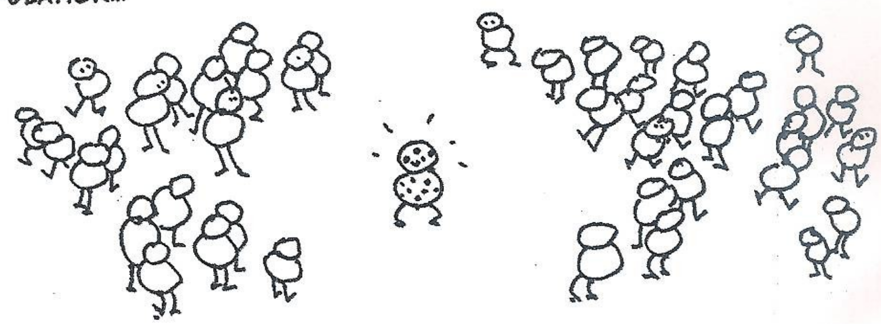
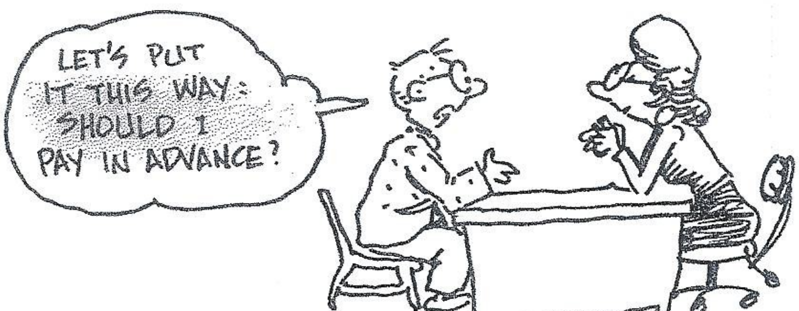
BAYES THEOREM and the case of the false positives...

FOR A MORE SERIOUS APPLICATION OF CONDITIONAL PROBABILITY, LET'S ENTER AN ARENA OF LIFE AND DEATH...

SUPPOSE A RARE DISEASE INFECTS ONE OUT OF EVERY 1000 PEOPLE IN A POPULATION...



AND SUPPOSE THAT THERE IS A GOOD, BUT NOT PERFECT, TEST FOR THIS DISEASE: IF A PERSON HAS THE DISEASE, THE TEST COMES BACK POSITIVE 99% OF THE TIME. ON THE OTHER HAND, THE TEST ALSO PRODUCES SOME *FALSE POSITIVES*. ABOUT 2% OF UNINFECTED PATIENTS ALSO TEST POSITIVE. AND YOU JUST TESTED POSITIVE. WHAT ARE YOUR CHANCES OF HAVING THE DISEASE?



WE HAVE TWO EVENTS TO WORK WITH:

A : PATIENT HAS THE DISEASE
B : PATIENT TESTS POSITIVE.

THE INFORMATION ABOUT THE TEST'S EFFECTIVENESS CAN BE WRITTEN

$P(A) = .001$ (ONE PATIENT IN 1000 HAS THE DISEASE)

$P(B|A) = .99$ (PROBABILITY OF A POSITIVE TEST, GIVEN INFECTION, IS .99)

$P(B|\text{NOT } A) = .02$ (PROBABILITY OF A FALSE POSITIVE, GIVEN NO INFECTION, IS .02)

AND WE ASK

$P(A|B) = \text{WHAT?}$ (PROBABILITY OF HAVING THE DISEASE, GIVEN A POSITIVE TEST)

SINCE THE TREATMENT FOR THIS DISEASE HAS SERIOUS SIDE EFFECTS, THE DOCTOR, HER LAWYER, AND HER LAWYER'S LAWYER CALL ON JOE BAYES, CP (CONSULTING PROBABILIST), FOR AN ANSWER. JOE DERIVES A THEOREM FIRST PROVED BY HIS ANCESTOR, THE REV. THOMAS *BAYES* (1744-1809).

JOE BEGINS WITH A 2X2 TABLE, WHICH DIVIDES THE SAMPLE SPACE INTO FOUR MUTUALLY EXCLUSIVE EVENTS. IT DISPLAYS EVERY POSSIBLE COMBINATION OF DISEASE STATE AND TEST RESULT.

	A	NOT A
B	A AND B	NOT A AND B
NOT B	A AND NOT B	NOT A AND NOT B

LET'S FIND THE PROBABILITIES OF EACH EVENT IN THE TABLE:

	A	NOT A	SUM
B	P(A AND B)	P(NOT A AND B)	P(B)
NOT B	P(A AND NOT B)	P(NOT A AND NOT B)	P(NOT B)
	P(A)	P(NOT A)	1

THE PROBABILITIES IN THE MARGINS ARE FOUND BY SUMMING ACROSS ROWS AND DOWN COLUMNS.

NOW COMPUTE:

$$P(A \text{ AND } B) = P(B|A)P(A) = (.99)(.001) = .00099$$

$$P(\text{NOT } A \text{ AND } B) = P(B|\text{NOT } A)P(\text{NOT } A) = (.02)(.999) = .01998$$

ALLOWING US TO FILL IN SOME ENTRIES:

	A	NOT A	SUM
B	.00099	.01998	.02097
NOT B	P(A AND NOT B)	P(NOT A AND NOT B)	P(NOT B)
	.001	.999	1

WE FIND THE REMAINING PROBABILITIES BY SUBTRACTING IN THE COLUMNS, THEN ADDING ACROSS THE ROWS.

