Stabilizing the Hénon Map with the OGY Algorithm

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Chaotic orbits are sometimes an undesirable behavior in real-world systems. However, these orbits can be stabilized in some cases if the system contains a parameter that is accessible to a control scheme. The OGY method is one way to stabilize an orbit using such a parameter. We explain this method and present two such controllers that stabilize a period-one and a period-two orbit for the Henon Map, a simple two-dimensional continuous map.

Introduction—Chaotic behavior occurs in many models of real-world systems, from simple one-dimensional maps to complex continuous-time systems with many variables. Sometimes this is a desirable result, for example in the case of random number generators. Often, however, it is not. An erratically oscillating machine may cause significant damage to itself and surrounding infrastructure, and a heart chaotically oscillating in the throes of arrhythmia does not pump blood[3]. Moreover, since the chaotic region for a system may contain many possible orbits, being able to select between several of these by changing control parameters allows multiple responses from the same system[1]. Thus, it is often useful to stabilize a chaotic orbit to one that is regular and predictable.

One method to do so involves the feedback-based OGY method. We aim to present an explanation and demonstration of this method targeted towards undergraduate students in engineering and mathematics. We begin by introducing the theory and proceed to implement it for the two-dimensional Hénon Map that demonstrates the successful stabilization of a period-1 and a period-2 orbit.

Stability—To understand what is involved in stabilizing a chaotic orbit, it is first necessary to lay context in the idea of periodic orbit. A period-n orbit of a map yields the same value after successive n iterations. Stable periodic orbits are surrounded by a region of points that get closer to that orbit with successive iterations, while unstable orbits are surrounded by a region of points that tend to diverge away. Thus, in the latter case, unless such an initial value is exactly on the unstable periodic orbit, it will not stabilize toward the fixed points because the surrounding region will tend to push it away. A chaotic orbit can be interpreted as an unstable period-infinity orbit.

The stability of a fixed point in a given map $\vec{F}$ can be calculated by taking the eigenvalues of the Jacobian of the map at the fixed point. For a periodic orbit, we use the matrix product of the Jacobian at each point in the orbit to calculate the eigenvalues. If all eigenvalues are greater than one, then the periodic orbit is unstable; even if a point starts very close to the periodic orbit, it will tend to move away with successive iterations. If the set of eigenvalues is mixed between values greater and less than 1, the orbit is a saddle point; the orbit will tend to diverge away from the fixed point along the eigenvectors of the eigenvalues greater than 1; hence saddle points are also unstable. If the absolute value of each of the eigenvalues is less then 1, then the periodic orbit is stable.

OGY Method— One way to generate such a periodic output from a chaotic system is to use the OGY method, a feedback-based approach published by Ott, Grebogi, and Yorke in 1990[1]. It can be explained as follows:

Let $\vec{F}$ be an N-dimensional map with an unstable period-1 fixed point $\vec{P}^*$ (the argument can be expanded to period-n orbits by replacing $\vec{F}$ with $\vec{F}^n$). Since we are interested in finding the motion of the orbit relative to the fixed point(s), we define another term, the "error" $\vec{p_n}$, equal to $\vec{P_n} - \vec{P}^*$ where $\vec{P}$ is the value of the nth iteration of the orbit and $\vec{P}^*$ is the value of the fixed point.

Assume that the value of the map at this fixed point is dependent on a system parameter Z that normally has value Q. That is, perturbing this parameter by some small amount z to a new value $Q - z$ results in a change of $\frac{\partial \vec{F}}{\partial z}$ in the map at this point. Near the fixed point, $\frac{\partial \vec{F}}{\partial z}$ can be approximated by a linear Taylor series expansion $\vec{C}$; hence, changing the parameter z affects the value of the map by $\vec{C}z$.

