LIE GROUPS EXERCISES

Gerard 't Hooft

Spinoza Instituut
Postbus 80.195
3508 TD Utrecht

e-mail: `g.thooft@phys.uu.nl`
`internet: http://www.phys.uu.nl/~thooft/`

Exercises with the lecture course Topics in Theoretical Physics, part II, Lie Groups 2007.

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1. Lecture April 23

Consider matrices A with two rows and columns(2×2 -matrices), and only real coefficients. We demand that A be orthogonal($A\tilde{A} = \mathbb{I}$), and that $\det(A) = 1$.

a. Show that these matrices form a group, and that one parameter (de rotation angle) suffices to describe the matrix. This group is called $SO(2)$.

b. Show that these matrices commute: two matrices of this type, A and B, obey $[A, B] = 0$. Such groups are called *Abelian*.

2. Lecture April 23

Consider the definition of a *group* as in the notes. A group obeys the 4 axioms listed there.

a. Derive, only using these axioms, that the "left-inverse" R^{-1} defined at point 4, also serves as a "right-inverse": $R\,R^{-1} = \mathbb{I}$
(Hint: Consider $R^{-1}\,R\,R^{-1}$ and make use of the fact that also R^{-1} has an inverse).

b. Prove that the unit element is also the "right-unit-element": $R = R\,\mathbb{I}$ for every R
(Hint: multiply at the right with $R^{-1}\,R$)

c. Prove that there is *exactly one* unit element.

3. Lecture April 23

We give a few elements of a representation of the three-dimensional rotation group $SO(3)$ that consists of 2 dimensional matrices:

$$R_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \tag{3.1}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} , \tag{3.2}$$

Compute the matrices belonging to some other elements of this group, such as

$$R_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{3.3}$$

4. Lecture May 7

Let $R_{(1)}$ be a rotation over 90° around the z-axis and $R_{(2)}$ a rotation over 45° around the x-axis.

a. Write the corresponding matrices.

b. Compute the product matrix $R_{(3)} = R_{(2)}\, R_{(1)}$.

c. The matrix $R_{(3)}$ may also be viewed as a rotation over an angle φ around an axis $(a_1,\, a_2,\, a_3)$. Show that

$$\mathrm{Tr}\,(R_{(3)}) = 1 + 2\cos\varphi\,, \tag{4.1}$$

and compute the angle φ.

d. The coordinates of the axis can be deduced from the *antisymmetric part* of $R_{(3)}$, see equation (3.10) of the notes. Find the orientation of this axis.

e. Check that the length of the vector indeed produces the sine of the rotation angle.

5. Lecture May 7

Consider the discussion of page 16—20 of the notes, but now applied to rotations in a *four* dimensional space. Thus, we have vectors $(x,\, y,\, z,\, u)$ that we rotate using orthogonal 4×4 matrices.

a. Show that there are 6 linearly independent generators. We may call them $L_1,\, \cdots,\, L_6$. Thus, a 4×4 matrix R can be written as

$$R = \exp\Big(i \sum_{k=1}^{6} \alpha_k\, L_k\Big) \tag{5.1}$$

b. Make a list of all structure constants $c_{ij}^{\,k}$.

6. Lecture May 7.

a. Derive equation (3.32) (see the notes) for the structure constants $c_{ij}^{\,m}$.

b. Show that, if we assume L_i to be hermitean, the structure constants are *purely imaginary.* Show furthermore that these constants are antisymmetric in the indices i en j: $c_{ij}^{\,m} = -c_{ji}^{\,m}$.

c. Now introduce the so-called *metric tensor* g_{ij} (using sommation convention):

$$g_{ij} \stackrel{\text{def}}{=} c_{ik}^{\ m} c_{mj}^{\ k} \ . \tag{6.1}$$

Show that g_{ij} is symmetric: $g_{ij} = g_{ji}$.

d. Show (with the use of (3.32)) that the 'normalized structure constants', $\tilde{c}_{kij} = g_{km} c_{ij}^{\ m}$ are completely antisymmetric:

$$\tilde{c}_{kij} = \tilde{c}_{ijk} = -\tilde{c}_{kji} \ . \tag{6.2}$$

Often, we *normealize* the L_i in such a way that $g_{ij} = \delta_{ij}$.