THE FINAL TABLE IS:

	A	NOT A		
B	.00099	.01998	.02097	P(B)
NOT B	.00001	.97902	.97903	P(NOT B)
	.001	.999	1	
	P(A)	P(NOT A)		

FROM WHICH WE DIRECTLY DERIVE

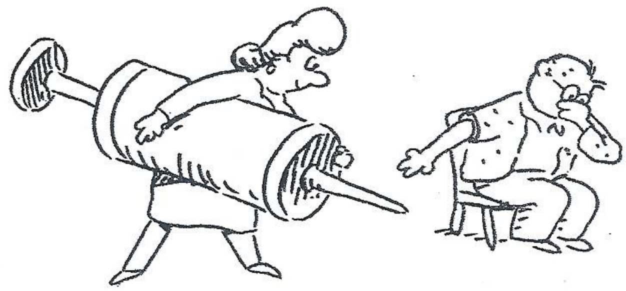
$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)} = \frac{.00099}{.02097} = .0472$$

DESPITE THE HIGH ACCURACY OF THE TEST, ***LESS THAN 5%*** OF THOSE WHO TEST POSITIVE ACTUALLY HAVE THE DISEASE! THIS IS CALLED THE ***FALSE POSITIVE PARADOX.***

THIS TABLE SHOWS WHAT HAPPENS IN A GROUP OF A THOUSAND PATIENTS. ON AVERAGE, ONLY 21 PEOPLE WILL TEST POSITIVE—AND ONLY ***ONE*** OF THEM HAS THE DISEASE! 20 FALSE POSITIVES COME FROM THE ***MUCH LARGER UNINFECTED GROUP.***

	DISEASE	NO DISEASE	
TESTS POSITIVE	1	20	21
TESTS NEGATIVE	0	979	979
	1	999	1000

WHAT'S THE PHYSICIAN TO DO? JOE BAYES ADVISES HER NOT TO START TREATMENT ON THE BASIS OF THIS TEST ALONE. THE TEST DOES PROVIDE INFORMATION, HOWEVER: WITH A POSITIVE TEST THE PATIENT'S CHANCE OF HAVING THE DISEASE INCREASED FROM 1 IN 1000 TO 1 IN 23. THE DOCTOR FOLLOWS UP WITH MORE TESTS.



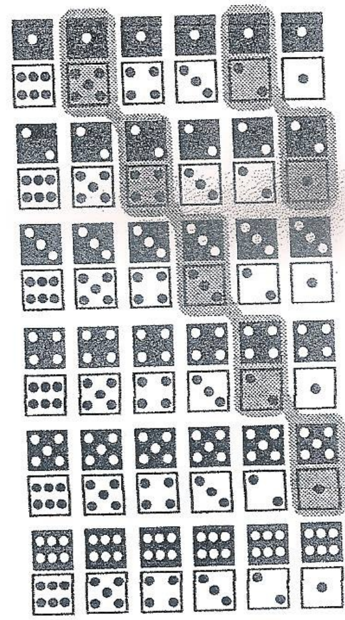
JOE BAYES COLLECTS HIS CONSULTING CHECK BEFORE ADMITTING THAT ALL THOSE STEPS HE WENT THROUGH CAN BE COMPRESSED INTO THE SINGLE FORMULA CALLED BAYES THEOREM:

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(\text{NOT } A)P(B|\text{NOT } A)}$$

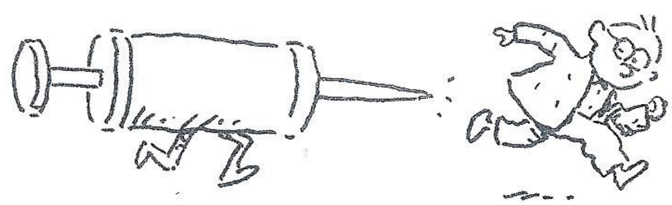
IT COMPUTES P(A|B) FROM P(A) AND THE TWO CONDITIONAL PROBABILITIES P(B|A) AND P(B|NOT A). YOU CAN DERIVE IT BY NOTING THAT THE BIG FRACTION CAN BE EXPRESSED AS

$$\frac{P(A \text{ and } B)}{P(A \text{ and } B)+P(\text{NOT } A \text{ and } B)} = \frac{P(A \text{ and } B)}{P(B)} = P(A|B)$$

IN THIS CHAPTER, WE COVERED THE BASICS OF PROBABILITY: ITS DEFINITION, SAMPLE SPACES AND ELEMENTARY OUTCOMES, CONDITIONAL PROBABILITY, AND SOME BASIC FORMULAS FOR COMPUTING PROBABILITIES. WE ILLUSTRATED THESE IDEAS USING A 2-DICE SAMPLE SPACE. FOR THE MODERN GAMBLER, PROBABILITY IS THE POWER TOOL OF CHOICE.



AND FINALLY, IN THE MEDICAL EXAMPLE, WE SHOWED HOW THESE ABSTRACT IDEAS COULD HELP TO MAKE GOOD DECISIONS IN THE FACE OF ***IMPERFECT INFORMATION*** AND ***REAL RISKS***—THE ***ULTIMATE GOAL*** OF STATISTICS.



BUT THIS IS JUST THE BEGINNING. FOR US, PROBABILITY IS ONLY A *TOOL*—AN ESSENTIAL TOOL, TO BE SURE—IN THE STUDY OF STATISTICS. IN THE CHAPTERS THAT FOLLOW, WE'LL EXPLORE THE SUBTLE RELATIONSHIP BETWEEN PROBABILITY, VARIATIONS IN STATISTICAL DATA, AND OUR CONFIDENCE IN INTERPRETING THE MEANING OF OUR OBSERVATIONS.