that this product is also antisymmetric,

$$\langle k\,p\rangle = -\langle p\,k\rangle \;. \tag{570}$$

Also, from eqs. (559) and (562), we have

$$\overline{u}_-(\mathbf{p})u_+(\mathbf{k}) = \langle p\,k\rangle \;. \tag{571}$$

Note that the other two possible spinor products vanish:

$$\overline{u}_+(\mathbf{p})u_+(\mathbf{k}) = \overline{u}_-(\mathbf{p})u_-(\mathbf{k}) = 0 \;. \tag{572}$$

The twistor products $\langle p\,k\rangle$ and $[p\,k]$ satisfy another important relation,

$$\begin{aligned}\langle p\,k\rangle[k\,p] &= (\phi^*_{\dot a}\kappa^{*\dot a})(\kappa^a\phi_a) \\ &= (\phi^*_{\dot a}\phi_a)(\kappa^a\kappa^{*\dot a}) \\ &= p_{\dot a a}k^{a\dot a} \\ &= -2p^\mu k_\mu \;,\end{aligned} \tag{573}$$

where the last line follows from $\bar\sigma^{\mu\dot a a}\sigma^\nu_{a\dot a} = -2g^{\mu\nu}$.

Let us apply this notation to the tree-level scattering amplitude for $e^-\varphi \to e^-\varphi$ in Yukawa theory, which we first computed in Section 44, and which reads

$$\mathcal{T}_{s's} = g^2\,\overline{u}_{s'}(\mathbf{p}')\Big[\tilde S(p{+}k) + \tilde S(p{-}k')\Big]u_s(\mathbf{p}) \;. \tag{574}$$

For a massless fermion, $\tilde S(p) = -\not p/p^2$. If the scalar is also massless, then $(p+k)^2 = 2p\cdot k$ and $(p-k')^2 = -2p\cdot k'$. Also, we can remove the $\not p$'s in the propagator numerators in eq. (574), because $\not p u_s(\mathbf{p}) = 0$. Thus we have

$$\mathcal{T}_{s's} = g^2\,\overline{u}_{s'}(\mathbf{p}')\left[\frac{-\not k}{2p\cdot k} + \frac{-\not k'}{2p\cdot k'}\right]u_s(\mathbf{p}) \;. \tag{575}$$

Now consider the case $s' = s = +$. From eqs. (562), (563), and

$$-\not{k} = \begin{pmatrix} 0 & \kappa_a \kappa^*_{\dot{a}} \\ \kappa^{*\dot{a}} \kappa^a & 0 \end{pmatrix}, \tag{576}$$

we get

$$\begin{aligned} \overline{u}_+(\mathbf{p}')(-\not{k})u_+(\mathbf{p}) &= \phi'^a \kappa_a \kappa^*_{\dot{a}} \phi^{*\dot{a}} \\ &= [p'\,k]\,\langle k\,p\rangle\ . \end{aligned} \tag{577}$$

Similarly, for $s' = s = -$, we find

$$\begin{aligned} \overline{u}_-(\mathbf{p}')(-\not{k})u_-(\mathbf{p}) &= \phi'^*_{\dot{a}} \kappa^{*\dot{a}} \kappa^a \phi_a \\ &= \langle p'\,k\rangle\,[k\,p]\ , \end{aligned} \tag{578}$$

while for $s' \neq s$, the amplitude vanishes:

$$\overline{u}_-(\mathbf{p}')(-\not{k})u_+(\mathbf{p}) = \overline{u}_+(\mathbf{p}')(-\not{k})u_-(\mathbf{p}) = 0\ . \tag{579}$$

Then, using eq. (573) on the denominators in eq. (575), we find

$$\begin{aligned} \mathcal{T}_{++} &= -g^2 \left(\frac{[p'\,k]}{[p\,k]} + \frac{[p'\,k']}{[p\,k']} \right), \\ \mathcal{T}_{--} &= -g^2 \left(\frac{\langle p'\,k\rangle}{\langle p\,k\rangle} + \frac{\langle p'\,k'\rangle}{\langle p\,k'\rangle} \right), \end{aligned} \tag{580}$$

while

$$\mathcal{T}_{+-} = \mathcal{T}_{-+} = 0\ . \tag{581}$$

Thus we have rather simple expressions for the fixed-helicity scattering amplitudes in terms of twistor products.

We can simplify the derivation of these results by setting up a bra-ket notation. Let

$$\begin{aligned} |p] &= u_-(\mathbf{p}) = v_+(\mathbf{p})\ , \\ |p\rangle &= u_+(\mathbf{p}) = v_-(\mathbf{p})\ , \\ [p| &= \overline{u}_+(\mathbf{p}) = \overline{v}_-(\mathbf{p})\ , \\ \langle p| &= \overline{u}_-(\mathbf{p}) = \overline{v}_+(\mathbf{p})\ . \end{aligned} \tag{582}$$

We then have

$$\begin{aligned}\langle k|\, |p\rangle &= \langle k\, p\rangle \ , \\ [k|\, |p] &= [k\, p] \ , \\ \langle k|\, |p] &= 0 \ , \\ [k|\, |p\rangle &= 0 \ . \end{aligned} \tag{583}$$

We also can write

$$-\not{p} = |p\rangle[p| + |p]\langle p| \ , \tag{584}$$

where p is any massless four-momentum. With this notation, we can easily reproduce the results of eqs. (577–579).

Problems

50.1a) Use eqs. (558) and (564) to verify eq. (553).

b) Show that $\not{p}u_-(\mathbf{p}) = p^{\dot{a}a}\phi_a$. Then use eq. (553) to show that that $p^{\dot{a}a}\phi_a = 0$.

c) Let the three-momentum $\mathbf{p}$ be in the $+\hat{\mathbf{z}}$ direction. Use eq. (218) in section 38 to compute $u_\pm(\mathbf{p})$ explicitly in the massless limit (corresponding to the limit $\eta \to \infty$, where $\sinh\eta = |\mathbf{p}|/m$. Verify that, when $\theta = 0$, your results agree with eqs. (557), (558), and (562). Hint: if a matrix M has eigenvalues ± 1 only, then $\exp(aM) = \cosh(a) + \sinh(a)M$.

