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Rational Choice and Erratic Behaviour

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1. INTRODUCTION

In this paper we show that rational choice in a stationary environment can lead to erratic behaviour when preferences depend on experience. We mean by erratic behaviour choice sequences that do not converge to a long-run stationary value or to any periodic pattern.

Early investigations of the effects of experience on choice goes back to Pareto (see also Benhabib (1979)); some Keynesian theories of the consumption function incorporate the effects of experience and habit formation on current levels of consumption (Duesenberry (1949), Modigliani (1949) and Brown (1952)); a general model of experience dependent choice was described in Day (1970) and investigated in Day and Kennedy (1971) while the specific issue of the existence of stable representable long-run consumer demand when tastes vary endogenously has been treated by Gorman (1967), Pollak (1970, 1976), Weizsäcker (1971), McCarthy (1974) and Hammond (1976). Pollak gave conditions for the representability of long-run choice when the equations of preference dependence are linear and utility additive. Hammond, in a generalization of these findings, provides conditions for stable, representable long run choice when preferences are acyclic, a condition which in essence assures convergence to a long-run choice.

Our concern here is with the contrary case when preferences are at least partially cyclic. In a stable environment (for example, fixed prices and incomes) acyclic preferences rule out feasible alternatives that have been rejected in previous periods. Thus sequences of alternating choices (consisting for instance of holidays in the mountains alternating with vacations at the sea) are impossible. While habit formation may explain the absence of such behaviour, diversity of behaviour and fluctuation in choices are too common to be assumed away. The surprising implication of some cyclicity in preferences is the occurrence of erratically fluctuating behaviour, which in our theory emerges endogenously from a completely deterministic structure in a stationary environment. This "chaos" contrasts with the usual explanations for such phenomena which are couched in terms of exogenously determined random shocks.

Many of the ideas which we present were already introduced, in heuristic form, in Georgescu-Roegen's (1950) analysis of consumer choice under endogenously changing preferences. (See also Georgescu-Roegen (1971, pp. 126–127).) He clearly realized the erratic nature of dynamic choice behaviour which led him to conclude the following:

> "Does the preceding analysis justify the negation of the constancy of economic laws? The right answer seems to be that, on the contrary, it eliminates the variability of consumer's behaviour as an eventual argument against such a constancy. However, the micro-approach is deprived to a large extent of any quantitative predictability over finite (i.e. important) changes."

Erratic behaviour of the type under investigation here was first shown to exist for equations of hydrodynamic flow applied to modelling turbulence in fluids or weather phenomenon by Lorenz (1963, 1964). Applications in biological population theory followed in the work of May (1975), May and Oster (1976) and Guckenheimer, Oster and Ipaktchi (1977) (to mention a few authors). A formal theory and existence conditions for the discrete case has been set forth by Li and Yorke (1975) whose results were generalized by Diamond (1976). Chaos in the sense of Lorenz and Li and Yorke has its continuous time counterpart in the phenomenon of "strange attractors" first discussed by Ruelle and Takens (1971). A highly readable survey will be found in Yorke and Yorke (1980).

In Section 2 of this paper we review the "chaos" theorems of Li and Yorke and Diamond and present a generalization suitable for application to the set-valued choice functions that arise in general economic theory. We then give, in Section 3, two examples of unique choice in $\mathbb{R}^2$ based on experience dependent utility to which the basic Li–Yorke theorem applies. This is followed in Section 4 by a general definition and analysis of tricyclic preferences in $\mathbb{R}^n$ and by the extension to dynamic, set-valued choice functions in Section 5. The paper concludes with remarks concerning interpretation.

From the point of view of empirical application the examples of Section 3 are highly suggestive. They show that erratic sequences of rational choices do not arise when income is low but can when wealth gets sufficiently high. Thus the "poor" might exhibit quite stable responses to price and income changes while the "rich" may be quite unstable appearing arbitrary or whimsical.

2. ERRATIC BEHAVIOUR OR "CHAOS"

The investigation of behaviour when preferences depend on experience boils down to the study of solutions of the dynamic choice functions. We will show that when preferences possess a special type of cyclic structure there exist periodic choice sequences (cycles) of every order and also choice sequences that are erratic in the well-defined sense that they are not stationary, periodic or asymptotically stationary or periodic. Before proceeding with the analysis we review some basic concepts, definitions and theorems.

The iterated map $C^k(\cdot\,;\alpha,s)$ is defined recursively by $C^{k+1}(\cdot\,;\alpha,s),\ \alpha,s]=C[C^k(\cdot\,;\alpha,s);\alpha,s]$, $k=0,1,2,\ldots$ where $C^0(\cdot\,;\alpha,s)=I$ (the identity map) and where $C^1(\cdot\,;\alpha,s)=C(\cdot\,;\alpha,s)$. A point $x\in X(s)$ is called k-periodic if $C^k(x;\alpha,s)=x$ and if $C^j(x\,;\alpha,s)\neq x$ for $0<j<k$. A set $A\in X(s)$ is called k-periodic if each of its elements are k-periodic. We note that if x is k-periodic then each of the points $x_j=C^j(x)$, $j=1,\ldots,k$ are distinct k-periodic points with $x_j=C^k(x_j)$. If x is a k-periodic point then the sequence $\{x(t)\}$ such that $x(t)=x_{(t \bmod k)}$ where $x(0)=x$ is called a k-periodic choice sequence.

