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M9302 Mathematical Models in Economics

2.5.Repeated Games

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Repeated Games

- The aim of the forth lecture is to describe a special subclass of dynamic games of complete and perfect information called repeated games
- Key question: Can threats and promises about future behavior influence current behavior in repeated relationships?

Repeated Games

- Let G = {$A_1,\ldots,A_n$; $u_1,\ldots,u_n$} denote a static game of complete information in which player 1 through player n simultaneously choose actions a_1 through a_n from the action spaces A_1 through A_n.

Respectively, the payoffs are u(a,…,a) through u(a,…,a)

Allow for any finite number of repetitions.

- Then, G is called the stage game of the repeated game

Finitely Repeated Game

- Finitely repeated game: Given a stage game G, let G(T) denote the finitely repeated game in which G is played T times, with the outcomes of all preceding plays observed before the next play begins.
- The payoffs for G(T) are simply the sum of the payoffs from the T stage games.

Finitely Repeated Game

- In the finitely repeated game G(T), a subgame beginning at stage t+1 is the repeated game in which G is played T-t times, denoted G(T-t).
- There are many subgames that begin in stage $t+1$, one for each of the possible histories of play through stage t.
- The t^{th} stage of a repeated game ($t<T$) is not a subgame of the repeated game.

Example: 2-stage Students' Dilemma

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a WEAK PASS (Grade C, or 3) and

who is under the average

FAIL (Grade F, or 5).

Example: 2-stage Students' Dilemma

Leisure Rule: HARD study schedule devotes twice more time (leisure = 1) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	3	**-1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	1	1	**0**
	All Hard	1	3	**-2**

Example: 2-stage Students' Dilemma

Bi-matrix of payoffs:

		OTHERS	
		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	-2,-2

Repeat the stage game twice!

Example: 2-stage Students’ Dilemma

Stage 2:

OTHERS

		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	-2,-2

Stage 1:

OTHERS

		Easy	Hard
YOU	Easy	-3,-3	-5,-2
	Hard	-2,-5	-4,-4

Finitely Repeated Game

- Proposition: If the stage game G has a unique Nash equilibrium then, for any finite T, the repeated game G(T) has a unique subgame-perfect outcome:
  - The Nash equilibrium of G is played in every stage.

Finitely Repeated Game

- What if the stage game has no unique solution?
  - If $G = \{A_1,\ldots,A_n; u_1,\ldots,u_n\}$ is a static game of complete information with multiple Nash equilibria, there may be subgame-perfect outcomes of the repeated game $G(T)$ in which the outcome in stage $t<T$ is not a Nash equilibrium in G.

Example: 2-stage Students' Dilemma - 2

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a **PASS (Grade B, or 2)** and

who is under the average

FAIL (Grade F, or 5).

Example: 2-stage Students' Dilemma - 2

- ☐ Leisure Rule: HARD study schedule devotes all their time (leisure = 0) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	2	**0**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	0	1	**-1**
	All Hard	0	2	**-2**

Example: 2-stage Students' Dilemma - 2

		OTHERS	
		Easy	Hard
YOU	Easy	$\underline{0},\underline{0}$	-3,-1
	Hard	-1,-3	$\underline{-2},\underline{-2}$

Suppose each player's strategy is:

- Play Easy in the 2nd stage if the 1st stage outcome is (Easy, Easy)
- Play Hard in the 2nd stage for any other 1st stage outcome

Example: 2-stage Students' Dilemma - 2

Stage 2:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

Stage 1:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-5,-3
	Hard	-3,-5	-4,-4

Example: 2-stage Students' Dilemma - 2

Stage 1:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-5,-3
	Hard	-3,-5	-4,-4

The threat of player *i* to punish in the 2nd stage player j's cheating in the 1st stage is not credible.

Example: 2-stage Students' Dilemma - 2

Stage 1:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

{Easy, Easy} Pareto-dominates {Hard, Hard} in the second stage. There is space for re-negotiation because punishment hurts punisher as well.