50.2) Prove the Schouten identity,

$$\langle p\, q\rangle\, \langle r\, s\rangle + \langle p\, r\rangle\, \langle s\, q\rangle + \langle p\, s\rangle\, \langle q\, r\rangle = 0 \ . \tag{585}$$

Hint: note that the left-hand side is completely antisymmetric in the three labels q, r, and s, and that each corresponding twistor has only two components.

50.3) Show that

$$\langle p\, q\rangle\, [q\, r]\, \langle r\, s\rangle\, [s\, p] = \mathrm{Tr}\, \tfrac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} \ , \tag{586}$$

and evaluate the right-hand side.

50.4a) Prove the useful identities

$$\langle p|\gamma^\mu|k] = [k|\gamma^\mu|p\rangle \ , \tag{587}$$

$$\langle p|\gamma^\mu|k]^* = \langle k|\gamma^\mu|p] \ , \tag{588}$$

$$\langle p|\gamma^\mu|p] = 2p^\mu \ , \tag{589}$$

$$\langle p|\gamma^\mu|k\rangle = 0 \ , \tag{590}$$

$$[p|\gamma^\mu|k] = 0 \ . \tag{591}$$

b) Extend the last two identies of part (a): show that the product of an odd number of gamma matrices sandwiched between either $\langle p|$ and $|k\rangle$ or $[p|$ and $|k]$ vanishes. Also show that the product of an even number of gamma matrices between either $\langle p|$ and $|k]$ or $[p|$ and $|k\rangle$ vanishes.

c) Prove the Fierz identities,

$$-\tfrac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = |q]\langle p| + |p\rangle[q| \ , \tag{592}$$

$$-\tfrac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = |q\rangle[p| + |p]\langle q| \ . \tag{593}$$

Now take the matrix element of eq. (593) between $\langle r|$ and $|s]$ to get yet another form of the Fierz identity,

$$[p|\gamma^\mu|q\rangle\,\langle r|\gamma_\mu|s] = 2\,[p\,s]\,\langle q\,r\rangle \ . \tag{594}$$

51: Loop Corrections in Yukawa Theory

Prerequisite: 19, 40, 48

In this section we will compute the one-loop corrections in Yukawa theory with a Dirac field. The basic concepts are all the same as for a scalar, and so we will mainly be concerned with the extra technicalities arising from spin indices and anticommutation.

First let us note that the general discussion of sections 18 and 29 leads us to expect that we will need to add to the lagrangian all possible terms whose coefficients have positive or zero mass dimension, and that respect the symmetries of the original lagrangian. These include Lorentz symmetry, the U(1) phase symmetry of the Dirac field, and the discrete symmetries of parity, time reversal, and charge conjugation.

The mass dimensions of the fields (in four spacetime dimensions) are $[\varphi] = 1$ and $[\Psi] = \frac{3}{2}$. Thus any power of φ up to φ^4 is allowed. But there are no additional required terms involving Ψ: the only candidates contain either γ_5 (e.g., $i\overline{\Psi}\gamma_5\Psi$) and are forbidden by parity, or $\mathcal{C}$ (e.g, $\Psi^{\mathrm{T}}\mathcal{C}\Psi$) and are forbidden by the U(1) symmetry.

Nevertheless, having to deal with the addition of three new terms (φ, φ^3, φ^4) is annoying enough to prompt us to look for a simpler example. Consider, then, a modified form of the Yukawa interaction,

$$Ł_1 = ig\varphi\overline{\Psi}\gamma_5\Psi \; . \tag{595}$$

This interaction will conserve parity if and only if φ is a pseudoscalar:

$$P^{-1}\varphi(\mathbf{x},t)P = -\varphi(-\mathbf{x},t) \; . \tag{596}$$

Then, φ and φ^3 are odd under parity, and so we will *not* need to add them to Ł. The one term we will need to add is φ^4.

Therefore, the theory we will consider is

$$\text{Ł} = \text{Ł}_0 + \text{Ł}_1 \ , \tag{597}$$

$$\text{Ł}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}M^2\varphi^2 \ , \tag{598}$$

$$\text{Ł}_1 = iZ_g g\varphi\overline{\Psi}\gamma_5\Psi - \tfrac{1}{24}Z_\lambda\lambda\varphi^4 + \text{Ł}_{\rm ct} \ , \tag{599}$$

$$\begin{aligned}\text{Ł}_{\rm ct} &= i(Z_\Psi{-}1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m{-}1)m\overline{\Psi}\Psi \\ &\quad - \tfrac{1}{2}(Z_\varphi{-}1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_M{-}1)M^2\varphi^2\end{aligned} \tag{600}$$

where λ is a new coupling constant. We will use an on-shell renormalization scheme. The lagrangian parameter m is then the actual mass of the electron. We will define the couplings g and λ as the values of appropriate vertex functions when the external four-momenta vanish. Finally, the fields are normalized according to the requirements of the LSZ formula. In practice, this means that the scalar and fermion propagators must have appropriate poles with unit residue.

We will assume that $M < 2m$, so that the scalar is stable against decay into an electron-positron pair. The exact scalar propagator (in momentum space) can be then written in Lehmann-Källén form as

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + M^2 - i\epsilon} + \int_{M_{\rm th}^2}^\infty ds\,\rho_\varphi(s)\,\frac{1}{k^2 + s - i\epsilon} \ , \tag{601}$$

where the spectral density $\rho_\varphi(s)$ is real and nonnegative. The threshold mass $M_{\rm th}$ is either $2m$ (corresponding to the contribution of an electron-positron pair) or $3M$ (corresponding to the contribution of three scalars; by parity, there is no contribution from two scalars), whichever is less.

We see that $\tilde{\mathbf{\Delta}}(k^2)$ has a pole at $k^2 = -M^2$ with residue one. This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

We can also write the exact scalar propagator in the form

$$\tilde{\mathbf{\Delta}}(k^2)^{-1} = k^2 + M^2 - i\epsilon - \Pi(k^2) \ , \tag{602}$$

where $i\Pi(k^2)$ is given by the sum of 1PI diagrams with two external scalar lines, and the external propagators removed. The fact that $\tilde{\mathbf{\Delta}}(k^2)$ has a pole at $k^2 = -M^2$ with residue one implies that $\Pi(-M^2) = 0$ and $\Pi'(-M^2) = 0$; this fixes the coefficients Z_φ and Z_M.