Let $\{x(t)\}$ be a choice sequence. Then, of course, $x(t)=C^t(x\,;\alpha,s)$ where $x(0)=x$. A solution $\{x(t)\}$ is quasi k-periodic (asymptotically k-periodic) if there exists a k-periodic solution $\{x_k(t)\}$ with $x(0)\neq x_k(0)$ such that

$$\limsup_{t\to\infty}|x_k(t)-x(t)|=0. \tag{1}$$

A solution $\{x(t)\}$ is aperiodic if it is neither periodic or quasi-periodic of any period k. A solution $x(t)$ will be called erratic or chaotic if it is aperiodic and remains in a bounded set, say S.

Erratic or chaotic solutions in the sense of Lorenz, Li and Yorke are highly unstable. Indeed, any two chaotic trajectories wander close to each other and, no matter how close such trajectories come to each other they wander apart. These facts are formalized in the expressions

$$\liminf_{t\to\infty}|x(t)-y(t)|=0, \tag{2}$$

$$\limsup_{t\to\infty}|x(t)-y(t)|>0, \tag{3}$$

where $x(t)$ and $y(t)$ are any two chaotic trajectories in S. In addition chaotic trajectories in S wander away from periodic cycles of any period so that the equality in (2) is changed to an inequality.

The existence of erratic or chaotic trajectories depends on an "over-shoot" or nonlinearity condition. This was established by Li and Yorke in a theorem which we present here without proof.

Theorem 1 (Li and Yorke). *Let J be an interval in $\mathbb{R}$ and consider the difference equation*

$$x_{t+1} = f(x_t) \tag{4}$$

in which f is a continuous mapping of $J \to J$. Suppose there exists a point $x \in J$ such that

$$f^3(x) \leqq x < f(x) < f^2(x).$$

Then

A. *For every $k = 1, 2, 3, \ldots$ there is a k-periodic solution of* (4) *in J; and*
B. *There is an uncountable set $S \in J$, which contains no periodic points, such that for every initial condition in S the solution of* (4) *is erratic, i.e. is aperiodic, and remains in S.*

It should be noted that a sufficient condition for erratic behaviour is the existence of a 3-period cycle, that is, a point x satisfying $x = f^3(x)$.

In the next section of this paper we present examples to which this theorem applies.

The extensions of the Li–Yorke theorem to $\mathbb{R}^n$ was accomplished by Phil Diamond (1976). In exploiting the proof of Li–Yorke, but extending it to the more general setting, Diamond lost the constructive quality of the sufficient conditions in the simpler setting. We show in Section 4 that it can nonetheless be applied to establish the existence of erratic economic behaviour.

Let f be a mapping from a subset A of $\mathbb{R}^n$ to $\mathbb{R}^n$. A subset P of A is k-periodic if $f^k(P) = P$ and $f^i(P) \cap f^j(P) = \varnothing$ for $1 \leqq i < j < k$. Then we have

Theorem 2 (Diamond). *Let A be a set in $\mathbb{R}^n$ and suppose $f: A \to \mathbb{R}^n$ is continuous. Assume that there is a non-empty compact set X in A satisfying*

(C1) $X \cup f(X) \subset f^2(X) \subset A$

(C2) $X \cap f(X) = \varnothing$.

Then:

(T1) *For every $k = 1, 2, 3, \ldots$ there is a k-periodic set in A,*
(T2) *There is an uncountable set S in A which contains no periodic set and for which*
  (i) $f(S) \subset S$
  (ii) *for distinct points $p, q \in S$*

$$\limsup_{k\to\infty} |f^k(p) - f^k(q)| > 0$$

  (iii) *for every p in S and periodic set P in A, then for all q in P*

$$\limsup_{k\to\infty} |f^k(p) - f^k(q)| > 0.$$

Diamond's simple extension of the Li–Yorke theorem brings out very nicely the fact that the proof of chaos rests exclusively on the combinatorial and topological character of the sequence of maps $f^1, f^2, \ldots$. No properties of $\mathbb{R}^n$ other than its metrical nature are exploited. For this reason it is possible to state a chaos theorem for set-valued maps. This enables us to extend the results on erratic economic behaviour to dynamic, set-valued choice functions. We state the theorem which we use in Section 5 here. For completeness the proof is stated in the Appendix.

Theorem 3. *Let F be an upper-semicontinuous, compact, set-valued map (correspondence) from a metric space $V \to \mathcal{P}(V)$ with metric d. Assume there exists a set $X \subset V$ such that*

(C1) $X \cup F(X) \subset F^2(X) \subset V$
(C2) $X \cap F(X) = \varnothing$.

