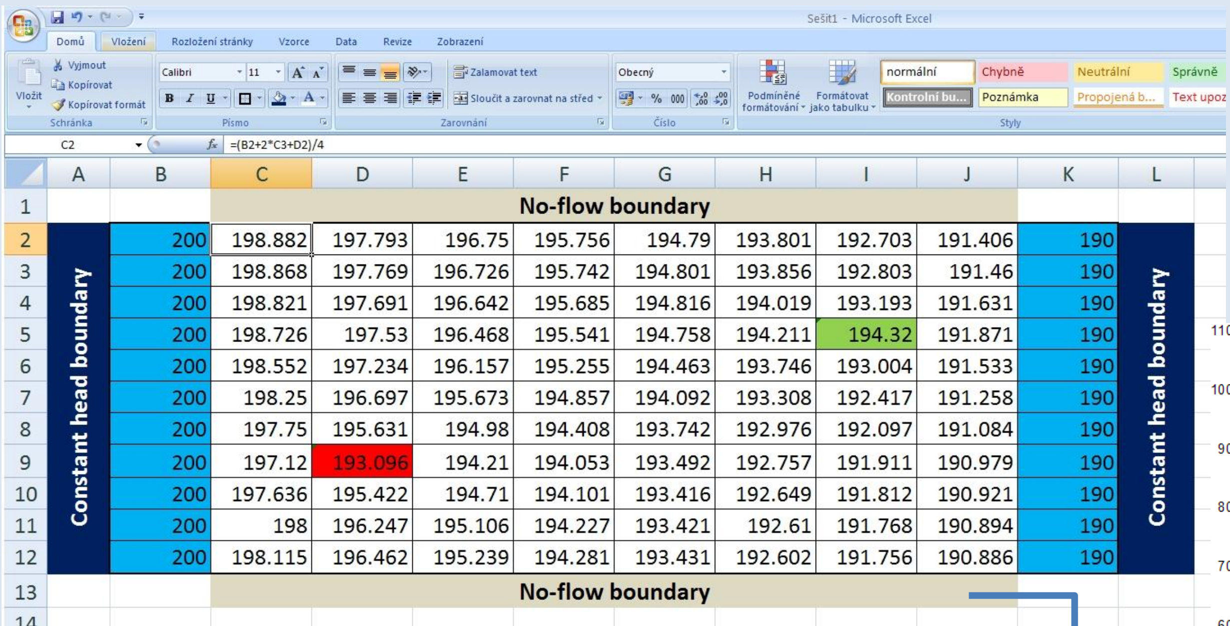
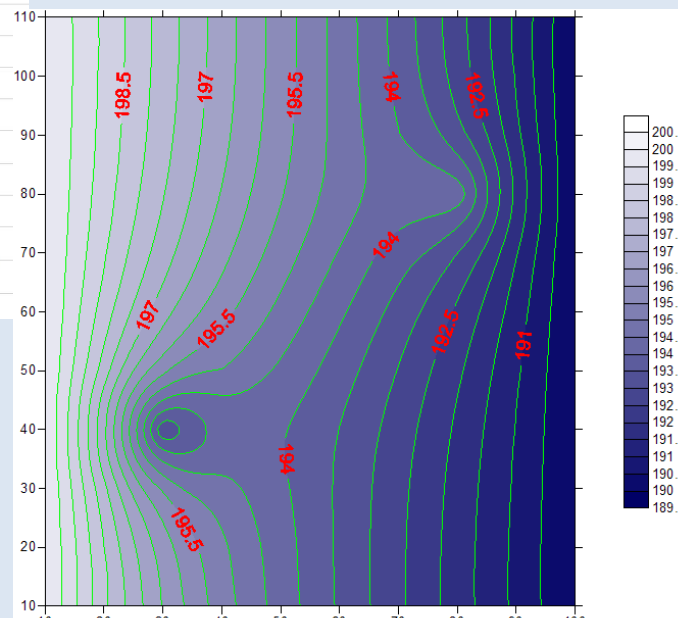
Applied Hydrogeology
Spreadsheet modeling