All of this is mimicked for the Dirac field. When parity is conserved, the exact propagator (in momentum space) can be written in Lehmann-Källén form as

$$\tilde{\mathbf{S}}(\not{p}) = \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon} + \int_{m_{\rm th}^2}^{\infty} ds\, \rho_\Psi(s)\, \frac{-\not{p} + \sqrt{s}}{p^2 + s - i\epsilon}\ , \qquad (603)$$

real and nonnegative. The threshold mass $m_{\rm th}$ is $m + M$ (corresponding to the contribution of a fermion and a scalar), which, by assumption, is less than $3m$ (corresponding to the contribution of three fermions; by Lorentz invariance, there is no contribution from two fermions).

Since $p^2 = -\not{p}\not{p}$, we can rewrite eq. (603) as

$$\tilde{\mathbf{S}}(\not{p}) = \frac{1}{\not{p} + m - i\epsilon} + \int_{m_{\rm th}^2}^{\infty} ds\, \rho_\Psi(s)\, \frac{1}{\not{p} + \sqrt{s} - i\epsilon}\ , \qquad (604)$$

with the understanding that $1/(\ldots)$ refers to the matrix inverse. However, since $\not{p}$ is the only matrix involved, we can think of $\tilde{\mathbf{S}}(\not{p})$ as an analytic function of the single variable $\not{p}$. With this idea in mind, we see that $\tilde{\mathbf{S}}(\not{p})$ has a pole at $\not{p} = -m$ with residue one. This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

We can also write the exact fermion propagator in the form

$$\tilde{\mathbf{S}}(\not{p})^{-1} = \not{p} + m - i\epsilon - \Sigma(\not{p})\ , \qquad (605)$$

where $i\Sigma(\not{p})$ is given by the sum of 1PI diagrams with two external fermion lines, and the external propagators removed. The fact that $\tilde{\mathbf{S}}(\not{p})$ has a pole at $\not{p} = -m$ with residue one implies that $\Sigma(-m) = 0$ and $\Sigma'(-m) = 0$; this fixes the coefficients Z_Ψ and Z_m.

We proceed to the diagrams. The Yukawa vertex carries a factor of $i(iZ_g g)\gamma_5 = -Z_g g\gamma_5$. Since $Z_g = 1 + O(g^2)$, we can set $Z_g = 1$ in the one-loop diagrams.

Consider first $\Pi(k^2)$, which receives the one-loop (and counterterm) corrections shown in fig. (12). The first diagram has a closed fermion loop. As we will see in problem 51.1 (and section 53), anticommutation of the fermion fields results in an extra factor of minus one for each closed fermion loop.

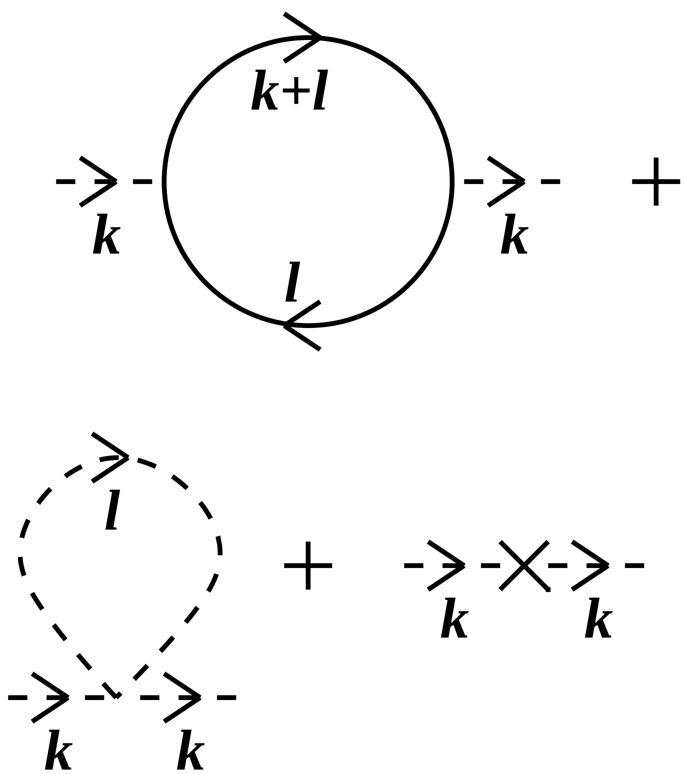

Figure 12: The one-loop and counterterm corrections to the scalar propagator in Yukawa theory.

The spin indices on the propagators and vertices are contracted in the usual way, following the arrows backwards. Since the loop closes on itself, we end up with a trace over the spin indices. Thus we have

$$i\Pi_{\Psi\,\text{loop}}(k^2) = (-1)(-g)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\text{Tr}\Big[\tilde{S}(\not\ell+\not k)\gamma_5\tilde{S}(\not\ell)\gamma_5\Big]\;, \tag{606}$$

where

$$\tilde{S}(\not p) = \frac{-\not p + m}{p^2 + m^2 - i\epsilon} \tag{607}$$

is the free fermion propagator in momentum space.

We now proceed to evaluate eq. (606). We have

$$\begin{aligned}\text{Tr}[(-\not\ell - \not k + m)\gamma_5(-\not\ell + m)\gamma_5] &= \text{Tr}[(-\not\ell - \not k + m)(+\not\ell + m)] \\ &= 4[(\ell + k)\ell + m^2] \\ &\equiv 4N\;. \end{aligned}\tag{608}$$

The first equality follows from $\gamma_5^2 = 1$ and $\gamma_5\not p\gamma_5 = -\not p$.

Next we combine the denominators with Feynman's formula. Suppressing the $i\epsilon$'s, we have

$$\frac{1}{(\ell{+}k)^2 + m^2}\,\frac{1}{\ell^2 + m^2} = \int_0^1 dx\,\frac{1}{(q^2 + D)^2}\;, \tag{609}$$

where $q = \ell + xk$ and $D = x(1{-}x)k^2 + m^2$.

We then change the integration variable in eq. (606) from ℓ to q; the result is

$$i\Pi_{\Psi\,\text{loop}}(k^2) = 4g^2\int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2 + D)^2}\;, \tag{610}$$

where now $N = (q+(1{-}x)k)(q-xk)+m^2$. The integral diverges, and so we analytically continue it to $d = 4 - \varepsilon$ spacetime dimensions. (Here we ignore a subtlety with the definition of γ_5 in d dimensions, and assume that $\gamma_5^2 = 1$ and $\gamma_5\not p\gamma_5 = -\not p$ continue to hold.) We also make the replacement $g \to g\tilde{\mu}^{\varepsilon/2}$, where $\tilde{\mu}$ has dimensions of mass, so that g remains dimensionless.

