Interactions between the Multiplier Analysis and the Principle of Acceleration
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INTERACTIONS BETWEEN THE MULTIPLIER ANALYSIS AND THE PRINCIPLE OF ACCELERATION

FEW economists would deny that the "multiplier" analysis of the effects of governmental deficit spending has thrown some light upon this important problem. Nevertheless, there would seem to be some ground for the fear that this extremely simplified mechanism is in danger of hardening into a dogma, hindering progress and obscuring important subsidiary relations and processes. It is highly desirable, therefore, that model sequences, which operate under more general assumptions, be investigated, possibly including the conventional analysis as a special case.[1]

In particular, the "multiplier," using this term in its usual sense, does *not* pretend to give the relation between total national income induced by governmental spending and the original amount of money spent. This is clearly seen by a simple example. In an economy (not necessarily our own) where any dollar of governmental deficit spending would result in a hundred dollars less of private investment than would otherwise have been undertaken, the ratio of total induced national income to the initial expenditure is overwhelmingly negative, yet the "multiplier" in the strict sense must be positive. The answer to the puzzle is simple. What the multiplier does give is the ratio of the total increase in the national income to the total amount of investment, governmental and private. In other words, it does *not* tell us how much is to be multiplied. The effects upon private investment are often regarded as tertiary influences and receive little systematic attention.

In order to remedy the situation in some measure, Professor Hansen has developed a new model sequence which ingeniously combines the multiplier analysis with that of the *acceleration* principle or *relation*. This is done by making additions to the national income consist of three components: (1) governmental deficit spending, (2) private consumption expenditure induced by previous public expenditure, and (3) induced private investment, assumed according to the familiar acceleration principle to be proportional to the time increase of consumption. The introduction of the last component accounts for the novelty of the conclusions reached and also the increased complexity of the analysis.

A numerical example may be cited to illuminate the assumptions made. We assume governmental deficit spending of one dollar per unit period, beginning at some initial time and continuing thereafter. The marginal propensity to consume, α, is taken to be one-half. This is taken to mean that the consumption of any period is equal to one-half the national income of the previous period. Our last assumption is that induced private investment is proportional to the increase in consumption between the previous and the current period. This factor of proportionality or *relation*, β, is provisionally taken to be equal to unity; i.e., a time increase in consumption of one dollar will result in one dollar's worth of induced private investment.

In the initial period when the government spends a dollar for the first time, there will be no consumption induced from previous periods, and hence the addition to the national income will equal the one dollar spent. This will yield fifty cents of consumption expenditure in the second period, an increase of fifty cents over the consumption of the first period, and so according to the *relation* we will have fifty cents worth of induced private investment. Finally, we must add the new dollar of expenditure by the government. The national income of the second period must therefore total two dollars. Similarly, in the third period the national income would be the sum of one dollar of consumption, fifty cents induced private investment, and one dollar current governmental expenditure. It is clear that given the values of the marginal propensity to consume, α, and the *relation*, β, all succeeding national income levels can be easily computed in succession. This is done in detail in Table 1 and illustrated in Chart 1. It will be noted that the introduction of the acceleration principle causes our series to reach a peak at the 3rd year, a trough at the 7th, a peak at the 11th, etc. Such oscil-

[1] The writer, who has made this study in connection with his research as a member of the Society of Fellows at Harvard University, wishes to express his indebtedness to Professor Alvin H. Hansen of Harvard University at whose suggestion the investigation was undertaken.