7. Lecture May 7.

Study Appendix D: The Campbell-Baker-Hausdorff formula. It implies that, given two matrices A and B, a matrix C exists, such that

$$e^A e^B = e^C \ , \tag{7.1}$$

and that C can be written as a power series in A and B consisting *exclusively of commutators*, and no direct products.

a. Admitting complex numbers, show that there are several matrices C_i such that (7.1) holds.
Hint: diagonalize C and add $2\pi i$ to one of its eigenvalues,

b. There are several ways to prove the theorem. Instead of equation (D.2) we can also define

$$e^{C(x)} = e^{xA} e^B \ . \tag{7.2}$$

Use the same method as in the Appendix to derive that

$$\frac{\mathrm{d}C(x)}{\mathrm{d}x} = \{A, \frac{C}{1-e^{-C}}\} = A + \tfrac{1}{2}[A, C] + \tfrac{1}{12}[[A, C], C] + \cdots \ . \tag{7.3}$$

c. Use this result to compute the coefficients in (D.29).

This exercise is a bit more difficult than what will be required at the test of this course. The proof of the CBH formula will not have to be reproduced in the test.

8. Lecture May 14

Consider again the group of orthogonal rotations in 4 dimensions. The elementary representation is formed by 4-vectors themselves. The matrices R en D then remain the same. This is the **4** representation. Write the components of the 4 vector as x^μ, $\mu = 1, 2, 3$ or 4.

a. Now consider the 16-dimensional space spanned by the product of two such vectors. Write $A^{\mu\nu}$, $\mu, \nu = 1, \cdots, 4$. Now give the matrix D if under a rotation

$$A^{\mu\nu} \to R^{\mu\alpha} R^{\nu\beta} A^{\alpha\beta} \, . \tag{8.1}$$

b. This **16** representation however is not irreducible. Show that the two subspaces spanned by the *symmetric* tensors $A^{\mu\nu}$, for which $A^{\mu\nu} = A^{\nu\mu}$, and those of the *antisymmetric* tensors, $A^{\mu\nu} = -A^{\nu\mu}$, are invariant subspaces, with dimensions 10 and 6, respectively.

c. Show that the 10-dimensional representation can be split up further (by looking at the *trace* $A^{\mu\mu}$: $\mathbf{10} = \mathbf{9} \oplus \mathbf{1}$.

d. But also the antisymmetric representation **6** can be split up. Consider two kinds of tensors:

$$A_L^{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} A_L^{\alpha\beta} \, , \tag{8.2}$$

$$A_R^{\mu\nu} = -\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} A_R^{\alpha\beta} \, , \tag{8.3}$$

to establish that there are two invariant subspaces: $\mathbf{6} = \mathbf{3}_L \oplus \mathbf{3}_R$. Later we will find out that the $\mathbf{3}_L$ en $\mathbf{3}_R$ are not unitarily equivalent.

9. Lecture May 14

Consider the generators L_i of the rotations as given in equation (3.14) of the notes.

a. They generate what we call the 'adjoint representation' of the group $SO(3)$. Find the eigenvalues $m = -1,\, 0,\, +1$ of L_3, and the associated eigenvectors (in the basis where (3.14) holds), which we will call $|m\rangle$.

b. Construct the matrices L_+ and L_-, and find out how they act on the states $|m\rangle$.

c. Check whether you indeed get $|m \pm 1\rangle$, with the norm factors $\sqrt{2 - m(m \pm 1)}$ as expected according to chapter 5.

d. Compute the Casimir operator $\sum_i L_i^2$ and check whether indeed the quantum number ℓ is 1 .

10. Lecture May 14 and May 21

This exercise is more difficult. It exhibits procedures that will be further elaborated on in the lectures.

A representation A of $SO(3)$ has basis elements ψ^A_α, $\alpha = 1, \cdots, N_A$, and a representation B has basis elements ψ^B_κ, $\kappa = 1, \cdots, N_B$. Consider the product representation $A \otimes B$ having the products $\psi^A_\alpha \psi^B_\kappa$ as its basis elements. The dimension of this representation is $N_A\, N_B$.

a. Show that the generators of this representation can be written as $I_i^{A\otimes B} = I_i^A + I_i^B$, where I_i^A only act on the indices α of representation A and I_i^B only on the indices κ of representation B.

b. Show that $[I_i^A,\, I_j^B] = 0$. We still have $[I_i^A,\, I_j^A] = i\varepsilon_{ijk} I_k^A$ and $[I_i^B,\, I_j^B] = i\varepsilon_{ijk} I_k^B$.

c. Show that the operators $I_i^{A\otimes B}$ obey the correct commutation rules. Omitting superscript $A \otimes B$: $[I_i,\, I_j] = i\varepsilon_{ijk} I_k$.

d. Write the basis elements of $A \otimes B$ as $|m^A, m^B\rangle$. How do the operators $I_\pm$ act on this basis?