Expanding out the numerator, we have

$$N = q^2 - x(1{-}x)k^2 + m^2 + (1{-}2x)kq\;. \tag{611}$$

The term linear in q integrates to zero. For the rest, we use the general result of section 14 to get

$$\tilde{\mu}^{\varepsilon}\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2}=\frac{i}{16\pi^2}\left[\frac{2}{\varepsilon}-\ln(D/\mu^2)\right]\,, \tag{612}$$

$$\tilde{\mu}^{\varepsilon}\int\frac{d^dq}{(2\pi)^d}\,\frac{q^2}{(q^2+D)^2}=\frac{i}{16\pi^2}\left[\frac{2}{\varepsilon}+\tfrac{1}{2}-\ln(D/\mu^2)\right](-2D)\,, \tag{613}$$

where $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$, and we have dropped terms of order ε. Plugging eqs. (612) and (613) into eq. (610) yields

$$\begin{aligned}\Pi_{\Psi\,\mathrm{loop}}(k^2) = -\frac{g^2}{4\pi^2}\Biggl[\frac{1}{\varepsilon}(k^2+2m^2)+\tfrac{1}{6}k^2+m^2 \\ -\int_0^1 dx\,\Bigl(3x(1{-}x)k^2+m^2\Bigr)\ln(D/\mu^2)\Biggr].\end{aligned} \tag{614}$$

We see that the divergent term has (as expected) a form that permits cancellation by the counterterms.

We evaluated the second diagram of fig. (12) in section 30, with the result

$$\Pi_{\varphi\,\mathrm{loop}}(k^2)=\frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon}+\tfrac{1}{2}-\tfrac{1}{2}\ln(M^2/\mu^2)\right]M^2\,. \tag{615}$$

The third diagram gives the contribution of the counterterms,

$$\Pi_{\mathrm{ct}}(k^2)=-(Z_\varphi{-}1)k^2-(Z_M{-}1)M^2\,. \tag{616}$$

Adding up eqs. (614–616), we see that finiteness of $\Pi(k^2)$ requires

$$Z_\varphi = 1-\frac{g^2}{4\pi^2}\Bigl(\frac{1}{\varepsilon}+\text{finite}\Bigr)\,, \tag{617}$$

$$Z_M = 1+\left(\frac{\lambda}{16\pi^2}-\frac{g^2}{2\pi^2}\frac{m^2}{M^2}\right)\Bigl(\frac{1}{\varepsilon}+\text{finite}\Bigr)\,, \tag{618}$$

plus higher-order (in g and/or λ) corrections. Note that, although there is an $O(\lambda)$ correction to Z_M, there is not an $O(\lambda)$ correction to Z_φ.

We can impose $\Pi(-M^2)=0$ by writing

$$\Pi(k^2)=\frac{g^2}{4\pi^2}\Biggl[\int_0^1 dx\,\Bigl(3x(1{-}x)k^2+m^2\Bigr)\ln(D/D_0)+\kappa_\Pi(k^2+M^2)\Biggr]\,, \tag{619}$$

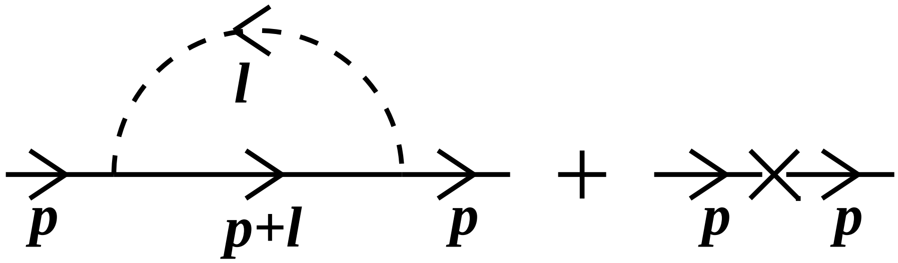

Figure 13: The one-loop and counterterm corrections to the fermion propagator in Yukawa theory.

where $D_0 = -x(1-x)M^2 + m^2$, and κ_Π is a constant to be determined. We fix κ_Π by imposing $\Pi'(-M^2) = 0$, which yields

$$\kappa_\Pi = \int_0^1 dx\, x(1-x)[3x(1-x)M^2 - m^2]/D_0 \; . \tag{620}$$

Note that, in this on-shell renormalization scheme, there is no $O(\lambda)$ correction to $\Pi(k^2)$.

Next we turn to the Ψ propagator, which receives the one-loop (and counterterm) corrections shown in fig. (13). The spin indices are contracted in the usual way, following the arrows backwards. We have

$$i\Sigma_{1\,\mathrm{loop}}(\not{p}) = (-g)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma_5 \tilde{S}(\not{p} + \not{\ell})\gamma_5\right] \tilde{\Delta}(\ell^2) \; , \tag{621}$$

where $\tilde{S}(\not{p})$ is given by eq. (607), and

$$\tilde{\Delta}(\ell^2) = \frac{1}{\ell^2 + M^2 - i\epsilon} \tag{622}$$

is the free scalar propagator in momentum space.

We evaluate eq. (621) with the usual bag of tricks. The result is

$$i\Sigma_{1\,\mathrm{loop}}(\not{p}) = -g^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{N}{(q^2 + D)^2} \; , \tag{623}$$

where $q = \ell + xp$ and

$$N = \not{q} + (1-x)\not{p} + m \; , \tag{624}$$

$$D = x(1-x)p^2 + xm^2 + (1-x)M^2 \; . \tag{625}$$

The integral diverges, and so we analytically continue it to $d = 4-\varepsilon$ spacetime dimensions, make the replacement $g \to g\tilde{\mu}^{\varepsilon/2}$, and take the limit as $\varepsilon \to 0$. The term linear in q in eq. (624) integrates to zero. Using eq. (612), we get

$$\Sigma_{1\,\text{loop}}(\not{p}) = -\frac{g^2}{16\pi^2}\left[\frac{1}{\varepsilon}(\not{p}+2m) - \int_0^1 dx\,\Big((1-x)\not{p}+m\Big)\ln(D/\mu^2)\right]. \tag{626}$$

We see that the divergent term has (as expected) a form that permits cancellation by the counterterms, which give

$$\Sigma_{\text{ct}}(\not{p}) = -(Z_\Psi - 1)\not{p} - (Z_m - 1)m\ . \tag{627}$$

Adding up eqs. (626) and (627), we see that finiteness of $\Sigma(\not{p})$ requires

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{628}$$

$$Z_m = 1 - \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{629}$$

plus higher-order corrections.