Then

A. *For all $k = 1, 2, 3, \ldots$ there exists a k-periodic set, say $X^k \subset V$ such that $F^k(X^k) = X^k$;*
B. *There exists an uncountable set $S \subset V$ which contains no periodic set, such that $F(S) \subset S$ and such that*
   (i) *for all $x, y \in S$, $x \neq y$, and $F(x) \cap F(y) = \varnothing$, there exist trajectories $x_{t+1} \in F(x_t)$, $x_0 = x$, and $y_{t+1} \in F(y_t)$, $y_0 = y$ such that*

$$\limsup_{t\to\infty} d(x_t, y_t) > 0.$$

   (ii) *for all $x \in S$ for any k-periodic set P in X, for all $y \in P$*

$$\limsup_{t\to\infty} d(x_t, y_t) > 0.$$

3. ERRATIC DEMAND WHEN TASTES ARE ENDOGENOUS: TWO EXAMPLES

In order to illustrate the existence of erratic sequences of rational choices, consider the familiar utility function

$$u(x, y; a) = x^a y^{1-a}, \tag{5}$$

in which x and y are amounts of two goods consumed within a given period and a is a utility weight with $0 < a < 1$. Maximizing utility subject to the usual budget constraint

$$px + qy = m \tag{6}$$

yields the demand equations

$$x = a\frac{m}{p}, \qquad y = (1-a)\frac{m}{q}. \tag{7}$$

The dependence of these functions upon experience is obtained by supposing that the parameter of the utility function representing preferences depends endogenously on past choices. Consider the case in which this dependence is upon the immediate past according to a function

$$a_{t+1} = g(x_t, y_t; \alpha). \tag{8}$$

The demand functions now become

$$x_{t+1} = \frac{m}{p} g(x_t, y_t; \alpha), \qquad y_{t+1} = \frac{m}{q}(1 - g(x_t, y_t; \alpha)), \tag{9}$$

or, using the budget restriction to eliminate y_t, we obtain a first-order difference equation in x_t,

$$x_{t+1} = C(x_t; \alpha, s) := \frac{m}{p} g(x_t, (m - px_t)/q : \alpha), \tag{10}$$

where $s := (m, p.)$

If this dynamic choice function is nonlinear enough so that a sufficient "overshoot" can occur then chaotic sequences of consumption pairs (x_t, y_t) will exist. To illustrate this fact we consider a specific form for (8). Thus, suppose the function g is defined by

$$a_{t+1} = \alpha x_t y_t. \tag{11}$$

This represents an ex post weight assigned to experienced consumption in which α is an "experience dependence" parameter. The function implies that the greater this ex post weight is, the more preferences will shift in favour of the good x.

Imagine, for example that we interpret x as leisure and y as consumption of a good obtained by work. Income, or m, would be the total available time to be allocated between leisure and work. Equation (11) now states that the higher the level of satisfaction obtained in one period the greater the weight placed on leisure in future choices. (One might call this complacency!).

Substituting (11) into (7) and exploiting the budget constraint (6) we get the short run (dynamic) demand function

$$x_{t+1} = \alpha m x_t (m - x_t), \tag{12}$$

which is analogous to the form originally investigated by Lorenz in a quite different context. A stationary (long-run) demand exists for $\bar{x} = (\alpha m^2 - 1)/\alpha m$ which is positive for $\alpha m^2 > 1$. The maximum consumption of good x (e.g. leisure) occurs when past experience is $x_t = x^* = m/2$. In this case $x_{t+1} = (\alpha m^3)/4$. If this is to satisfy the budget constraint then $\alpha m^2 \leqq 4$. Consequently, we need only consider a system with income endowment m and experience dependence parameter α such that

$$1 < \alpha m^2 \leqq 4.$$

When αm^2 is close to 1, the stationary state is stable. As αm^2 increases past 3 cycles emerge. At some combination αm^2 such that

$$(\alpha m^2)^2 (r - \alpha m^2) < 8 < 4\alpha m^2$$

a three period cycle occurs with period 3 points, $\alpha m^2(4 - \alpha m^2)/16$, $m/2$ and $m^2/4$. Let us call this point c, the chaos point. Hence, by Theorem 1, for $c < \alpha m^2 \leqq 4$, there exists an uncountable number of initial conditions in the interval $[0, m]$ which lead to erratic solutions. The value of c is approximately 3·57. There are also periodic solutions of every order.

Equation (11) is not the only specification of experience dependent preferences that will generate chaotic behaviour. Many other functions will do. Another example is provided by

$$a_{t+1} = x_t e^{\alpha(1-x_t)}. \tag{13}$$

In this specification x_t may be interpreted as a stimulating factor: "the more leisure one experiences the more one tends to prefer it". The term $e^{\alpha(1-x_t)}$ may be interpreted as a depressing factor: "the more leisure one has experienced the less attractive it appears". The weight on leisure that determines preferences for the imminent period is then the product of those two contending factors.

Using this definition for $g(\cdot)$ we get

$$x_{t+1} = m x_t e^{\alpha(1-x_t)}. \tag{14}$$

A detailed analysis of this equation for $m = 1$ has been carried out by May and Oster (1976) who find that chaos emerges when α is approximately 2·6924.

An interesting relationship occurs between m and α in these two examples. The smaller the experience dependence parameter α the greater the income endowment m must be to generate chaos. The two models then characterize experience dependent demand as converging to a stable long run pattern for relatively low incomes, but exhibit increasing instability and eventually become completely erratic as income grows (reflecting the whimsical, seemingly arbitrary behaviour of the complacent, or the very rich!).

