Lecture 5 21.04.2011

# M9302 Mathematical Models in Economics

## 5.1.Static Games of Incomplete Information

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# Revision

- When a combination of strategies $\left(s_1^*, \ldots, s_n^*\right)$ is a Nash equilibrium?
  - If for any player *i*, is player *i*'s best response to the strategies of the n-1 other players
- Following this definition we could easily find game that have no Nash equilibrium:
  - Example: Penny Game

# Example: Penny Game

| | | **P2** | |
|---|---|---|---|
| | | Heads | Tails |
| **P1** | Heads | -1,$\underline{1}$ | $\underline{1}$,-1 |
| | Tails | $\underline{1}$,-1 | -1,$\underline{1}$ |

No pair of strategies can satisfy N.E.:

If match (H,H), (T,T) – P1 prefers to switch

If no match (H,T), (T,H) – P2 prefers to switch

# Extended definition of Nash Equilibrium

- In the 2-player normal-form game $G=\{S_1,S_2;u_1,u_2\}$, the **MIXED** strategies $\left(p_1^*, p_2^*\right)$ are a Nash equilibrium if each player's mixed strategy is a best response to the other player's **MIXED** strategy
- Hereafter, let's refer to the strategies in $S_i$ as player *i*'s **pure strategies**
- Then, a **mixed strategy** for player *i* is a probability distribution over the strategies in $S_i$

# Example: Penny Game

- In Penny Game, $S_i$ consists of the two **pure strategies** H and T
- A **mixed strategy** for player *i* is the probability distribution ($q,1-q$), where q is the probability of playing H, and $1-q$ is the probability of playing T, $0 \leq q \leq 1$
- Note that the mixed strategy (0,1) is simply the pure strategy T, likewise, the mixed strategy (1,0) is the pure strategy H

# Example: Penny Game

- Computing P1’s best response to a mixed strategy by P2 represents P1’s uncertainty about what P2 will do.
- Let (q,1-q) denote the mixed strategy in which P2 plays H with probability q.
- Let (r, 1-r) denote the mixed strategy in which P1 plays H with probability r.

# Example: Penny Game

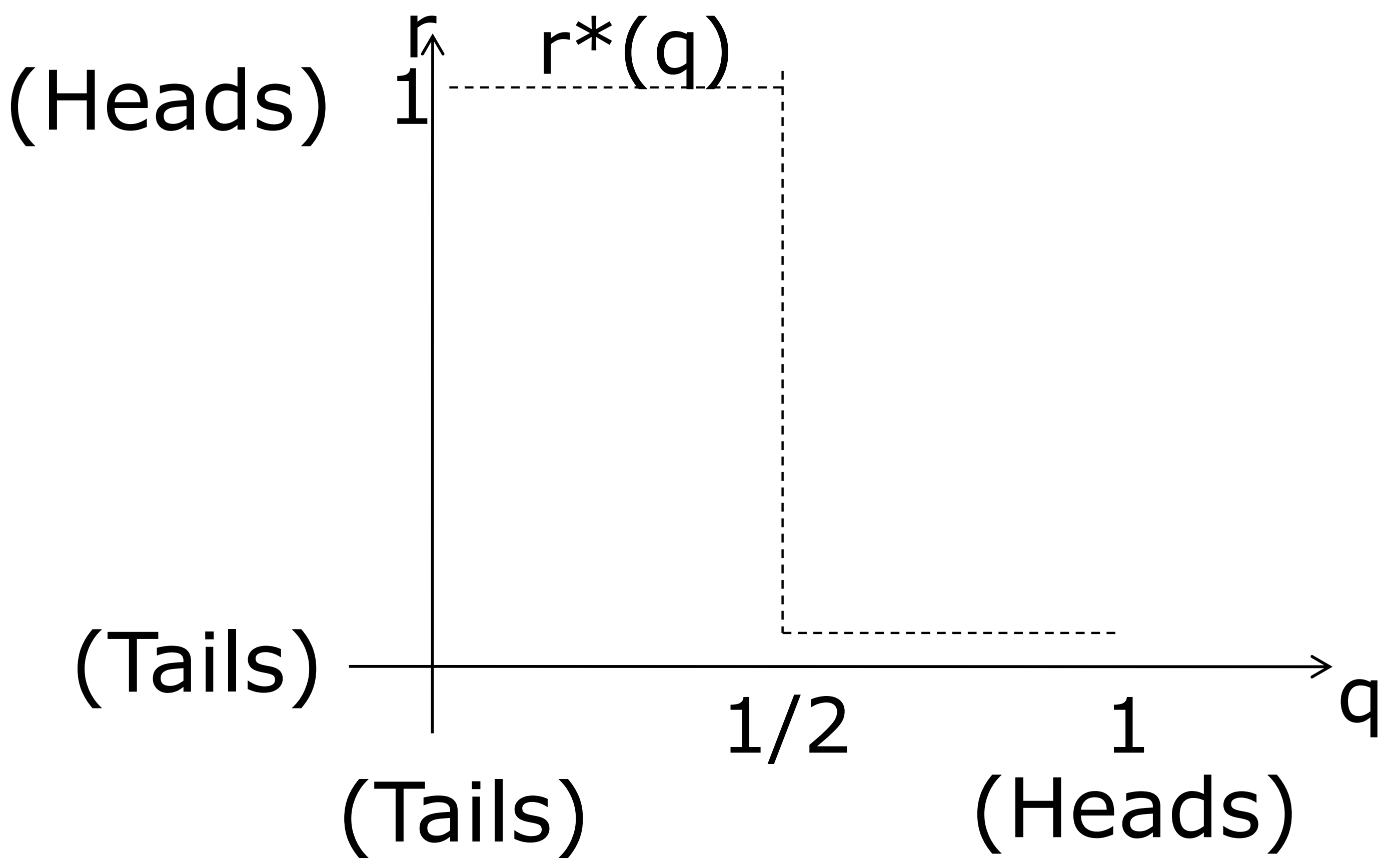
- P1's expected payoff from playing (r,1-r) when P2 plays (q,1-q) is:

$$rq\cdot(-1)+r(1-q)\cdot 1+(1-r)(1-q)\cdot(-1)+(1-r)\cdot q=$$

$$=(2q-1)+r(2-4q)$$

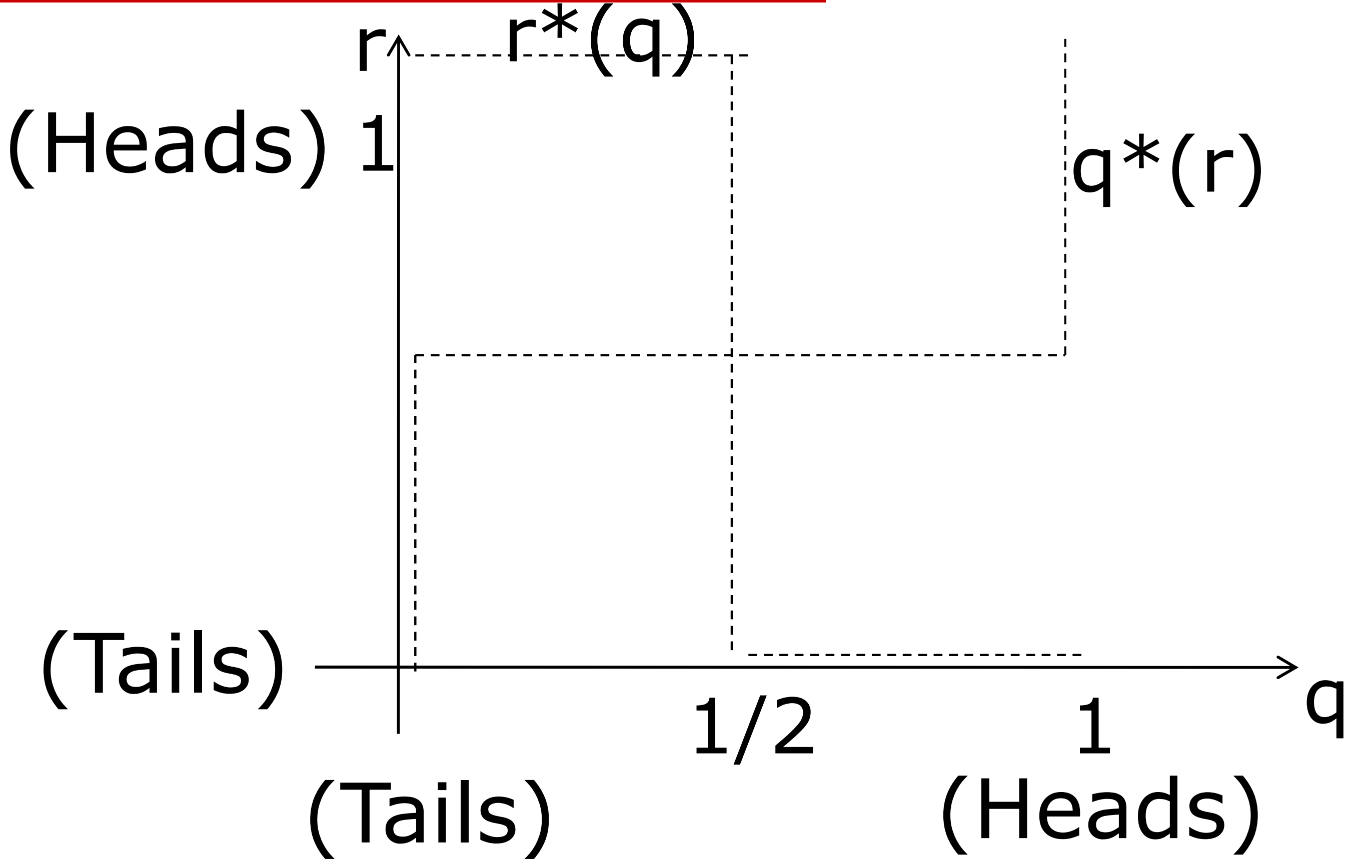
- which is increasing in r for q<1/2 (i.e. P1's best response is r=1) and decreasing in r for q>1/2 (i.e. P1's best response is r=0).
- P1 is indifferent among all mixed strategies (r,1-r) when q=1/2.