We can impose $\Sigma(-m) = 0$ by writing

$$\Sigma(\not{p}) = \frac{g^2}{16\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not{p}+m\Big)\ln(D/D_0) + \kappa_\Sigma(\not{p}+m)\right], \tag{630}$$

where D_0 is D evaluated at $p^2 = -m^2$, and κ_Σ is a constant to be determined. We fix κ_Σ by imposing $\Sigma'(-m) = 0$. In differentiating with respect to $\not{p}$, we take the p^2 in D, eq. (625), to be $-\not{p}^2$; we find

$$\kappa_\Sigma = -2\int_0^1 dx\, x^2(1-x)m^2/D_0\ . \tag{631}$$

Next we turn to the correction to the Yukawa vertex. We define the vertex function $i\mathbf{V}_Y(p',p)$ as the sum of one-particle irreducible diagrams with one incoming fermion with momentum p, one outgoing fermion with momentum p', and one incoming scalar with momentum $k = p' - p$. The original vertex $-Z_g g\gamma_5$ is the first term in this sum, and the diagram of fig. (14) is the second. Thus we have

$$i\mathbf{V}_Y(p',p) = -Z_g g\gamma_5 + i\mathbf{V}_{Y,\,1\,\text{loop}}(p',p) + O(g^5)\ , \tag{632}$$

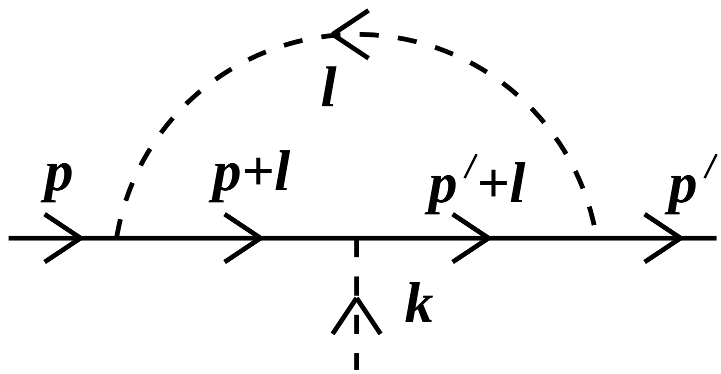

Figure 14: The one-loop correction to the scalar-fermion-fermion vertex in Yukawa theory.

where

$$i\mathbf{V}_{Y,1\,\text{loop}}(p',p) = (-g)^3\left(\tfrac{1}{i}\right)^3 \int \frac{d^d\ell}{(2\pi)^d} \left[\gamma_5 \tilde{S}(\not{p}'+\not{\ell})\gamma_5 \tilde{S}(\not{p}+\not{\ell})\gamma_5\right]\tilde{\Delta}(\ell^2)\ . \quad (633)$$

The numerator can be written as

$$N = (\not{p}' + \not{\ell} + m)(-\not{p} - \not{\ell} + m)\gamma_5\ , \quad (634)$$

and the denominators combined in the usual way. We then get

$$i\mathbf{V}_{Y,1\,\text{loop}}(p',p)/g = -ig^2 \int dF_3 \int \frac{d^4q}{(2\pi)^4}\, \frac{N}{(q^2+D)^3}\ , \quad (635)$$

where the integral over Feynman parameters was defined in section 16, and now

$$q = \ell + x_1 p + x_2 p'\ , \quad (636)$$

$$N = [\not{q} - x_1\not{p} + (1{-}x_2)\not{p}' + m][-\not{q} - (1{-}x_1)\not{p} + x_2\not{p}' + m]\gamma_5\ , \quad (637)$$

$$D = x_1(1{-}x_1)p^2 + x_2(1{-}x_2)p'^2 - 2x_1x_2\,p{\cdot}p' + (x_1{+}x_2)m^2 + x_3M^2\ . \quad (638)$$

Using $\not{q}\not{q} = -q^2$, we can write N as

$$N = q^2\gamma_5 + \widetilde{N} + (\text{linear in } q)\ , \quad (639)$$

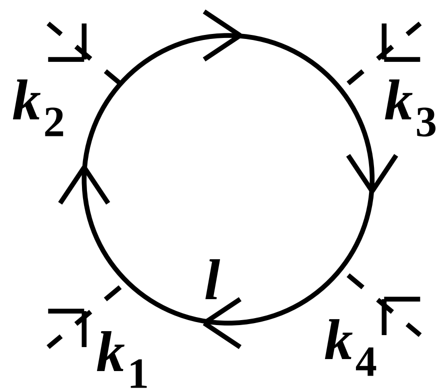

Figure 15: One of six diagrams with a closed fermion loop and four external scalar lines; the other five are obtained by permuting the external momenta in all possible inequivalent ways.

where

$$\widetilde{N} = [-x_1\not{p} + (1-x_2)\not{p}' + m][-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma_5 \; . \tag{640}$$

The terms linear in q in eq. (639) integrate to zero, and only the first term is divergent. Performing the usual manipulations, we find

$$i\mathbf{V}_{Y,\,1\,\text{loop}}(p',p)/g = -\frac{g^2}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - \tfrac{1}{4} - \tfrac{1}{2}\int dF_3\,\ln(D/\mu^2)\right)\gamma_5 + \tfrac{1}{4}\int dF_3\,\frac{\widetilde{N}}{D}\right] . \tag{641}$$

From eq. (632), we see that finiteness of $\mathbf{V}_Y(p',p)$ requires

$$Z_g = 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) , \tag{642}$$

plus higher-order corrections.

To fix the finite part of Z_g, we need a condition to impose on $\mathbf{V}_Y(p',p)$. One possibility is to mimic what we did in φ^3 theory in section 16: require $\mathbf{V}_Y(0,0)$ to have the tree-level value $ig\gamma_5$. As in φ^3 theory, this is not well motivated physically, but has the virtue of simplicity, and this is a good enough reason for us to adopt it. We leave the details to problem 51.2.