e. The new representation $A \otimes B$ is not irreducible. Consider the values that $m^{A\otimes B}$ can have and use that to derive that the maximal value of $\ell^{A\otimes B}$ must be equal to $\ell^A + \ell^B$.

f. Take the case $\ell^A = \frac{5}{2}$ and $\ell^B = \frac{3}{2}$. Show that there is only one state with $m^{A\otimes B} = 4$. It hust have $\ell^{A\otimes B} = 4$ (convince yourself of this by letting I_+ act on this state). It is indicated as $|\ell = 4,\ m = 4\rangle$. How many basis elements do we have with $m^{A\otimes B} = 3$, with $m^{A\otimes B} = 2$, etc. ?

g. Now let I_- act on them. Construct the state $|\ell = 4,\ m = 3\rangle$. Show that it is equal to $\sqrt{\frac{5}{8}}\,|\frac{3}{2},\, \frac{3}{2}\rangle + \sqrt{\frac{3}{8}}\,|\frac{5}{2},\, \frac{1}{2}\rangle$. Why do we still have $\ell = 4$?

h. There is an other state with $m^{A\otimes B} = 3$ orthogonal to the one found above. Construct this state and show that it is $|\ell = 3,\ m = 3\rangle$.

i. This way we can find all states $|\ell,\, m\rangle$. Indicating the representations by their dimension $N = 2\ell + 1$, show that $\mathbf{4} \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{5} \oplus \mathbf{7} \oplus \mathbf{9}$.

j. In general: $\ell^{A\otimes B} = |\ell^A - \ell^B|,\ |\ell^A - \ell^B| + 1,\ \cdots,\ \ell^A + \ell^B$. Show that the sums of the dimensions of these irreducible parts match:

$$\sum_{\ell=|\ell^A-\ell^B|}^{\ell^A+\ell^B} (2\ell + 1) = (2\ell^A + 1)(2\ell^B + 1)\ . \tag{10.1}$$

11. Lecture May 21.

a. Show that 2×2-matrices X of the form $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, where a an b are arbitrary complex numbers, form a group because they obey the axioms of page 8.

b. Show that this remains to be true if we impose on a en b the condition that $|a|^2 + |b|^2 = 1$.

c. Show that this condition corresponds to $\det(X) = 1$.

12. Lecture May 21.

Consider the *four*-dimensinal vector space spanned by the 4 matrix elements of the 2×2 matrices.

a. Show that the three τ matrices and the identity matrix, which we could write as

$$(\tau_0)^\alpha_\beta = \delta^\alpha_\beta$$

form orthonormal basis elements in this vector space, apart from a normalizing coefficient:

$$(\tau_i)^{*\,\beta}_\alpha (\tau_j)^\beta_\alpha = N\delta_{ij}\ , \qquad i,j = 0,\cdots,3\ .$$

Compute the coefficient N and then normalize these basis elements.

b. In this basis, therefore,

$$\sum_{i=0}^{3} (\tau_i)^*_\mu\, (\tau_i)_\nu = N\delta_{\mu\nu}\ , \quad \mu,\nu = 1,\cdots,4\ .$$

This must hold in any basis. Show now that

$$\sum_{i=0}^{3} (\tau_i)^{*\ \beta}_\alpha (\tau_i)^\gamma_\delta = N\delta_{\alpha\delta}\delta_{\beta\gamma}\ .$$

Use this to prove Eq. (6.41) in the lecture notes (6.42 in the new English version):

$$\sum_{i=1}^{3} (\tau_i)^\alpha_\beta (\tau_i)^\gamma_\delta = -\delta^\alpha_\beta\delta^\gamma_\delta + 2\delta^\alpha_\delta\delta^\gamma_\beta\ .$$

This argument may sound complicated at first sight, but it is in fact the fastest way to prove this equation.

13. Lecture May 21.

We consider analytical functions f of two complex variables φ^1, φ^2. The operators L_i, $i = 1, 2, 3,$ are defined as in eq. (6.38), or (6.39) in the English version:

$$L_i = -\tfrac{1}{2}(\tau_i)^\alpha{}_\beta \varphi^\beta \frac{\partial}{\partial \varphi^\alpha} \ . \tag{13.1}$$

Show that they obey the commutation rules of $SU(2)$ (and $SO(3)$):

$$[L_i,\, L_j] = i\,\varepsilon_{ijk} L_k \ . \tag{13.2}$$

Note the minus sign in the definition!