Example: 2-stage Students' Dilemma - 2

		OTHERS			
		Easy	Hard	C	O
YOU	Easy	0,0	-3,-1	-3,-3	-3,-3
	Hard	-1,-3	-2,-2	-3,-3	-3,3
	C	-3,-3	-3,-3	0,-2.5	-3,-3
	O	-3,-3	-3,-3	-3,-3	-2.5,0

Add 2 more actions and suppose each player's strategy is:

- Play Easy in the 2nd stage if the 1st stage outcome is (E, E)
- Play C in the 2nd stage if the 1st stage outcome is (E, w≠E)
- Play O in the 2nd stage if the 1st stage is (y≠E,z=E/H/C/O)
- Outcomes{C,C} and {O,O} are on the **Pareto frontier**.

Finitely Repeated Game

- Conclusion: Credible threats or promises about future behavior which leave no space for negotiation (Pareto improvement) in the final stage can influence current behavior in a finite repeated game.

Infinitely Repeated Game

- Given a stage-game G, let G(∞,δ) denote the infinitely repeated game in which G is repeated forever and the players share the discount factor δ.
- For each *t*, the outcomes of the t-1 preceding plays of G are observed.
- Each player's payoff in G(∞,δ) is the present value of the player's payoffs from the infinite sequence of stage games

Infinitely Repeated Game

- The history of play through stage *t* – in the finitely repeated game G(T) or the infinitely repeated game G(∞,δ) – is the record of the player’s choices in stages 1 through *t*.

Infinitely Repeated Game

- Strategy /in a repeated game/ - the sequence of actions the player will take in each stage, for each possible history of play through the previous stage.
- Subgame /in a repeated game/ - the piece of the game that remains to be played beginning at any point at which the complete history of the game thus far is common knowledge among the players.

Infinitely Repeated Game

- As in the finite-horizon case, there are as many subgames beginning at stage $t+1$ of G(∞,δ) as there are possible histories through stage t.
- In the infinitely repeated game G(∞,δ), each subgame beginning at stage t+1 is identical to the original game.

Infinitely Repeated Game

- How to compute the player's payoff of an infinitely repeated game?
  - Simply summing the payoffs of all stage-games does not provide a useful measure
  - Present value of the infinite sequence of payoffs:

$$\Pi_1 + \delta\Pi_2 + \delta^2\Pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1}\Pi_t$$

Infinitely Repeated Game

- Key result: Even when the stage game has a unique Nash equilibrium it does not need to be present in every stage of a SGP outcome of the infinitely repeated game.
- The result follows the argument of the analysis of the 2-stage repeated game with credible punishment.

Infinitely Repeated Game

		OTHERS			
		Easy	Hard	C	O
YOU	Easy	0,0	-3,-1	-3,-3	-3,-3
	Hard	-1,-3	-2,-2	-3,-3	-3,3
	C	-3,-3	-3,-3	0,-2.5	-3,-3
	O	-3,-3	-3,-3	-3,-3	-2.5,0

Instead of adding artificial equilibria that brings higher payoff tomorrow, the Pareto dominant action is played.

Example: Infinite Students' Dilemma

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

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Example: Infinite Students' Dilemma

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Example: Infinite Students' Dilemma

Bi-matrix of payoffs:

OTHERS

		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	-2,-2

Repeat the stage game infinitely!

Example: Infinite Students' Dilemma

- Consider the following trigger strategy:
  - Play Easy in the 1st stage.
  - In the tth stage if the outcome of all t-1 preceding stages has been (E, E) then play Easy,
  - Otherwise, play Hard in the tth stage.
- Need to define δ for which the trigger strategy is SGPNE.