Next we turn to the corrections to the φ^4 vertex $i\mathbf{V}_4(k_1,k_2,k_3,k_4)$; the tree-level contribution is $-iZ_\lambda\lambda$. There are diagrams with a closed fermion loop, as shown in fig. (15), plus one-loop diagrams with φ particles only that we evaluated in section 30. We have

$$\begin{aligned} i\mathbf{V}_{4,\,\Psi\,\text{loop}} = (-1)(-g)^4\Big(\tfrac{1}{i}\Big)^4 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[&\tilde{S}(\ell\!\!/)\gamma_5\tilde{S}(\ell\!\!/-k\!\!\!/_1)\gamma_5 \\ &\times \tilde{S}(\ell\!\!/+k\!\!\!/_2+k\!\!\!/_3)\gamma_5\tilde{S}(\ell\!\!/+k\!\!\!/_2)\gamma_5\Big] \\ &+ 5 \text{ permutations of } (k_2, k_3, k_4)\ . \end{aligned} \tag{643}$$

Again we can employ the standard methods; there are no unfamiliar aspects. This being the case, let us concentrate on obtaining the divergent part; this will give us enough information to calculate the one-loop contributions to the beta functions for g and λ.

To obtain the divergent part of eq. (643), it is sufficient to set $k_i = 0$. Then the numerator in eq. (643) becomes simply $\mathrm{Tr}\,(\ell\!\!/\gamma_5)^4 = 4(\ell^2)^2$, and the denominator is $(\ell^2 + m^2)^4$. Then we find, after including the identical contributions from the other five permutations of the external momenta,

$$\mathbf{V}_{4,\,\Psi\,\text{loop}} = -\frac{3g^4}{\pi^2}\Big(\frac{1}{\varepsilon} + \text{finite}\Big)\ . \tag{644}$$

From section 30, we have

$$\mathbf{V}_{4,\,\varphi\,\text{loop}} = \frac{3\lambda}{16\pi^2}\Big(\frac{1}{\varepsilon} + \text{finite}\Big)\ . \tag{645}$$

Then, using

$$\mathbf{V}_4 = -Z_\lambda\lambda + \mathbf{V}_{4,\,\Psi\,\text{loop}} + \mathbf{V}_{4,\,\varphi\,\text{loop}} + \ldots\ , \tag{646}$$

we see that finiteness of $\mathbf{V}_4$ requires

$$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\Big(\frac{1}{\varepsilon} + \text{finite}\Big)\ , \tag{647}$$

plus higher-order corrections.

Problems

51.1) Derive the one-loop correction to the scalar propagator by working through eq. (452), and show that it has an extra minus sign (corresponding to the closed fermion loop).

51.2) Prove eq. (603). Hints: Given a multiparticle state $|p, s, q, n\rangle$ with four momentum p^μ and mass $M^2 = -p^2$, $J_z = \frac{1}{2}s$, charge q, and other attributes specified by n, show that $\langle 0|\Psi(x)|p, s, q, n\rangle$ vanishes unless $s = \pm 1$ and $q = +1$. Argue that this is enough information to fix $\langle 0|\Psi(x)|p, s, q, n\rangle \propto u_s(p)$, a spinor of mass M.

51.3) Finish the computation of $\mathbf{V}_Y(p', p)$, imposing the condition

$$\mathbf{V}_Y(0, 0) = ig\gamma_5 \ . \tag{648}$$

52: Beta Functions in Yukawa Theory

Prerequisite: 27, 51

In this section we will compute the beta functions for the Yukawa coupling g and the φ^4 coupling λ in Yukawa theory, using the methods of section 27.

The relations between the bare and renormalized couplings are

$$g_0 = Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g \tilde{\mu}^{\varepsilon/2} g \ , \tag{649}$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^{\varepsilon} \lambda \ . \tag{650}$$

Let us define

$$\ln\Big(Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g\Big) = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n} \ , \tag{651}$$

$$\ln\Big(Z_\varphi^{-2} Z_\lambda\Big) = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n} \ . \tag{652}$$

From our results in section 51, we have

$$G_1(g,\lambda) = \frac{5g^2}{16\pi^2} + \ldots \ , \tag{653}$$

$$L_1(g,\lambda) = \frac{3\lambda}{16\pi^2} + \frac{g^2}{2\pi^2} - \frac{3g^4}{\pi^2\lambda} + \ldots \ , \tag{654}$$

where the ellipses stand for higher-order (in g^2 and/or λ) corrections.

Taking the logarithm of eqs. (649) and (650), and using eqs. (651) and (652), we get

$$\ln g_0 = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n} + \ln g + \tfrac{1}{2}\varepsilon \ln \tilde{\mu} \ , \tag{655}$$

$$\ln \lambda_0 = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n} + \ln \lambda + \varepsilon \ln \tilde{\mu} \ . \tag{656}$$

We now use the fact that g_0 and λ_0 must be independent of μ. We differentiate eqs. (655) and (656) with respect to $\ln\mu$; the left-hand sides vanish, and we multiply the right-hand sides by g and λ, respectively. The result is

$$0 = \sum_{n=1}^{\infty}\left(g\frac{\partial G_n}{\partial g}\frac{dg}{d\ln\mu} + g\frac{\partial G_n}{\partial\lambda}\frac{d\lambda}{d\ln\mu}\right)\frac{1}{\varepsilon^n} + \frac{dg}{d\ln\mu} + \tfrac{1}{2}\varepsilon g\ , \tag{657}$$

$$0 = \sum_{n=1}^{\infty}\left(\lambda\frac{\partial L_n}{\partial g}\frac{dg}{d\ln\mu} + \lambda\frac{\partial L_n}{\partial\lambda}\frac{d\lambda}{d\ln\mu}\right)\frac{1}{\varepsilon^n} + \frac{d\lambda}{d\ln\mu} + \varepsilon\lambda\ . \tag{658}$$

In a renormalizable theory, $dg/d\ln\mu$ and $d\lambda/d\ln\mu$ must be finite in the $\varepsilon\to 0$ limit. Thus we can write

$$\frac{dg}{d\ln\mu} = -\tfrac{1}{2}\varepsilon g + \beta_g(g,\lambda)\ , \tag{659}$$

$$\frac{d\lambda}{d\ln\mu} = -\varepsilon\lambda + \beta_\lambda(g,\lambda)\ . \tag{660}$$

Substituting these into eqs. (657) and (658), and matching powers of ε, we find

$$\beta_g(g,\lambda) = g\left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\right)G_1\ , \tag{661}$$

$$\beta_\lambda(g,\lambda) = \lambda\left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\right)L_1\ . \tag{662}$$

The coefficients of all higher powers of $1/\varepsilon$ must also vanish, but this gives us no more information about the beta functions.