4. TRICYCLIC PREFERENCES

With these examples before us we now turn to the phenomenon of partial tricyclicity of preferences. In this section we confine attention to uniquely determined choice in $\mathbb{R}^n$. In Section 5 we show that similar results follow when rational choice is non-unique and choice functions are set-valued.

Suppose preferences in a choice space $X \subset R^n$ are represented by a utility function $\varphi(x_{t+1}; x_t, \alpha)$, continuous in $(x_{t+1}; x_t)$, strictly quasiconcave in x_{t+1}, and locally non-satiable, where x_t is the immediately past choice which has been experienced and "α" is a parameter (or vector of parameters) upon which preferences depend. Let $X(s)$ be a compact, convex choice set or feasible region that depends on a parameter or vector of parameters s representing an environmental situation. For example, represent prices by p and income by m and define the choice set by the usual budget constraint:

$$X(s) := \{x | px \leqq m, x \geqq 0\},$$

where $s := (p, m)$.

Given our assumptions there exists a unique optimal choice x_{t+1} that solves

$$\pi(x_t, \alpha, s) := \max_{x \in X(s)} \varphi(x; x_t, \alpha) \tag{15}$$

so that

$$\{x_{t+1}\} = X(s) \cap \{x | \varphi(x; x_t, \alpha) \geqq \pi(x_t, \alpha, s)\}. \tag{16}$$

In Section 5 below we consider the case in which φ may be locally satiable or is not strictly quasi-concave so that choice need not be unique. At this point, because we temporarily assumed the strict quasi-concavity and non-satiability of the utility function, $\{x_{t+1}\}$ in (16) is a singleton and we obtain the dynamic choice function,

$$x_{t+1} = C(x_t; \alpha, s) \tag{17}$$

where C is a single-valued function, continuous in x_t. Our problem is to determine characteristics of the preferences representable by φ that will imply that this choice function will satisfy the sufficient conditions for the existence of chaotic choice sequences.

Before proceeding we note that a "long-run" or stationary choice $\bar{x}$ satisfies $\bar{x} = C(\bar{x}; \alpha, s)$ while a long-run or stationary choice function is a continuous single valued function $D(\alpha, s)$ such that $D(\alpha, s) = C[D\alpha, s); \alpha, s]$. Such a function exists by virtue of the continuity of C and the compactness and convexity of $X(s)$. In the endogenous taste literature the representability and stability of such long-run choice functions is examined. Hammond (1976) shows that the answers hinge on the acyclicity of preferences; a requirement that prevents the selection of a feasible choice at a given time that was previously feasible but rejected. This condition precludes oscillations in behaviour within a stationary environment and is equivalent to strong habit formation. Individuals will converge to a stationary mode of behaviour and will reveal consistent preferences in a sequence of choices.

While consistent choices are often observed and while habit is no doubt a powerful stabilizing force in human behaviour it cannot be denied that individuals, in the pursuit of novelty and variety, choose life styles in which activities may be pursued from time to time while others are favoured in-between-times so to speak. Such complex patterns cannot reveal consistency within a stable environment but they can nonetheless be perfectly rational.

Consider the following extreme yet quite common example. An individual, having read a book (such as Shōgun) decides in a later period of leisure not to read the book again but to watch the story enacted on T.V. on the basis of which experience the person is "motivated" to read the book a second time. Or consider the family that alternates ski holidays with vacations at a favourite beach resort. Such consumers, having experienced a given consumption bundle, explicitly shift their preferences away from it at given income and price levels but may well shift back in its favour after experiencing alternative activities.

We now provide a formal definition of experience dependent preferences of this type. Let the upper contour sets of $\varphi(\cdot, x, \alpha)$ when experience is x and optimal utility is $\pi(x; \alpha, s)$ be denoted by

$$\Phi(x; \alpha, s) := \{y \mid \varphi(y, x) \geqq \pi(x, \alpha, s)\} \tag{18}$$

and define

$$\Phi(A; \alpha, s) = \bigcup_{x \in A} \Phi(x; \alpha, s). \tag{19}$$

Definition 1. The preferences represented by $\varphi(\cdot, x, \alpha)$ are partially tricyclic on $X(s)$ if there is some set $A \subset X(s)$ such that

(a) $A \cap \Phi(A; \alpha, s) = \varnothing$, and
(b) $A \cup \Phi(A; \alpha, s) \subset \Phi(\Phi(A; \alpha, s) \cap X(s); \alpha, s)$.

Partially tricyclic preferences are illustrated in Figure 1. It is seen that they imply an oscillation in choice.