TABLE 1.—THE DEVELOPMENT OF NATIONAL INCOME AS A RESULT OF A CONTINUOUS LEVEL OF GOVERNMENTAL EXPENDITURE WHEN THE MARGINAL PROPENSITY TO CONSUME EQUALS ONE-HALF AND THE RELATION EQUALS UNITY

(*Unit: one dollar*)

Period	Current governmental expenditure	Current consumption induced by previous expenditure	Current private investment proportional to time increase in consumption	Total national income
1	1.00	0.00	0.00	1.00
2	1.00	0.50	0.50	2.00
3	1.00	1.00	0.50	2.50
4	1.00	1.25	0.25	2.50
5	1.00	1.25	0.00	2.25
6	1.00	1.125	−0.125 *	2.00
7	1.00	1.00	−0.125	1.875
8	1.00	0.9375	−0.0625	1.875
9	1.00	0.9375	0.00	1.9375
10	1.00	0.96875	0.03125	2.00
11	1.00	1.00	0.03125	2.03125
12	1.00	1.015625	0.015625	2.03125
13	1.00	1.015625	0.00	2.015625
14	1.00	1.0078125	−0.0078125	2.00
....	...	....	....	....

* Negative induced private investment is interpreted to mean that for the system as a whole there is *less* investment in this period than there otherwise would have been. Since this is a marginal analysis, superimposed implicitly upon a going state of affairs, this concept causes no difficulty.

latory behavior could not occur in the conventional model sequences, as will soon become evident.

For other chosen values of α and β similar model sequences can be developed. In Table 2 national income totals are given for various selected values of these coefficients. In the first column, for example, the marginal propensity to consume is assumed to be one-half, and the *relation* to be equal to zero. This is of special interest because it shows the conventional multiplier sequences to be special cases of the more general Hansen analysis. For this case no oscillations are possible. In the second column the oscillations in the national income are undamped and regular. In column three things are still worse; the oscillations are explosive, becoming larger and larger but always fluctuating around an "average value." In the fourth column the behavior is no longer oscillatory but is explosive upward approaching a compound interest rate of growth.

By this time the investigator is inclined to feel somewhat disorganized. A variety of qualitatively different results emerge in a seemingly capricious manner from minor changes in hypotheses. Worse than this, how can we be sure that for still different selected values of our coefficients new and stronger types of behavior will not emerge? Is it not even possible that if Table 2 were extended to cover more periods, new types of behavior might result for these selected coefficients?

Fortunately, these questions can be given a definite negative answer. Arithmetical methods cannot do so since we cannot try all possible values of the coefficients nor compute the endless terms of each sequence. Nevertheless, comparatively simple algebraic analysis can be applied which will yield all possible qualitative types of behavior and enable us to unify our results.

The national income at time t, Y_t, can be written as the sum of three components: (1) governmental expenditure, g_t, (2) consumption expenditure, C_t, and (3) induced private investment, I_t.

$$Y_t = g_t + C_t + I_t.$$

But according to the Hansen assumptions

$$C_t = \alpha Y_{t-1}$$

$$I_t = \beta[C_t - C_{t-1}] = \alpha\beta Y_{t-1} - \alpha\beta Y_{t-2}$$

and

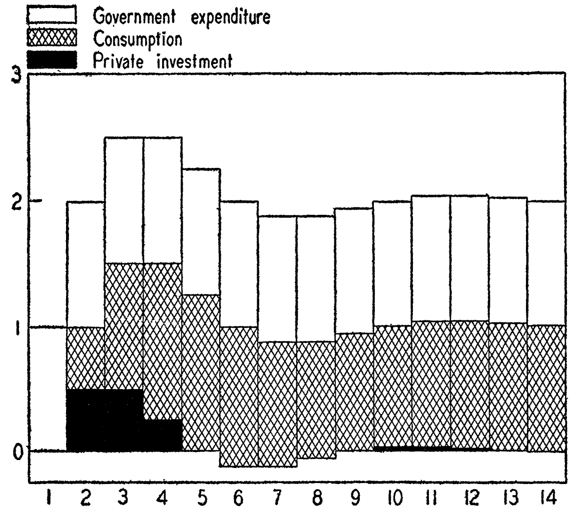
$$g_t = 1.$$

Therefore, our national income can be rewritten

$$Y_t = 1 + \alpha[1+\beta]Y_{t-1} - \alpha\beta Y_{t-2}.$$

CHART 1.—GRAPHIC REPRESENTATION OF DATA IN TABLE 1

(*Unit: one dollar*)