Using eqs. (653) and (654) in eqs. (661) and (662), we get

$$\beta_g(g,\lambda) = \frac{5g^3}{16\pi^2} + \ldots\ , \tag{663}$$

$$\beta_\lambda(g,\lambda) = \frac{1}{16\pi^2}\Big(3\lambda^2 + 8\lambda g^2 - 48g^4\Big) + \ldots\ . \tag{664}$$

The higher-order corrections have extra factors of g^2 and/or λ.

53: Functional Determinants

Prerequisite: 44, 45

In the section we will explore the meaning of the *functional determinants* that arise when doing gaussian path integrals, either bosonic or fermionic. We will be interested in situations where the path integral over one particular field is gaussian, but generates a functional determinant that depends on some other field. We will see how to relate this functional determinant to a certain infinite set of Feynman diagrams. We will need the technology we develop here to compute the path integral for nonabelian gauge theory in section 70.

We begin by considering a theory of a complex scalar field χ with

$$\mathcal{L} = -\partial^\mu \chi^\dagger \partial_\mu \chi - m^2 \chi^\dagger \chi + g\varphi \chi^\dagger \chi \ , \tag{665}$$

where φ is a real scalar *background field*. That is, $\varphi(x)$ is treated as a fixed function of spacetime. Next we define the path integral

$$Z(\varphi) = \int \mathcal{D}\chi^\dagger \, \mathcal{D}\chi \; e^{i\int d^4x \, \mathcal{L}} \ , \tag{666}$$

where we use the ϵ trick of section 6 to impose vacuum boundary conditions, and the normalization $Z(0) = 1$ is fixed by hand.

Recall from section 44 that if we have n complex variables z_i, then we can evaluate gaussian integrals by the general formula

$$\int d^n z \, d^n \bar{z} \, \exp\left(-i\bar{z}_i M_{ij} z_j\right) \propto (\det M)^{-1} \ . \tag{667}$$

In the case of the functional integral in eq. (666), the index i on the integration variable is replaced by the continuous spacetime label x, and the "matrix" M becomes

$$M(x,y) = [-\partial_x^2 + m^2 - g\varphi(x)]\delta^4(x-y) \ . \tag{668}$$

In order to apply eq. (667), we have to understand what it means to compute the determinant of this expression.

To this end, let us first note that we can write $M = M_0\widetilde{M}$, which is shorthand for

$$M(x,z) = \int d^4y\, M_0(x,y)\widetilde{M}(y,z)\ , \tag{669}$$

where

$$M_0(x,y) = (-\partial_x^2 + m^2)\delta^4(x-y)\ , \tag{670}$$

$$\widetilde{M}(y,z) = \delta^4(y-z) - g\Delta(y-z)\varphi(z)\ . \tag{671}$$

Here $\Delta(y-z)$ is the Feynman propagator, which obeys

$$(-\partial_y^2 + m^2)\Delta(y-z) = \delta^4(y-z)\ . \tag{672}$$

After various integrations by parts, it is easy to see that eqs. (669–671) reproduce eq. (668).

Now we can use the general matrix relation

$$\det AB = \det A\,\det B \tag{673}$$

to conclude that

$$\det M = \det M_0\,\det \widetilde{M}\ . \tag{674}$$

The advantage of this decomposition is that M_0 is independent of the background field φ, and so the resulting factor of $(\det M_0)^{-1}$ in $Z(\varphi)$ can simply be absorbed into the overall normalization. Furthermore, we have $\widetilde{M} = I - G$, where

$$I(x,y) = \delta^4(x-y) \tag{675}$$

is the identity matrix, and

$$G(x,y) = g\Delta(x-y)\varphi(y)\ . \tag{676}$$

Thus, for $\varphi(x) = 0$, we have $\widetilde{M} = I$ and so $\det\widetilde{M} = 1$. Then, using eq. (667) and the normalization condition $Z(0) = 1$, we see that for nonzero $\varphi(x)$ we must have simply

$$Z(\varphi) = (\det\widetilde{M})^{-1}\ . \tag{677}$$

Next, we need the general matrix relation

$$\det A = \exp \operatorname{Tr} \ln A \; , \tag{678}$$

which is most easily proven by remembering that the determinant and trace are both basis independent, and then working in a basis where A is in Jordan form (that is, all entries below the main diagonal are zero). Thus we can write

$$\begin{aligned} \det \widetilde{M} &= \exp \operatorname{Tr} \ln \widetilde{M} \\ &= \exp \operatorname{Tr} \ln(I - G) \\ &= \exp \operatorname{Tr} \biggl[-\sum_{n=1}^{\infty} \frac{1}{n} G^n \biggr]. \end{aligned} \tag{679}$$

Combining eqs. (677) and (679) we get

$$Z(\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} G^n \; , \tag{680}$$

where

$$\operatorname{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \, \Delta(x_1{-}x_2)\varphi(x_2) \ldots \Delta(x_n{-}x_1)\varphi(x_1) \; . \tag{681}$$

This is our final result for $Z(\varphi)$.