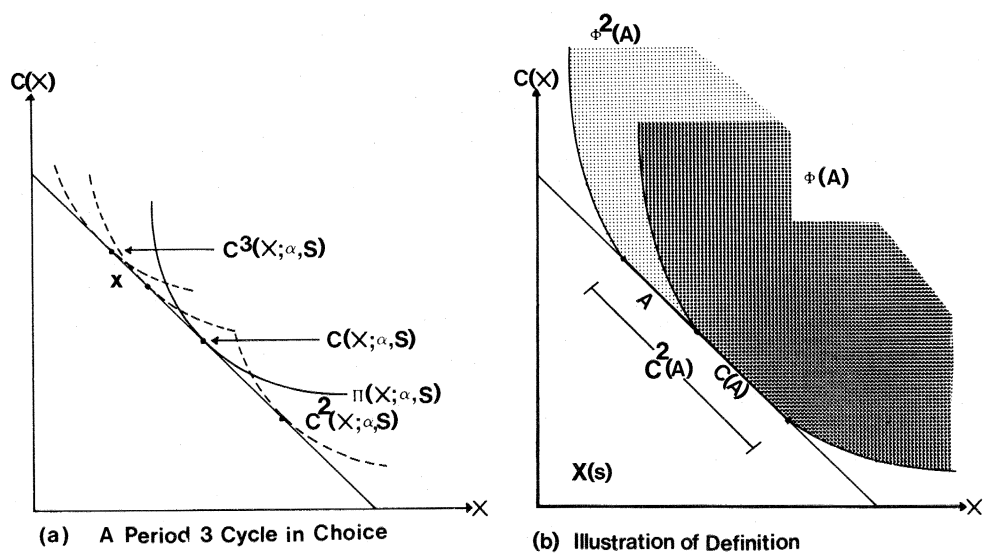
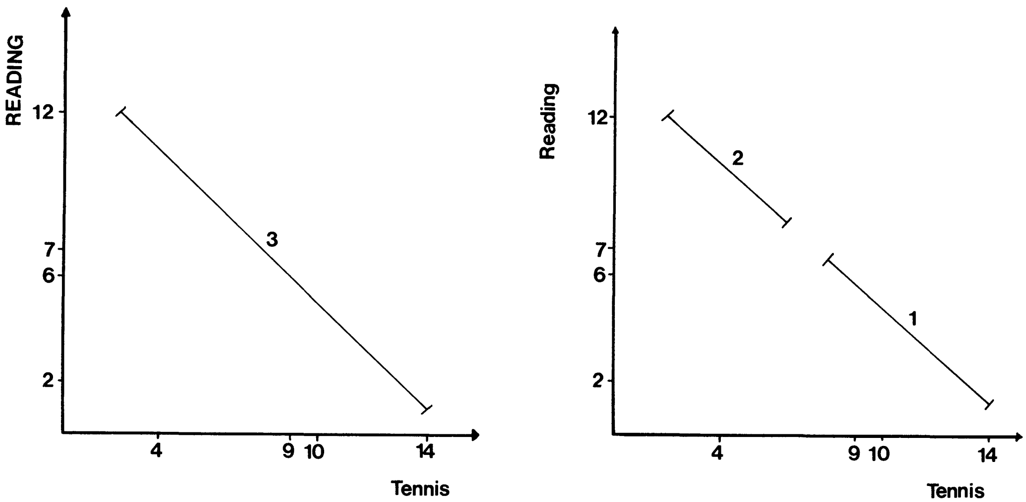


FIGURE 1

Consider the allocation of an individual's time between tennis, reading, and a composite activity which we call sleep, over a three day weekend. Let the individual always sleep eight hours. If he plays tennis between 10 to 14 hours the first day, let him play tennis between 4 to 9 hours the second day depending on his choice the first day, and having done so the second day, let his choice for tennis vary between 4 to 14 hours for the third day depending on his choice for the second day. Then his choice function will be partially tricyclical. This is illustrated in Figure 2.

It should be emphasized that the conditions of partial tricyclicity in Definition 1 has to hold for some but not every subset A of $X(s)$ for a given s. Furthermore, our results will require partial tricyclicity for some s, not over the whole range of s. When $X(s)$ is "small" (for example when income is low) preferences may be acyclic, but when $X(s)$ is "large" (for example when income is large) tricyclicity may emerge. Indeed, this latter possibility was precisely the character of the examples of Section 3.



2nd day to 3rd day: Clearly 2 maps to 3 which includes 1.

1st day to 2nd day: Cross-Section for 8 hours of sleep. The intersection is empty.

FIGURE 2

It is immediately apparent that the tricyclicity of preferences imply that the dynamic choice function has a similar tricyclic character. For completeness, we note this fact in the following lemma.

Lemma. *Suppose that preferences are tricyclic on a compact, convex choice set $X(s)$ in the sense of Definition* 1. *Then the choice function $C(y;\alpha,s)$ is tricyclic in the same sense, that is, there exists a set $A\subset X(s)$ such that*

(a) $A\cap C(A;\alpha,s)=\varnothing$,
(b) $A\cup C(A;\alpha,s)\subset C^2(A;\alpha,s)$.