If we strategically modify z in response to the error $\vec{p}$, we may be able to use $\vec{C}z$ to cancel the effect of $\mathbf{J}$, the tendency of the orbit to move away from the fixed point. Modifying z affects each component of the vector $\vec{p}_{n+1}$; to find a global optimally solution, the change to z must be determined with a mind to its effects on all components of the vector $\vec{P}_{n+1}$. Thus, we define a set of weights $\{k_1, k2_, \ldots, k_N\}$ which we can express as a row vector $\vec{K}$ such that $z = \vec{K}\vec{C}$. In equation 1, we examine the equation for the error of the next value of the orbit in terms of K to solve for the proper values for $\vec{K}$.

$$\vec{p}_{n+1} = (\mathbf{J} - \vec{C}\vec{K})\vec{p_n} \tag{1}$$

In equation 1, $\vec{p}_{n+1}$ is the error of the next point in the orbit, $\vec{P}_{n+1} - \vec{P}^*$, p_n is the error of the current point in the orbit, $\vec{P}_n - \vec{P}^*$. $\mathbf{J}$ is the Jacobian of the map which is the local sensitivity of p_{n+1} to the error p of the previous value of the orbit. $\vec{C}$ is the sensitivity of the map to some accessible system parameter Z that is displaced by

z from its normal value.

The stability of the map thus depends on the matrix $\mathbf{J} - \vec{C}\vec{K}$, which is the Jacobian of the controlled map. Thus if $\mathbf{J} - \vec{C}\vec{K}$ has eigenvalues with modulus less than 1, the control system can successfully stabilize the system. This requires an appropriate choice of $\vec{K}$ such that the magnitude of each eigenvalue is less than 1. To determine the range of appropriate values for $k_1, k_2, \ldots, k_N$, one can solve $\det(\mathbf{J} - \vec{C}\vec{K} - \lambda\vec{I}) = 0$ for $\lambda_1, \lambda_1, \ldots, \lambda_N$ (the characteristic equation of the matrix) in terms of $k_1, k_2, \ldots, k_N$. Then, solving the equations with the boundary conditions for each lambda value ($\lambda_n = 1$, $\lambda_n = -1$) allows us to find the set of boundary conditions for stability.

Applying to the Hénon map— We can apply the OGY algorithm to a two-dimensional map in the form $\vec{p}_{n+1} = \vec{F}(\vec{p}_n)$, where $\vec{p}$ is a two-dimensional vector, $\vec{F}$ is a vector function, and n is the iteration count. One example of a two-dimensional map is the Hénon map, defined in equation 2. At certain parameter values, the Hénon map presents chaotic behavior [2]. In our examples, we use the parameters $a = 1.4$ and $b = 0.3$ because those were the parameters Hénon [2] showed demonstrated chaotic behavior, but our analysis is symbolic and should work with any given parameters the has chaotic behavior.

$$F(\vec{P_{n+1}}) = \left\{ \begin{array}{l} X_{n+1} = 1 + Y_n - aX_n^2 \\ Y_{n+1} = bX_n \end{array} \right\} \tag{2}$$

We first perform a change of variables such that $X_n = \frac{1}{a}x_n$ and $Y_n = \frac{b}{a}y_n$. By changing the variables, the revised Hénon map, in equation 3, has the parameter a which is not attached to either the variable x_n or y_n, making that system parameter easily adjustable. Our control system consists of creating perturbations a_0 in the parameter a.

$$\vec{F}(\vec{P}_{n+1}) = \left\{ \begin{array}{l} x_{n+1} = a + by_n - ax_n^2 \\ y_{n+1} = x_n \end{array} \right\} \tag{3}$$

We can find the fixed points by solving the equation $\vec{P}_n = \vec{F}(\vec{P}_n)$, where $\vec{F}(\vec{P}_n)$ is defined in equation 3. The solutions to this system of equation for both x and y are the roots to the polynomial $0 = x^2 + (1-b)x - a$. Only one of these fixed points is located in the chaotic attractor [4]. Because the OGY algorithm only makes a fixed point locally stable, the control system work only when the chaotic orbit passes close to that fixed point. Therefore, we want to stabilize the orbit around the fixed point located in the chaotic attractor, which we call the stabilization fixed point x^*. In equation 4, we use the quadratic formula to find x^*.

$$x^* = y^* = \frac{b - 1 + \sqrt{(1-b)^2 + 4a}}{2} \tag{4}$$

Characteristic Polynomial and Control Parameters— We proceed to apply the control algorithm, a perturbation in the parameter a when the system is close to the fixed point, defined in general in equation 1. The total perturbation, in this case a_0, is proportional to the error. In this specific case, $\vec{C}$, the partial derivatives of the map with regards to the controlled system parameter a, equals the column vector $(1, 0)$. The row vector $\vec{K}$, also written as $[k_1, k_2]$, is the control parameter that varies according to the map parameters. The chaotic orbit can be controlled only by choosing the correct values for $[k_1, k_2]$.