To better understand what it means, we will rederive it in a different way. Consider treating the $g\varphi\chi^\dagger\chi$ term in Ł as an interaction. This leads to a vertex that connects two χ propagators; the associated vertex factor is $ig\varphi(x)$. According to the general analysis of section 9, we have $Z(\varphi) = \exp i\Gamma(\varphi)$, where $i\Gamma(\varphi)$ is given by a sum of connected diagrams. (We have called the exponent Γ rather than W because it is naturally interpreted as a quantum action for φ after χ has been integrated out.) The only diagrams we can draw with these Feynman rules are those of fig. (16), with n insertions of the vertex, where $n \geq 1$. The diagram with n vertices has an n-fold cyclic symmetry, leading to a symmetry factor of $S = n$. The factor of i associated with each vertex is canceled by the factor of $1/i$ associated with each propagator. Thus the value of the n-vertex diagram is

$$\frac{1}{n} \, g^n \int d^4x_1 \ldots d^4x_n \, \Delta(x_1{-}x_2)\varphi(x_2) \ldots \Delta(x_n{-}x_1)\varphi(x_1) \; . \tag{682}$$

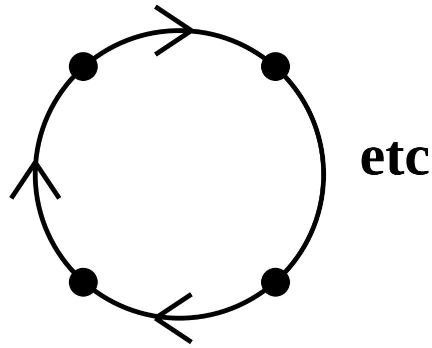

Figure 16: All connected diagrams with $\varphi(x)$ treated as an external field. Each of the n dots represents a factor of $ig\varphi(x)$, and each solid line is a χ or Ψ propagator.

Summing up these diagrams, and using eq. (681), we find

$$i\Gamma(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} G^n \ . \tag{683}$$

This neatly reproduces eq. (680). Thus we see that a functional determinant can be represented as an infinite sum of Feynman diagrams.

Next we consider a theory of a Dirac fermion Ψ with

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + g\varphi\overline{\Psi}\Psi \ , \tag{684}$$

where φ is again a real scalar background field. We define the path integral

$$Z(\varphi) = \int \mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\ e^{i\int d^4x\,\mathcal{L}} \ , \tag{685}$$

where we again use the ϵ trick to impose vacuum boundary conditions, and the normalization $Z(0) = 1$ is fixed by hand.

Recall from section 44 that if we have n complex Grassmann variables ψ_i, then we can evaluate gaussian integrals by the general formula

$$\int d^n\bar{\psi}\, d^n\psi\ \exp\Big(-i\bar{\psi}_i M_{ij}\psi_j\Big) \propto \det M \ . \tag{686}$$

In the case of the functional integral in eq. (685), the index i on the integration variable is replaced by the continuous spacetime label x plus the spinor index

α, and the "matrix" M becomes

$$M_{\alpha\beta}(x,y) = [-i\partial\!\!\!/_x + m - g\varphi(x)]_{\alpha\beta}\delta^4(x-y)\ . \tag{687}$$

In order to apply eq. (686), we have to understand what it means to compute the determinant of this expression.

To this end, let us first note that we can write $M = M_0\widetilde{M}$, which is shorthand for

$$M_{\alpha\gamma}(x,z) = \int d^4y\, M_{0\alpha\beta}(x,y)\widetilde{M}_{\beta\gamma}(y,z)\ , \tag{688}$$

where

$$M_{0\alpha\beta}(x,y) = (-i\partial\!\!\!/_x + m)_{\alpha\beta}\delta^4(x-y)\ , \tag{689}$$

$$\widetilde{M}_{\beta\gamma}(y,z) = \delta_{\beta\gamma}\delta^4(y-z) - gS_{\beta\gamma}(y-z)\varphi(z)\ . \tag{690}$$

Here $S_{\beta\gamma}(y-z)$ is the Feynman propagator, which obeys

$$(-i\partial\!\!\!/_y + m)_{\alpha\beta}S_{\beta\gamma}(y-z) = \delta_{\alpha\gamma}\delta^4(y-z)\ . \tag{691}$$

After various integrations by parts, it is easy to see that eqs. (688–690) reproduce eq. (687).

Now we can use eq. (674). The advantage of this decomposition is that M_0 is independent of the background field φ, and so the resulting factor of $\det M_0$ in $Z(\varphi)$ can simply be absorbed into the overall normalization. Furthermore, we have $\widetilde{M} = I - G$, where

$$I_{\alpha\beta}(x,y) = \delta_{\alpha\beta}\delta^4(x-y) \tag{692}$$

is the identity matrix, and

$$G_{\alpha\beta}(x,y) = gS_{\alpha\beta}(x-y)\varphi(y)\ . \tag{693}$$

Thus, for $\varphi(x) = 0$, we have $\widetilde{M} = I$ and so $\det\widetilde{M} = 1$. Then, using eq. (686) and the normalization condition $Z(0) = 1$, we see that for nonzero $\varphi(x)$ we must have simply

$$Z(\varphi) = \det\widetilde{M}\ . \tag{694}$$

Next, we use eqs. (679) and (694) to get

$$Z(\varphi) = \exp\, -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} G^n \;, \tag{695}$$

where now

$$\operatorname{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \operatorname{tr} S(x_1{-}x_2)\varphi(x_2) \ldots S(x_n{-}x_1)\varphi(x_1) \;, \tag{696}$$

and "tr" denotes a trace over spinor indices. This is our final result for $Z(\varphi)$.

To better understand what it means, we will rederive it in a different way. Consider treating the $g\varphi\overline{\Psi}\Psi$ term in Ł as an interaction. This leads to a vertex that connects two Ψ propagators; the associated vertex factor is $ig\varphi(x)$. According to the general analysis of section 9, we have $Z(\varphi) = \exp i\Gamma(\varphi)$, where $i\Gamma(\varphi)$ is given by a sum of connected diagrams. (We have called the exponent Γ rather than W because it is naturally interpreted as a quantum action for φ after Ψ has been integrated out.) The only diagrams we can draw with these Feynman rules are those of fig. (16), with n insertions of the vertex, where $n \geq 1$. The diagram with n vertices has an n-fold cyclic symmetry, leading to a symmetry factor of $S = n$. The factor of i associated with each vertex is canceled by the factor of $1/i$ associated with each propagator. The closed fermion loop implies a trace over the spinor indices. Thus the value of the n-vertex diagram is

$$\frac{1}{n}\, g^n \int d^4x_1 \ldots d^4x_n \operatorname{tr} S(x_1{-}x_2)\varphi(x_2) \ldots S(x_n{-}x_1)\varphi(x_1) \;. \tag{697}$$

Summing up these diagrams, we find that we are missing the overall minus sign in eq. (695). The appropriate conclusion is that we must associate an extra minus sign with each closed fermion loop.