In words, if we know the national income for two periods, the national income for the following period can be simply derived by taking a weighted sum. The weights depend, of course, upon the values chosen for the marginal propensity to consume and for the *relation.*

This is one of the simplest types of difference equations, having constant coefficients and being of the second order. The mathematical details of its solution need not be entered upon here. Suffice it to say that its solution depends upon the roots — which in turn depend upon the coefficients α and β — of a certain equation.[1]

TABLE 2.—MODEL SEQUENCES OF NATIONAL INCOME FOR SELECTED VALUES OF MARGINAL PROPENSITY TO CONSUME AND RELATION

(*Unit: one dollar*)

Period	$\alpha=.5$ $\beta=0$	$\alpha=.5$ $\beta=2$	$\alpha=.6$ $\beta=2$	$\alpha=.8$ $\beta=4$
1	1.00	1.00	1.00	1.00
2	1.50	2.50	2.80	5.00
3	1.75	3.75	4.84	17.80
4	1.875	4.125	6.352	56.20
5	1.9375	3.4375	6.6256	169.84
6	1.9688 *	2.0313	5.3037	500.52
7	1.9844	.9141	2.5959	1,459.592
8	1.9922	− .1172	− .6918	4,227.704
9	1.9961	.2148	−3.3603	12,241.1216
.. ..	.....	....	....	....

* Table is correct to four decimal places.

It can be easily shown that the whole field of possible values of α and β can be divided into four regions, each of which gives qualitatively different types of behavior. In Chart 2 these regions are plotted. Each point in this diagram represents a selection of values for the marginal propensity to consume and the *relation.* Corresponding to each point there will be a model sequence of national income through time. The qualitative properties of this sequence depend upon whether the point is in Region A, B, C, or D.[2] The properties of each region can be briefly summarized.

Region A (relatively small values of the *relation*)

If there is a constant level of governmental expenditure through time, the national income will approach asymptotically a value $\frac{1}{1-\alpha}$ times the constant level of governmental expenditure. A single impulse of expenditure, or any amount of expenditure followed by a complete cessation, will result in a gradual approach to the original zero level of national income. (It will be noted that the asymptote approached is identically that given by the Keynes-Kahn-Clark formula. Their analysis applies to points along the α axis and is subsumed under the more general Hansen analysis.) Perfectly periodic net governmental expenditure will result eventually in perfectly periodic fluctuations in national income.

Region B

A constant continuing level of governmental expenditure will result in damped oscillatory movements of national income, gradually approaching the asymptote $\frac{1}{1-\alpha}$ times the constant level of government expenditure. (Cf. Table 1.) Governmental expenditure in a single or finite number of periods will result eventually in damped oscillations around the level of income zero. Perfectly regular periodic fluctuations in government expenditure will result eventually in fluctuations of income of the same period.

Region C

A constant level of governmental expenditure will result in *explosive,* ever increasing oscillations around an asymptote computed as above. (Cf. column 3 of Table 2.) A single impulse of expenditure or a finite number of expenditure impulses will result eventually in explosive oscillations around the level zero.

Region D (large values of the marginal propensity to consume and the *relation*)

A constant level of governmental expenditure will result in an ever increasing national income, eventually approaching a compound interest rate of growth. (Cf. column 4 of Table 2.) A