Proof. Let A be the set satisfying Definition 1 which exists by hypothesis. Then (a) follows from the facts that $A\cap X(s)=A$ and $\Phi(A;\alpha,s)\cap X(s)=C(A;\alpha,s)$. To prove (b) note that $\Phi(\Phi(A;\alpha,s)\cap X(s);\alpha,s)\cap X(s)=\Phi(C(A;\alpha,s);\alpha,s)\cap X(s)=C^2(A;\alpha,s)$. Also, $(A;\alpha,s)\cup A\cap X(s)=C(A;\alpha,s)\cup A$. Using (b) of Definition 1, the result follows. ||

The existence of erratic sequences of experience dependent choices now follows immediately from Diamond's theorem. We state our finding as:

Proposition 1. *Let preferences be partially tricyclic on a choice set $X(s)\subset R^n$ as in Definition* 1. *Then for the dynamic choice function* (15)

(a) *there exist periodic choice sequences of every period $k=1,2,3,\ldots$*; and
(b) *there exists an uncountable set $S\subset X(s)$ such that for every initial condition in S the choice sequence remains in S and is aperiodic, i.e. is erratic.*

The relationship between partially tricyclic choice functions as defined in the Lemma and the conditions for erratic behaviour originally stated by Li and Yorke (Theorem 1 in Section 2) can be seen in Figure 3. Here

$$x^3=C^3(x^0)<x^0<x^1=C(x^0)<x^2<C^2(x^0)$$

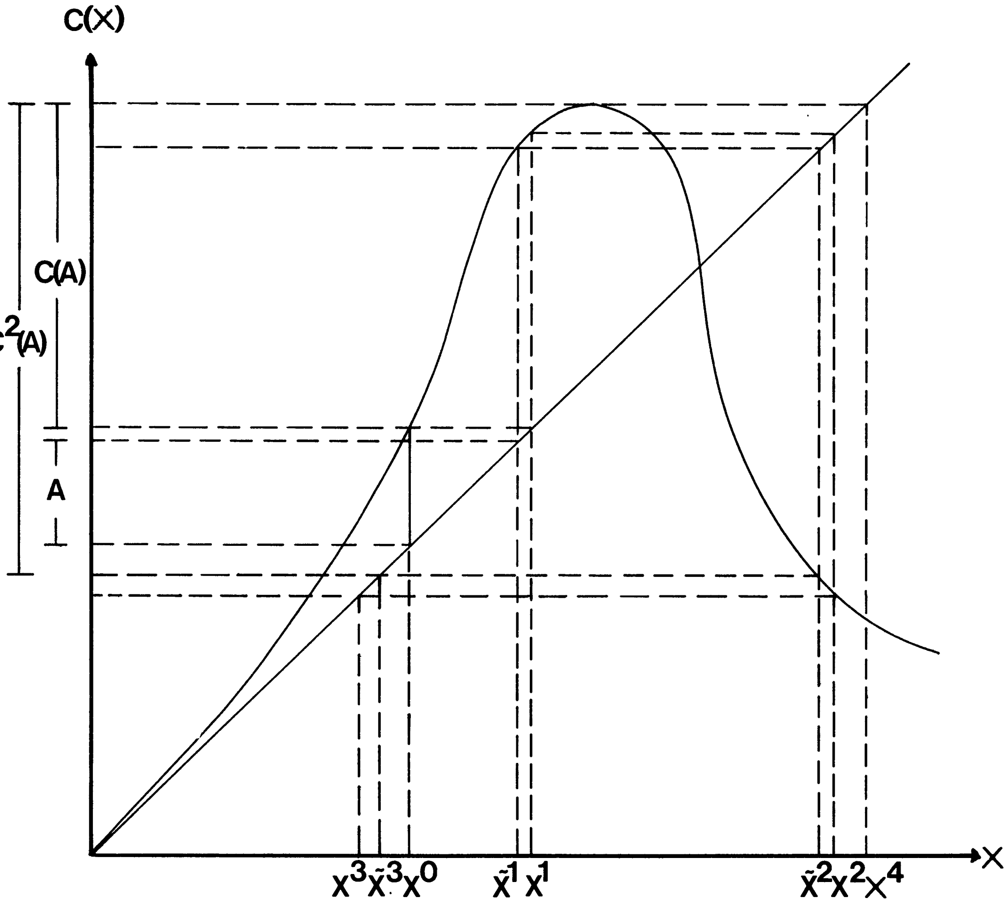


FIGURE 3

Relationship between Theorem 1 and Theorem 2 for the conditions of partial tricyclicity

are the points of the Proposition. We assume that $C(\cdot)$ is quasi-concave. The set $A=[x^0, \tilde{x}^1]$ is the set required in Theorem 2. Evidently, $[x^0, \tilde{x}^1]\cup[x^1, \tilde{x}^2]\subset C^2(A)$ where $\tilde{x}^3<x^0<\tilde{x}^1<x^4$ and $C^2(A)=[\tilde{x}^3, x^4]$.

Remark. An example, in the spirit of those in the previous section but in two dimensions, can easily be generated. Let utility be given by $C_1^{\alpha}C_2^{\beta}C_3^{\gamma}$ subject to the budget constraint $C_1+C_2+C_3=1$ where prices and income are taken as unity. First order conditions yield $x=C_2/C_1=\beta/\alpha$, $y=C_3/C_1=\gamma/\alpha$. Letting $\alpha_{t+1}=4x_t^2y_t^2(1-x_t)(1-y_t)$, $\beta_{t+1}=x_ty_t^2(1-y_t)$, $\gamma_{t+1}=x_t^2y_t(1-x_t)$ we obtain the two difference equations:

$$x_t=4x_{t-1}(1-x_{t-1}),$$

$$y_t=4x_{t-1}(1-y_{t-1}),$$

for $0\leqq x\leqq 1, 0\leqq y\leqq 1$. Conditions for Theorem 2 are satisfied by the set $S=\{(x, y):0{\cdot}18\leqq x\leqq 0{\cdot}58, 0\leqq y\leqq 1\}$. See P. Diamond (1976).