Adam Říčka



	A	B	C	D	E	F	G	H	I	J	K	L
1			No-flow boundary									
2	Constant head boundary	200	198.882	197.793	196.75	195.756	194.79	193.801	192.703	191.406	190	Constant head boundary
3	Constant head boundary	200	198.868	197.769	196.726	195.742	194.801	193.856	192.803	191.46	190	Constant head boundary
4	Constant head boundary	200	198.821	197.691	196.642	195.685	194.816	194.019	193.193	191.631	190	Constant head boundary
5	Constant head boundary	200	198.726	197.53	196.468	195.541	194.758	194.211	194.32	191.871	190	Constant head boundary
6	Constant head boundary	200	198.552	197.234	196.157	195.255	194.463	193.746	193.004	191.533	190	Constant head boundary
7	Constant head boundary	200	198.25	196.697	195.673	194.857	194.092	193.308	192.417	191.258	190	Constant head boundary
8	Constant head boundary	200	197.75	195.631	194.98	194.408	193.742	192.976	192.097	191.084	190	Constant head boundary
9	Constant head boundary	200	197.12	193.096	194.21	194.053	193.492	192.757	191.911	190.979	190	Constant head boundary
10	Constant head boundary	200	197.636	195.422	194.71	194.101	193.416	192.649	191.812	190.921	190	Constant head boundary
11	Constant head boundary	200	198	196.247	195.106	194.227	193.421	192.61	191.768	190.894	190	Constant head boundary
12	Constant head boundary	200	198.115	196.462	195.239	194.281	193.431	192.602	191.756	190.886	190	Constant head boundary
13			No-flow boundary									
14												



Iterative methods

Iterative methods need initial values at iteration level *m*, $h_{i,j}^{m}$ and the purpose is to calculate $H_{i,j}^{m+1}$.

Jacobi iteration

Resulted values from previous level of iteration are used in actual iteration – not often used

$$h_{i,j}^{m+1} = (h_{i+1,j}^{m} + h_{i-1,j}^{m} + h_{i,j+1}^{m} + h_{i,j-1}^{m})/4$$

Gauss-Seidel iteration

Newly computed values in the iteration formula are used: iteration level *m+1* values are available for nodes *(i-1,j)* and *(i,j-1)* when calculating *h* for node *(i,j)* - more efficient than Jacobi iteration

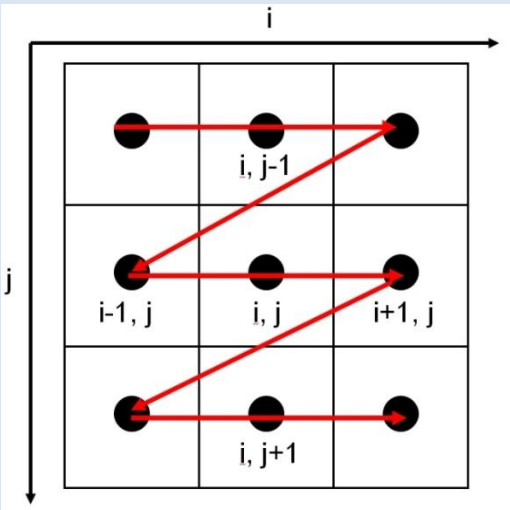
$$h_{i,j}^{m+1} = (h_{i+1,j}^{m+1} + h_{i-1,j}^{m+1} + h_{i,j+1}^{m} + h_{i,j-1}^{m})/4$$

Succesive over relaxation (SOR)

Convergence rate of Gauss-Seidel iteration method can be improved by relaxation factor ω, which is obtained by trial and error and optimal value is between 1,5 - 1,8

$$h_{i,j}^{m+1} = (1-\omega)h_{i,j}^{m}\omega(h_{i-1,j}^{m+1} + h_{i,j-1}^{m+1} + h_{i+1,j}^{m} + h_{i,j+1}^{m})/4$$

Iterative methods



Gauss-Seidel Iteration

$$h_{i,j}{}^{m} = \frac{h_{i+1,j}{}^{m} + h_{i-1,j}{}^{m} + h_{i,j+1}{}^{m} + h_{i,j-1}{}^{m}}{4}$$

initial guesses *m*

$$h_{i,j}{}^{m+1} = \frac{h_{i+1,j}{}^{m} + h_{i-1,j}{}^{m+1} + h_{i,j+1}{}^{m} + h_{i,j-1}{}^{m+1}}{4}$$

initial guesses *m* and next step *m+1*

$$\left\| h^{m} - h^{m+1} \right\| < \varepsilon$$

Iteration reached Convergence criteria

Let's try iterative method in spreadsheet by hand

	function x=1/x+3		
	Convergetion critaria	0.01	
	iteration indices		
Initial vlaue	1	1	
	4.00	2	
	3.25	3	
	3.31	4	
	3.30	**5**	Convergetion criteria is reached in 5th iteration
	3.30	6	
	3.30	7	
	3.30	8	
	3.30	9	
	3.30	10	