[1] Actually, the solution can be written in the form

$$Y_t=\frac{1}{1-\alpha}+a_1[x_1]^t+a_2[x_2]^t$$

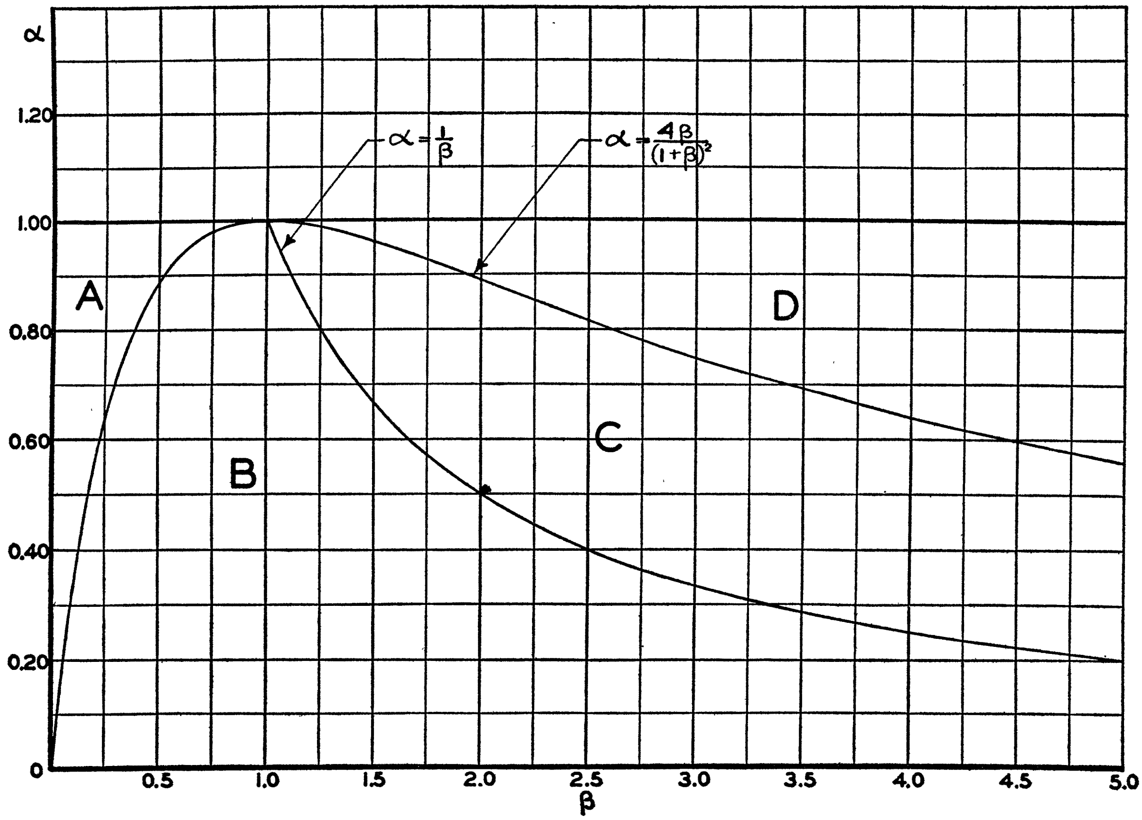
where x_1 and x_2 are roots of the quadratic equation

$$x^2-\alpha[1+\beta]x+\alpha\beta=0,$$

and a_1 and a_2 are constants dependent upon the α's and β's chosen.

[2] Mathematically, the regions are demarcated by the conditions that the roots of the equation referred to in the previous footnote be real or complex, greater or less than unity in absolute value.

CHART 2.—DIAGRAM SHOWING BOUNDARIES OF REGIONS YIELDING DIFFERENT QUALITATIVE BEHAVIOR OF NATIONAL INCOME



single impulse of net investment will likewise send the system up to infinity at a compound interest rate of growth. On the other hand, a single infinitesimal unit of disinvestment will send the system ever downward at an increasing rate. This is a highly unstable situation, but corresponds most closely to the pure case of pump-priming, where the total increase in national income bears no finite ratio to the original stimulus.

The limitations inherent in so simplified a picture as that presented here should not be overlooked.[1] In particular, it assumes that the marginal propensity to consume and the *relation* are constants; actually these will change with the level of income, so that this representation is strictly a *marginal* analysis to be applied to the study of small oscillations. Nevertheless, it is more general than the usual analysis. Contrary to the impression commonly held, mathematical methods properly employed, far from making economic theory more abstract, actually serve as a powerful liberating device enabling the entertainment and analysis of ever more realistic and complicated hypotheses.

[1] It may be mentioned in passing that the formal structure of our problem is identical with the model sequences of Lundberg, and the dynamic theories of Tinbergen. The present problem is so simple that it provides a useful introduction to the mathematical theory of the latter's work.

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