14. Lecture June 4.

Show, using the arguments on the last pages of section 6 of the Notes, that *all* representations of $SO(3)$ with ℓ = integer $+\frac{1}{2}$ have the property that the rotation over 2π is mapped onto the matrix $-\mathbb{I}$. Give an argument why, consequently, orbital wave functions of quantum particles always have integral angular momentum.

15. Lectures until June 4

Consider again the rotation group in 4 dimensions, $SO(4)$. Consider the 6 generators found in exercise 5. Instead of writing them as $L_1, \cdots, L_6$, we write them as $L_k^{(1)} = (L_{23},\, L_{31},\, L_{12})$, $L_k^{(2)} = (L_{14},\, L_{24},\, L_{34})$, with $k = 1, 2, 3$. Here, L_{ij} generates a rotation in the (ij) plane.

a) Compute the commutators $[L_i^{(a)}, L_j^{(b)}]$. Hint: first discover the general rule for the commutators $[L_{ij}, L_{kl}]$ by looking at some examples.

Consider two sets of vectors, $L_k^{(+)}$ and $L_k^{(-)}$, defined by: $L_k^{(\pm)} = \frac{1}{2}(L_k^{(1)} \pm L_k^{(2)})$.

b) Show that $[L_i^{(+)}, L_j^{(+)}] = i\varepsilon_{ijk} L_k^{(+)}$ and similarly for $L_i^{(-)}$.

c) Show that $[L_i^{(+)}, L_j^{(-)}] = 0$.

d) This is the algebra for $SU(2)_L \otimes SU(2)_R$, where L, R refer to the $+$ and the $-$ components of the generators. Now argue that the 4-vector x^μ is in the $\mathbf{2}_L \otimes \mathbf{2}_R$ representation of this group. Hint: compute how $L_k^{(1)}$ acts on $\mathbf{2}_L \otimes \mathbf{2}_R$, and note that this representation indeed contains a 3-vector and a scalar (the 4th component).

e) Now take the skew symmetric (=antisymmetric) product of two of these: $A_{\mu\nu} = -A_{\nu\mu}$, as in exercise 8. Explain that this representation decomposes into two $SU(2)_L \otimes SU(2)_R$ representations: $(\mathbf{2}_L \otimes \mathbf{2}_R) \otimes (\mathbf{2}_L \otimes \mathbf{2}_R)_{\text{antisymm}} = \mathbf{3}_L \oplus \mathbf{3}_R$.

f) Similarly, show that the symmetric part of this product, $A_{\mu\nu} = A_{\nu\mu}$, obeys $(\mathbf{2}_L \otimes \mathbf{2}_R) \otimes (\mathbf{2}_L \otimes \mathbf{2}_R)_{\text{symm}} = \mathbf{3}_L \otimes \mathbf{3}_R \oplus \mathbf{1}$. Indeed, the $\mathbf{1}$ is the trace, and the $\mathbf{9}$ represents the rest.

16. Lecture June 11.

Derive Equations (7.18)—(7.20) from Eq. (7.17) in the notes.

17. Lecture June 11.

Consider an experiment as in Chapter 7, but now the scattering of a particle with spin 1 against a spherically symmetric target. We write the representation in the basis where L_z is diagonal, so

$$\psi^{+1} = f(\vec{r}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \psi^{0} = f(\vec{r}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \psi^{-1} = f(\vec{r}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{17.1}$$

Now define the scattering function $F(\theta)$ as the 3×3 matrix

$$\begin{pmatrix} f_{++}(\theta) & f_{+0}(\theta) & f_{+-}(\theta) \\ f_{0+}(\theta) & f_{00}(\theta) & f_{0-}(\theta) \\ f_{-+}(\theta) & f_{-0}(\theta) & f_{--}(\theta) \end{pmatrix} . \tag{17.2}$$

a. Construct the matrix that generates infinitesimal rotations around the x axis in this basis: $L_x = \frac{1}{2}(L_+ + L_-)$.

b. We find the rotation over a finite angle ξ, or $\exp(i\xi L_x)$, by applying (3.10):

$$\exp(i\xi L_x) = \mathbb{I} + (\cos\xi - 1)L_x^2 + i\sin\xi\, L_x \ . \tag{17.3}$$

Show that this follows from $L_x^{2n+1} = L_x$ and compute $\exp(i\xi L_x)$.

c. Find out how $F(\theta)$ transforms under this rotation.

d. What is the most general angle dependence to be expected for f_{++}, f_{00} en f_{--}?

e. Now compute the angular distribution for scattering experiments with particles having $S_z = 1, 0,$ en -1. Arguments using parity symmetry may be used to find further restrictions for the parameters in this result, which otherwise would be allowed to be anything.

f. So, how do we distinguish particles with spin $\frac{1}{2}$ from particles with spin 1?