# Example: Penny Game




Because there is a value of q such that r*(q) has more than one value, r*(q) is called P1's **best-response correspondence**.

# Example: Penny Game




The intersection of the best-response correspondences r*(q) and q*(r)yields the mixed-strategy N.E. in Penny Game.

# General Definition of Mixed Strategy

- Suppose that player i has K pure strategies, $S_i=\{s_{i1},\ldots, s_{iK}\}$
- Then, a **mixed strategy** for player *i* is a probability distribution $(p_{i1},\ldots, p_{iK})$, where $p_{ik}$ is the probability that player *i* will play strategy $s_{ik}$, k=1,…,K
- Respectively, $0 \leq p_{ik} \leq 1$ for k=1,…,K and $p_{i1} + \ldots + p_{iK} = 1$
- Denote an arbitrary mixed strategy by $p_i$

# General Definition of Nash Equilibrium

- Consider 2-player case where strategy sets of the two players are $S_1=\{s_{11},\ldots, s_{1J}\}$ and $S_1=\{s_{11},\ldots, s_{1K}\}$, respectively
- P1's expected payoff from playing the mixed strategies $p_1 = (p_{11},\ldots,p_{1J})$ is:

$$v_1\left(p_1^*, p_2^*\right) = \sum_{j=1}^{J}\sum_{k=1}^{K} p_{1j}\cdot p_{2k} u_1\left(s_{1j}, s_{2k}\right)$$

- P2's expected payoff from playing the mixed strategies $p_2 = (p_{21},\ldots,p_{2K})$ is:

$$v_2\left(p_1^*, p_2^*\right) = \sum_{j=1}^{J}\sum_{k=1}^{K} p_{1j}\cdot p_{2k} u_2\left(s_{1j}, s_{2k}\right)$$

# General Definition of Nash Equilibrium

- For the pair of mixed strategies $\left(p_1^*, p_2^*\right)$ to be a Nash equilibrium, $p_1^*$ must satisfy:

$$v_1\left(p_1^*, p_2^*\right) \geq v_1\left(p_1, p_2^*\right)$$

- for every probability distribution $p_1$ over $S_1$, and $p_2^*$ must satisfy:

$$v_2\left(p_1^*, p_2^*\right) \geq v_2\left(p_1^*, p_2\right)$$

- for every probability distribution $p_2$ over $S_2$.

# Existence of Nash Equilibrium

- Theorem (Nash 1950): In the n-player normal-form game $G=\{S_1,\ldots,S_n;u_1,\ldots,u_n)$, if $n$ is finite and $S_i$ is finite for every $i$ then there exists at least one Nash equilibrium, possibly involving mixed strategies.
- Proof consists of 2 steps:
  - Step1: Show that any fixed point of a certain correspondence is a N.E.
  - Step 2: Use an appropriate fixed-point theorem to show that the correspondence must have a fixed point.

# Revision

- What is a strictly dominated strategy?
  - If a strategy $s_i$ is strictly dominated then there is no belief that player *i* could hold such that it would be optimal to play $s_i$.
- The converse is also true when mixed strategies are introduced
  - If there is no belief that player i could hold such that it would be optimal to play $s_i$, then there exists another strategy that strictly dominates $s_i$.