5. DYNAMIC SET-VALUED CHOICE FUNCTIONS

Rational choice does not determine behaviour when, for example, preferences are represented by a continuous utility function that is merely quasi-concave or that is locally satiable. In this case the set of optimizers is not (in general) unique so that some selection criterion other than rational choice must be invoked. Random selection for example could determine behaviour which would then have an erratic appearance. We can show that there is room for erratic behaviour even for completely non-stochastic, deterministic selection criteria such as that of "conservative selection" (Day and Kennedy (1971)) which minimizes the distance to the previous choice.

Under the more general conditions now assumed equation (16) must be rewritten as an inclusion,

$$x_{t+1} \in C(x_t; \alpha, s) = X(s) \cap \{x \mid \varphi(x; x_t, \alpha) \geqq \pi(x_t, \alpha, s)\}. \tag{20}$$

The dynamic choice mapping $C(x_t; \alpha, s)$ is now a correspondence from $X(s) \to \mathscr{P}[X(s)]$. Suppose $\varphi(x, y, \alpha)$ is quasi-concave for each $y \in X(s)$ and continuous with respect to both x and y. Suppose also that $X(s)$ is compact and convex as before. Then $C(x_t; \alpha, x)$ is upper semicontinuous with compact convex images.

Because the Definition and Lemma still apply, neither having exploited strict quasi-concavity of φ, we have:

Proposition 2. *Let preferences be represented by a utility function $\varphi(x, y; \alpha)$ continuous in x and y and quasi-concave in x for each y. Suppose further these preferences are tricyclic in the sense of Definition* 1. *Then for the dynamic choice correspondence* (20)

(a) *there exist periodic choice sequences for every period $k = 1, 2, 3, \ldots$;*
(b) *there exists an uncountable set $S \subset X(s)$ such that choice sequences with initial conditions in this set are erratic and remain in S.*

Proof. By the maximum theorem $C(\cdot)$ is upper semicontinuous and by the Lemma tricyclic. Hence by Theorem 3 the result holds. (For the proof of Theorem 3 see the Appendix). ||

6. REMARKS

1. Could an individual, by observing how his preferences have varied in the past, discover the deterministic mechanism generating his choice sequence? For choice sequences that have a very large period or that are totally aperiodic this seems very unlikely. The dynamics within the aperiodic sets may be indistinguishable from a stochastic process, sometimes even from a simple Bernoulli process (see Lorenz (1963), May and Oster (1976), and Guckenheimer, Oster and Ipaktchi (1977)). It seems quite likely then that the individual would accept the forces governing his choices over time as being subject to unpredictable shocks. He may on the other hand, attempt a statistical description of the dynamics by partitioning his choice set and observing the density of his choices to obtain information on the underlying mechanism. However, such procedures require a large number of observations, especially if the dimension of the system is large. If time is limited and experimentation is costly the individual is likely to resort to myopic choice which, if he discounts the future heavily enough, would appear to be inter-temporally optimal anyway.

2. For a discussion of the statistical characterization of the chaotic dynamics of some deterministic systems from experimental data see the article by Guckenheimer, Oster and Ipaktchi (1977) and its references.

3. Aperiodic behaviour seems to be a ubiquitous feature of non-linear difference equations. (See Lorentz (1964) and Li and Yorke (1975).) Such aperiodic behaviour is robust (structurally stable) to small perturbations in the functional form of the difference equations. (See Guckenheimer, Oster and Ipaktchi (1977), Kloeden (1976), Butler and

Pianigiani (1978).) Furthermore, such behaviour is not limited to difference equations. A number of studies have exhibited aperiodic behaviour in continuous systems of order 3 or more (see Ruelle and Takens (1971) and Lorentz (1963)).

4. The results for nondetermined choice presented in the last section can easily be extended to the case in which the choice set depends on experience also. This would yield the existence of erratic behaviour for a quite general class of recursive decision systems of the type investigated by Day and Kennedy (1971). One would therefore expect the possibility of erratic behaviour for a wide variety of dynamic economic models involving rational decision-making with feedback.

APPENDIX

Proof of Theorem 3 for set-valued dynamical systems

We follow Diamond in using two lemmas that derive from the finite intersection property of compact sets. The first follows Diamond (p. 954) with slight modification.