With the control algorithm in place, the stability of the fixed point is determined by the eigenvalues of the matrix $[\mathbf{J} - \vec{C}\vec{K}]$, defined below. $\mathbf{J}$ is the Jacobian of the uncontrolled Hénon map, defined in equation 3. The error grows if the modulus of the eigenvalues were greater than 1 and decay if the modulus is less than 1.

$$[\mathbf{J} - \vec{C}\vec{K}] = \begin{pmatrix} -2x^* - k_1 & b - k_2 \\ 1 & 0 \end{pmatrix}$$

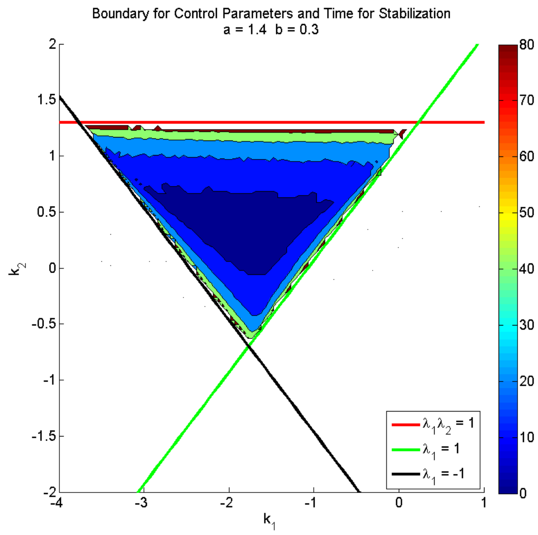

FIG. 1: The boundary on the k_1, k_2 plane that can stabilize the chaotic orbit defined in equation 6. If we choose parameters within the region bounded by the triangle and apply the control algorithm, the system will stabilize around the fixed point. In general, control parameters close to the boundary causes the system to stabilize more slowly because the modulus of their eigenvalues is closer to 1, causing a slower rate of decay in the error.

The characteristic equation to find the eigenvalues is defined below in equation 5, where x^* is the fixed point defined in equation 4

$$0 = \lambda^2 + \lambda(2x^* + k_1) + k_2 - b \tag{5}$$

To stabilize the system about the fixed points through decaying the error, we must choose values of k_1 and k_2 such that both eigenvalues, λ_1 and λ_2 have magnitudes less than 1. Another way to interpret this is that the the product of both eigenvalues must be less than 1, and that $\lambda_1 < 1$ and $\lambda_1 > -1$.

To meet the condition that $\lambda_1\lambda_2 < 1$, we can factor out the characteristic equation 5 into $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$. This means $\lambda_1\lambda_2 = k_2 - b$. Thus, the boundary when $\lambda_1\lambda_2$ transitions from below to above 1 equals $k_2 = 1 + b$.

We proceed to find conditions to ensure $\lambda_1 < 1$; to do this, we can find the parameter ranges such that $\lambda_1 = 1$, which form the boundary for when λ_1 transitions from below 1 to above 1. We can locate the boundary by trying to factor out $(\lambda - 1)$ from the characteristic equation 5. Through long division, we can factor out $(\lambda - 1)$ if and only if $k2 - b + 2x^* + k_1 + 1 = 0$. Thus, the boundary for $\lambda_1 = 1$ on the k_1, k_2 plane is a line defined by $k_2 = -k_1 - 2x_0 - 1 + b$.

We use a similar approach in factoring out $(\lambda + 1)$ to find the boundary for $\lambda_1 = -1$. The boundary where λ_1 transitions from below -1 to above is a line defined by $k_2 = k_1 + b + 2x^* - 1$. Equation 6 states the boundaries.

$$\begin{aligned} \lambda_1\lambda_2 < 1 : \quad & k_2 = 1 + b \\ \lambda_1 < 1 : \quad & k_2 = -k_1 - 2x^* - 1 + b \\ \lambda_1 > -1 : \quad & k_2 = k_1 + 2x^* - 1 + b \end{aligned} \tag{6}$$

Figure 1 plots the three boundary, a triangle defined by equation 6. To stabilize the system about the fixed point, we must choose a value of k_1 and k_2 within that triangle. A thorough discussion of the inconsistencies is given right before the conclusion.