# Example /mixed strategy dominance/:

**P2**

| | | B1 | B2 |
|---|---|---|---|
| **P1** | A1 | 3,— | 0,— |
| | A2 | 0,— | 3,— |
| | A3 | 1,— | 1,— |

For any belief of P1, A3 is not a best response even though it is not strictly dominated by any pure strategy. A3 is strictly dominated by a mixed strategy (½ , ½, 0)

# Example /mixed strategy best response/:

| | | P2 | |
|---|---|---|---|
| | | B1 | B2 |
| P1 | A1 | 3,— | 0,— |
| | A2 | 0,— | 3,— |
| | A3 | 2,— | 2,— |

For any belief of P1, A3 is not a best response to any pure strategy but it is the best response to mixed strategy (q,1-q) for $1/3<q<2/3$.

# Introduction to Incomplete Information

- What is complete information?
- What must be incomplete information then?

# Introduction to Incomplete Information

A game in which one of the players does not know for sure the payoff function of the other player is a game of INCOMPLETE INFORMATION

<u>Example:</u>

Cournot Duopoly with Asymmetric Information about Production

# Static Games of Incomplete Information

- The aim of this lecture is to show:
- How to represent a static game of incomplete information in normal form?
- What solution concept is used to solve a static game of incomplete information?

# Normal-form Representation

- ADD a TYPE parameter $t_i$ to the payoff function -> $u_i(a_1,\ldots,a_n;\ t_i)$

A player is uncertain about

**{other player's payoff function} = {other player's type $t_{-i}$}**

where $t_{-i} = (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$

# Normal-form Representation

- ADD probability measure of types to account for uncertainty:
- $p_i(t_{-i}|t_i)$ - player *i*'s belief about the other players' types ($t_{-i}$) given player *i*'s knowledge of her own type, $t_i$.
- Bayesian Theorem

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)}$$

# Normal-form Representation

- PLAYERS
- ACTIONS – $A_1, \ldots, A_n; A_i = \{a_{i1}, \ldots, a_{in}\}$
- TYPES – $T_i = \{t_{i1}, \ldots, t_{in}\}$
- System of BELIEFS - $p_i(t_{-i} / t_i)$
- PAYOFFS - $u_i(a_1, \ldots, a_n; t_i)$

which is briefly denoted as

$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$$

# Timing of the Bayesian Games (Harsanyi, 1967)

- Stage 1: Nature draws a type vector

$t = (t_1,\ldots,t_n)$, where $t_i$ is drawn from the set of possible types $T_i$.

- Stage 2: Nature reveals $t_i$ to player $i$ but not necessarily to the other players.
- Stage 3: Players simultaneously choose actions i.e. player $i$ chooses $a_i$ from the feasible set $A_i$.
- Stage 4: Payoffs $u_i(a_1,\ldots,a_n;\ t_i)$ are received.

# Strategy in a Bayesian Game

- In a static Bayesian game, a strategy for player *i* is a function , where for each type $t_i$ in $Ti$, $s_i(t_i)$ specifies the action from the feasible set $A_i$ that type $t_i$ would choose if drawn by nature.
- In a ***separating strategy***, each type $t_i$ in $T_i$ chooses a different action $a_i$ from $A_i$.
- In a **pooling strategy**, in contrast, all types choose the same action.

# How to solve a Bayesian game?

- Bayesian Nash Equilibrium:

*In the static Bayesian game*

$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$$

*the strategies* $s^* = (s_1^*, \ldots, s_n^*)$ *are a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each of i's types ti in Ti,* $s_i^*(t_i)$ *solves:*

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n); t\right) p_i\left(t_{-i} \middle| t_i\right)$$

*That is, no player wants to change his or her strategy, even if the change involves only one action by one type.*

# Existence of a Bayesian Nash Equilibrium

- In a finite static Bayesian game (i.e., where n is finite and $(A_1,\ldots,A_n)$ and $(T_1,\ldots,T_n)$ are all finite sets), there exists a Bayesian Nash equilibrium, perhaps in mixed strategies.

Mixed-strategy in a Bayesian game:

Player *i* is uncertain about player *j's* choice not because it is random but rather because of **incomplete information about j's payoffs**.

Examples: Battle of Sexes; Cournot Competition with Asymmetric Information

# Summary

- Game Theory distinguishes between pure and mixed strategy
- Mixed strategy is a probability distribution over the strategy set
- To be efficient in solving games including uncertainty, N.E. concept needs to be extended and defined for mixed strategies
- Games with uncertainty are called Bayesian games and their solution concept – Bayesian N.E.