Lemma A1. *Let X_t, $t = 0, 1, 2, \ldots$ be a decreasing sequence of nonempty compact subsets of X in the sense that $X_{t+1} \subset F(X_t)$, all t. Then there exists a nonempty compact subset $Q \subset X_0$ such that for all $x \in Q$, there exists a trajectory $\{x_t\}_t \in \mathcal{P}(x)$ = subset of trajectories starting at x such that $x_t \in X_t$.*

Proof. Let $G_t := F|X_t$ be the restriction of F to X_t. Then the assumption that $X_{t+1} \subset F(X_t)$ implies that X_{t+1} is contained in the domain of G_t^{-1}. As X_t is compact G_t^{-1} is upper semi-continuous (because it has a closed graph). Consequently, $Q_t = G_0^{-1} \cdot G_1^{-1} \cdots G_t^{-1}(X_{t+1})$ is compact, nonempty, contained in X_0 and forms a decreasing sequence of sets. Hence $Q := \bigcap_{t \geqq 0} Q_t \subset X_0$. ||

Lemma A2. *Suppose $W \subset X$ is a compact, nonempty set such that $W \subset G(W)$ where $G: X \to \mathcal{P}(X)$ is an upper semi-continuous correspondence with compact images. Then there exists a compact nonempty subset $T \subset X$ such that $T = G(T)$.*

Proof. See Berge (1963, p. 113). ||

The importance of Lemma 1 lies in its implication that trajectories of a discrete dynamical system can be "filtered" to choose those that lie in a prescribed sequence of sets. That of Lemma 2 is used to establish periodic sets which are invariant with respect to the iterated mappings F^k.

We proceed now to the proof:

Proof of Theorem 3. The proof presented by Li and Yorke is based on the sequences $\{X_t\}$ constructed according to the rules, (i) X_t is either X or $F(X)$, and (ii) any X in the sequence is followed by two successive sets $F(X)$. Since $X \cup F(X) \subset F^2(X)$ such sequences satisfy the requirement that $X_{t+1} \subset F(X_t)$. In Part A we show that a countable number of sequences of periodic sets can be constructed in the present more general setting; in Part B we show that an uncountable number of aperiodic sets can be constructed that satisfy the rules.

Part A. By hypothesis there exists a set X such that $X \cap F(X) = \emptyset$ and $X \cup F(X) \subset F^2(X)$. Let k be a positive integer and define a sequence

$$
\begin{aligned}
&X_0 = F(X)\\
&X_1 = F(X) \subset F^2(X) = F(X_0)\\
&\vdots\\
&X_{k-2} = F(X) \subset F^2(X) = F(X_{k-1})\\
&X_{k-1} = X \subset F^2(X) = F(X_{k-2})\\
&X_k = F(X) \subset F(X_{k-1})\\
&\vdots\\
&X_{n+k} = X_n, \qquad n = 0, 1, 2, \ldots.
\end{aligned}
$$

Let $G_t := F|_{X_t}$ be the restriction of F to X_t as described in Lemma 1. By construction $G_0^{-1} \cdot G_1^{-1}, \ldots, G_{k-1}^{-1}(X_k)$ maps $F(X)$ into $F(X)$. Since $F(X)$ is compact the $G_0^{-1} \cdots G_{k-1}^{-1}$ is upper semi-continuous there exists (by Lemma A2) a set T_k in X such that $G_0^{-1} \cdots G_{k-1}^{-1}(T_k) = T_k$. But this is equivalent to $F^{-k}(T_k) = T_k$.

We must now guarantee that T_k is not invariant for some iterate less than k. Suppose the contrary. Then for some $l, F^l(T_k) = T_k \subset Q \subset X_0$. Let $l = k-1-m$ for $m = 0, \ldots, k-2$. Now $F^{k-1}(T_k) = F^{m+l}(T_k) = F^m F^l(T_k) = F^m(T_k) \subset F^m(Q) \subset X_m = F(X)$. Hence $F^{k-1}(T_k) \subset F(X)$ when $m < k-2$. But $F^{k-1}(T_k) \subset F^{k-1}(Q) \subset X_{k-1} = X$. But as $X \cap F(X) = \emptyset$, this is a contradiction. Hence T_k is k-periodic.

Part B. Let $r \in (0, 1)$ and consider sequences $\{X_t^r\}$ such that when n is large and there exist approximately nr elements $X_t = X$ before elements X_{n^2} in the sequence. Such a sequence cannot contain any k-period sets because in that case there would be at least n^2/k elements X in the sequence before X_{n^2} contradicting our assumptions. By Lemma A1 for each r there exists an x^r such that there exists a trajectory $\{x_t^r\}_{t \geq 0}$ satisfying $x_{t+1}^r \in F(x_t^r)$ and $x_t^r \in X_t$ for all t. Let $S : \{x^r\}_r$. This set is uncountable because the map $r \to \{X_t^r\}$ is one-one. Suppose $r \neq s \in (0, 1)$ and consider the distinct sequences $\{X_t^r\}, \{X_t^s\}$. There exist infinite subsequences $\{X_{t_p}^r\}, \{X_{t_p}^s\}$ such that $X_{t_p}^r \neq X_{t_p}^s$ (either $X_{t_p}^s = X$ while $X_{t_p}^s = F(X)$ or vice versa). Let $\delta_0 = \inf_{x \in X, y \in F(X)} d(x, y)$, $\delta_0 > 0$ because $X \cap F(X) = \emptyset$. Hence, there exist trajectories $\{x_t^r\}, \{y_t^s\}$ such that $d(x_t^r, y_t^s) > 0$ for an infinite subsequence. Thus $\lim_{t \to \infty} \sup d(x_t^r, y_t^s) > 0$. The remainder of the proof follows analogously. ||

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