In figure 2, we choose control parameters within the stable region from figure 1 and iterate the controlled map. Because we made linear approximations, our method should work when $\vec{P}$ is close to the fixed point, or when the error is small. Thus, our control algorithm activates whenever it detects that x_n and y_n are close to the fixed points. Another reason the control algorithm activates when the error is small is because the power of the control might be limited such that it can only work with small error.

Stabilizing on a Period-2 Orbit— We can apply the OGY algorithm to stabilize a chaotic orbit to a period-2 orbit.

We find the period-2 orbits by solving the equation $\vec{P} = \vec{F}^2(\vec{P})$, where the function is defined in equation 3. Because we aim to have the system oscillate between two points, we can remove the fixed points from the set

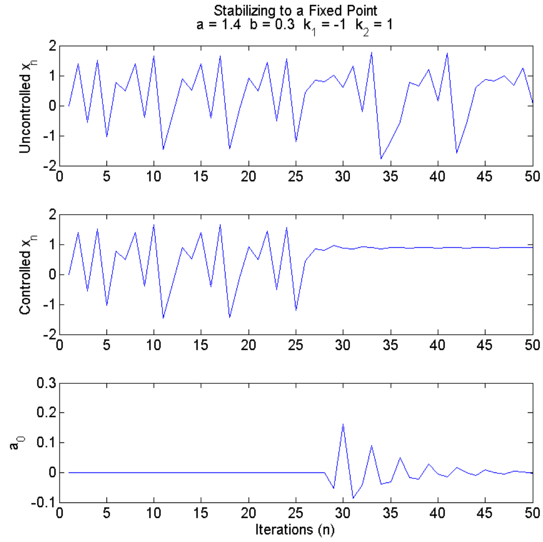

FIG. 2: The top graph represents the chaotic orbit of an uncontrolled Hénon map. With the control algorithm activating around the 28th iteration in the middle graph, we see that the map stabilizes around the fixed point. The control parameters k_1 and k_2 is the blue dot inside the teal region in Figure 1, ensuring the system stabilizes within the fixed point. The bottom graph illustrates the size of the perturbations of the control parameter; it is strong when it first starts activating but eventually decreases when the map is stabilized around the fixed point.

of period-2 orbits. The two remaining period-2 orbits are designated as x_α and x_β. y_α and y_β equal x_α and x_β according to equation 3. If a chaotic attractor exists, the period-2 orbit would be located within that attractor because a chaotic attractor contains all periodic orbits.

We can apply the OGY algorithm at x_α, x_β, or both. We chose the final option because applying perturbations to both orbit points stabilizes the system most quickly.

Using the chain rule, the stability of a period-2 orbit is governed by the eigenvalues of the product of the respective Jacobian matrix for x_α and x_β. To find proper control parameters, we try to find the eigenvalues of the matrix product of $[\mathbf{J}|_{x=x_\alpha} - \vec{C}|_{x=x_\alpha}\vec{K}]$ and $[\mathbf{J}|_{x=x_\beta} - \vec{C}|_{x=x_\beta}\vec{K}]$ since control is applied at both x_α and x_β. We notice that the characteristic equation for the eigenvalues of is defined below:

$$\begin{aligned} 0 = & -4x_1x_2\lambda - 2k_1x_1\lambda - 2x_2k_1\lambda + b^2 - \\ & 2bk_2 + k_2^2 - k_1^2\lambda - 2b\lambda + 2k_2\lambda + \lambda^2 \end{aligned}$$

To find control parameters for a stable period-2 orbit, we can again solve for the boundary of the region on the k_1, k_2 plane such that $\lambda_1\lambda_2 < 1$, $\lambda_1 < 1$, and $\lambda_1 > -1$. The approach and logic is the same as the ones used to derive the parameter boundaries in equation 6. Because of various nonlinear terms in the characteristic equation, the boundary is no longer explicitly defined; we can, however, describe the boundary as the roots of the polynomial equations defined in equation 7. Figure 3 plots the boundary for control parameters that can stabilize the chaotic orbit. Figure 4 gives an example of stabilizing a chaotic orbit onto a period-2 orbit by choosing a control parameter within the stable region.