18. Lecture June 11.

Just like the ordinary nucleons $N = (p, n)$, the N^* has isospin $\frac{1}{2}$. Its mass is sufficient to decay, just like the Δ particles, into a nucleon and a pion. The resulting states $|N, \pi\rangle$ now must have total isospin $\frac{1}{2}$. Of all states occurring in formula (8.6), we now search for those combinations that have $I^{\rm tot} = \frac{1}{2}$ and $I_3 = +\frac{1}{2}$.

a. Find the two basis elements with $I_3 = +\frac{1}{2}$. One combination must have $I^{\rm tot} = \frac{3}{2}$ and one combination, orthogonal to that, has $I^{\rm tot} = \frac{1}{2}$. Why are these combinations orthogonal to one another?

b. Show that the state $I^{\rm tot} = \frac{1}{2}$ must obey $I_+|\psi\rangle = 0$. Now find this state.

c. The state $I^{\rm tot} = \frac{1}{2}$, $I_3 = -\frac{1}{2}$ is now obtained from the result of (b) by letting I_- act on it. Compute this state.

d. An alternative method is to solve the equation $I_-|\psi\rangle = 0$. Show that this produces the same result, apart from a possible phase factor (or minus sign).

e. Now compute the decay probabilities for

$$\begin{aligned} N^{*+} &\rightarrow p + \pi^0 , \qquad & N^{*+} &\rightarrow n + \pi^+ , \\ N^{*0} &\rightarrow p + \pi^- , \qquad & N^{*0} &\rightarrow n + \pi^0 . \end{aligned} \tag{18.1}$$

Observe the difference with the Δ decay, Eq. (8.14).

19. Lecture June 18

It is not difficult to show that

$$\begin{aligned} [L_i,\, r_j] &= i\hbar\varepsilon_{ijk} r_k \\ \text{and} \qquad [L_i,\, p_j] &= i\hbar\varepsilon_{ijk} p_k . \end{aligned} \tag{19.1}$$

Now show that, if two arbitrary vectors $\vec{A}$ and $\vec{B}$ both obey the commutation rules with L_i as in (19.1), and if an other operator R commutes with all L_i, then the same commutation relations are obeyed by vectors $\vec{C}$ and $\vec{D}$ defined according to $\vec{C} = R\,\vec{A}$, and $\vec{D} = \vec{A} \times \vec{B}$, so

$$\begin{aligned} [L_i,\, C_j] &= i\hbar\varepsilon_{ijk} C_k \\ \text{and} \qquad [L_i,\, D_j] &= i\hbar\varepsilon_{ijk} D_k . \end{aligned} \tag{19.2}$$

This is the so-called Wigner-Eckhart theorem. With this, we prove both (9.18) and (9.21) in one stroke.

20. Lecture June 18

a. Check the calculations that lead to the equations $[\vec{K},\, H] = 0$ (9.24) and $\vec{K} \cdot \vec{L} = 0$ (9.29) for the quantum mechanical Runge-Lenz vector (9.16).

b. Show that

$$(\vec{L} \times \vec{p}) \cdot (\vec{p} \times \vec{L}) = -p^2 L^2 \ . \tag{20.1}$$

Make sure that no extra terms arise by correctly manipulating the commutation rules

c. We now calculate $\vec{K}^2$ by writing $\vec{K}$ as

$$\vec{K} = \frac{1}{\mu e^2}(\vec{L} \times \vec{p} - \hbar i \vec{p}) + \frac{\vec{r}}{r} \ = \ \frac{1}{\mu e^2}(-\vec{p} \times \vec{L} + \hbar i \vec{p}) + \frac{\vec{r}}{r} \ , \tag{20.2}$$

after which we multiply one expression with the other, making use of b. Now prove (9.30).

21. Lecture June 18

Consider the quantum states of the hydrogen atom at given n.

a. Show that both L_3^+ and L_3^- take the values $-\frac{1}{2}(n-1),\ -\frac{1}{2}(n-1)+1,\ \cdots,\ \frac{1}{2}(n-1)$.

b. What are now the possible values of $L_3 = L_3^+ + L_3^-$, and what is the degree of degeneracy at these values?

c. Use this to derive that ℓ takes the integral values $0,\, 1,\, \cdots\, n-1$ all exactly once, according to equation (9.45).
(This was explained in the lecture; make sure that the argument is well understood)