Example: Infinite Students' Dilemma

- Subgames could be grouped into 2 classes:
  - Subgames in which the outcome of at least one earlier stage differs from (E,E) – trigger strategy fails to induce cooperation
  - Subgames in which all the outcomes of the earlier stages have been (E,E) – trigger strategy induces cooperation

Example: Infinite Students' Dilemma

- If HARD is played in the 1st stage total payoff is:

$$0+(-2\delta)+(-2\delta^{2})+\ldots=0-\frac{2\delta}{1-\delta}$$

- If EASY is played in the 1st stage, let the present discounted value be V:

$$V=-1+\delta V \Rightarrow V=-\frac{1}{1-\delta}$$

Example: Infinite Students’ Dilemma

- In order to have a SGPE where (E, E) is played in all the stages till infinity the following inequality must hold:

$$-\frac{2\delta}{1-\delta} \leq V$$

- After substituting for V we get the following condition on δ:

$$\delta \geq \frac{1}{2}$$

Folk’s Theorem

- In order to generalize the result of the SD to hold for all infinitely repeated games, several key terms need to be introduced:
- The payoffs $(x_1,\ldots,x_n)$ are called ***feasible*** in the stage game G if they are a convex (i.e. weighted average, with weights from 0 to1) combination of the pure-strategy payoffs of G.

Folk’s Theorem

- The average payoff from an infinite sequence of stage-game payoffs is the payoff that would have to be received in every stage so as to yield the same present value as the player’s infinite sequence of stage-game payoffs.
- Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\Pi_1, \Pi_2, \Pi_3 \ldots$ is: $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pi_t$

Folk's Theorem

- Folk's Theorem (Friedman 1971): Let G be a finite, static game of complete information. Let $(e_1,\dots,e_n)$ denote the payoffs from a Nash Equilibrium of G, and let $(x_1,\dots,x_n)$ denote any other feasible payoffs from G.
- If $x_i > e_i$ for every player *i* and if δ is sufficiently close to 1, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty,\delta)$ that achieves $(x_1,\dots,x_n)$ as the average payoff.

Folk's Theorem

- Reservation payoff r_i – the largest payoff player *i* can guarantee receiving, no matter what the other players do.
- It must be that $r_i \leq e_i$, since if r_i were greater than e_i, it would not be a best response for player *i* to play her Nash equilibrium strategy.
- In SD, $r_i = e_i$ but in the Cournot Duopoly Game (and typically) $r_i < e_i$

Folk’s Theorem

- Folk’s Theorem (Fudenber & Maskin 1986): If (x_1, x_2) is a feasible payoff from G, with $x_i > r_i$ for each *i*, then for δ sufficiently close to 1, there exists a SGPNE of $G(\infty,\delta)$ that achieves (x_1, x_2) as the average payoff even if $x_i < e_i$ for one or both of the players.

Folk’s Theorem

- What if δ is close to 0?
  - 1st Approach: After deviation follow the trigger strategy and play the stage-game equilibrium.
  - 2nd Approach (Abreu 1988): After deviation play the N.E. that yields the lowest payoff of all N.E. Average strategy can be lower than the one of the 1st approach if switching to stage game is not the strongest credible punishment.

Summary

- Key question that stays behind repeated games is whether threats or promises about future behavior can affect current behavior in repeated relationships.
- In finite games, if the stage game has a unique Nash Equilibrium, repetition makes the threat of deviation credible.
- If stage game has multiple equilibria however there could be a space for negotiating the punishment in the next stage after deviation.

Summary

- In infinitely repeated games, even when the stage game has a unique Nash equilibrium it does not need to be present in every stage of a SGP outcome of the infinitely repeated game.
- Folk's theorem implies that if there is a set of feasible payoffs that are larger than the payoffs from the stage game Nash equilibrium, and the discount factor is close to one, there is a SGPNE at which the set of higher feasible payoffs is achieved as an average payoff.

Summary

- Extension of the Folk's theorem implies that for 2-player infinitely repeated game if there is a set of feasible payoffs that exceed the reservation ones, the outcome that yields these feasible payoffs as an average payoff could constitute a SGPNE even if they are smaller than the payoffs from the stage game N.E., provided that the discount factor is close to 1.
- If the discount factor is close to 0, an alternative strategy to the trigger one (where the stage game equilibrium is played after deviation) is to play instead the N.E. that yields the lowest payoff of all N.E. This might be stronger credible punishment.