$$\begin{aligned} \lambda_1\lambda_2 < 1: \quad 0 =& k_2^2 - k_2(2b) + b^2 - 1 \\ \lambda_1 < 1: \quad 0 =& k_2^2 - k_2(-2b-2) + k_1^2 + 2k_1x_1 \\ &+ 2k_1x_2 + 4x_1x_2 + b^2 + 2b + 1 \\ \lambda_1 > -1: \quad 0 =& k_2^2 - k_2(-2b+2) - k_1^2 - 2k_1x_1 \\ &- 2k_1x_2 - 4x_1x_2 + b^2 - 2b + 1 \end{aligned} \tag{7}$$

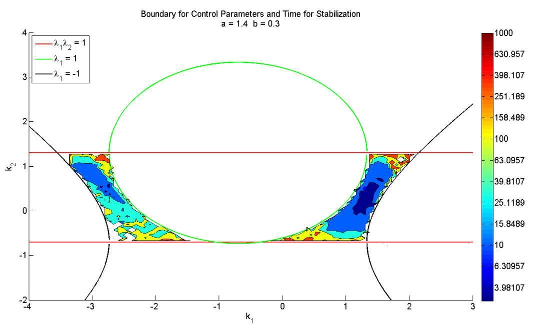

FIG. 3: The boundary of the control parameters, with the boundary defined in equation 7. Choosing parameters within the region stabilizes the Hénon map around a period-2 orbit. Inside the region, locations closer to the boundaries in general takes more iterations to stabilize, defined as when the error is reduced by 99%.

Figure 3 also plots the number of iterations the system takes to stabilize the chaotic orbit to a period-2 orbit once the control system is activated. The time is calculated through numerical iteration like that shown in figure 4. Locations closer to the boundaries generally take more time to stabilize because the modulus of their eigenvalues are closer to 1. The eigenvalues are an approximate measure the rate of growth or decay of the error from the desired position (the period-2 orbit); eigenvalues closer to 0 means the error decays quicker.

In both figure 6 and figure 7, there exists some regions inside the boundary that do not stabilize. This is because the simulation has not run enough iterations for such points to stabilize, defined as when 99% of the error is eliminated. Furthermore, the contours for the number of iterations are not smooth and well-defined. One reason is this occurs because the boundaries do not give direct information about the λ_2 or their corresponding eigenvector. Even if λ_1 has a large modulus, λ_2 may be small, resulting in a faster decay. In addition, there exists roundoff error which amplifies because the results from each iteration affects the results of the next iteration.

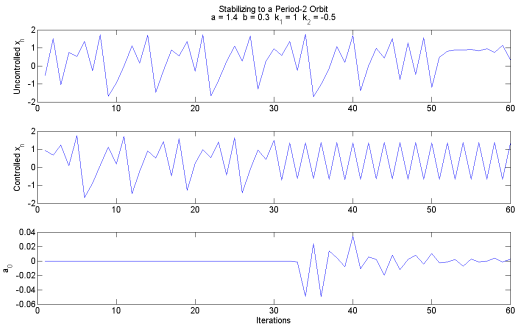

FIG. 4: The top graph represents the chaotic orbit of an uncontrolled Hénon map. With the control algorithm activating around the 90th iteration in the middle graph, we see that the map stabilizes by oscillating around the period-2 orbits. The control parameters k_1 and k_2 is the blue dot in figure 3. The bottom graph again describes the relative size of the perturbations made to stabilize the system.

Conclusion — In many cases, chaotic behavior is not desirable and needs to be controlled to achieve a predictable steady or periodic behavior. The OGY control algorithm is a feedback algorithm that creates small perturbations in system parameters to stabilize the system to periodic orbits. We applied the OGY algorithm to the Hénon map with chaotic system parameters. We found parameter ranges that the control algorithm is able to stabilize a system to a fixed point and a periodic orbit. In addition, the amount of "work" the control algorithm must do is directly linked to the eigenvalues of the controlled matrix.

We applied the OGY algorithm to stabilize the Hénon map about the period-1 and period-2 orbit. Preliminary results show that the same algorithm can be used to stabilize higher-period orbits; a further area of investigation is determining whether the OGY algorithm can stabilize a chaotic Hénon map about any period-n orbit.

We are also interested in other chaotic maps the OGY algorithm can control. Preliminary investigations into the Baker map have yielded promising results.

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