Steady-state flow without leaves and/or enters

Darcy's law in *x* and *y* direction

$$q_x = k_x \frac{\partial h}{\partial x}$$

$$q_y = k_y \frac{\partial h}{\partial y}$$

+

Water balance equation

$$\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} = 0$$

=

K is constant

$$\frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y}\right) = 0$$

→

Laplace's equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

Finite difference approximation of Laplace's equation

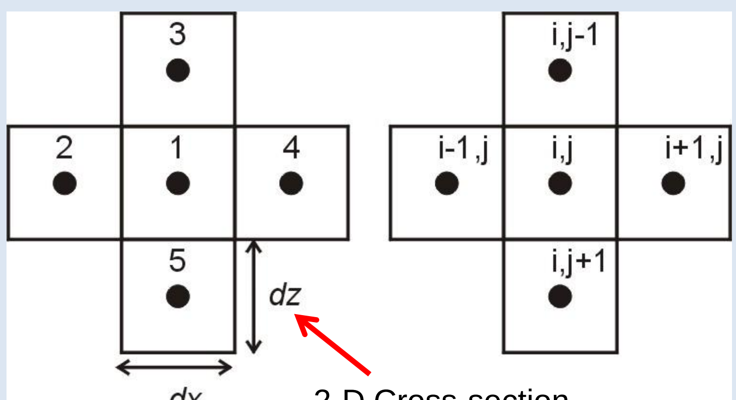
$$q_{1\,in} = q_{21} + q_{31} + q_{41} + q_{51} \longrightarrow K \frac{h_2 - h_1}{\Delta x}\Delta z + K \frac{h_3 - h_1}{\Delta z}\Delta x + K \frac{h_4 - h_1}{\Delta x}\Delta z + K \frac{h_5 - h_1}{\Delta z}\Delta x = 0 \longrightarrow$$

$$\longrightarrow h_1 = \frac{(\Delta z)^2 (h_2 + h_4) + (\Delta x)^2 (h_3 + h_5)}{2(\Delta z)^2 + 2(\Delta x)^2}$$

providing *dx* and *dz* is the same →

Central difference

$$\rightarrow h_1 = \frac{(h_2 + h_3 + h_4 + h_5)}{4}$$



$$h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}}{4}$$

Boundary conditions – mesh centered grid

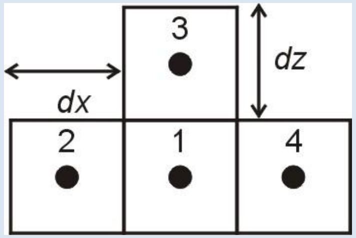
No-flow boundary

$$\frac{\partial h}{\partial x} = 0 \qquad \frac{\partial h}{\partial z} = 0$$

$$h_{i+1,j} + h_{i-1,j} = 0$$

providing *dx* and *dz* is the same

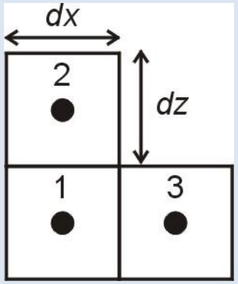
$$h_1 = \frac{(2h_2 + h_3 + h_4)}{4}$$



providing *dx* and *dz* is the same

corner No-flow boundary

$$h_1 = \frac{(2h_2 + 2h_3)}{4}$$



Constant head boundary

Simply enter appropriate head value in the boundary cell and use **central difference**

Steady-state flow with leaves and/or enters

$$\frac{\partial}{\partial x}\left(kb\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(kb\frac{\partial h}{\partial y}\right)=-R$$

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{R}{T}$$

Poisson equation
Confined aquifer

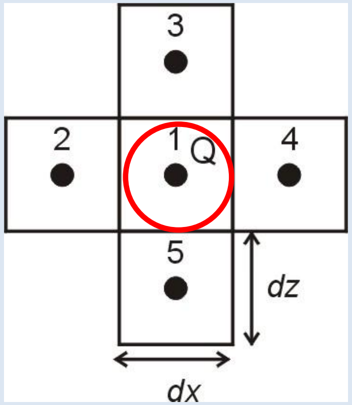
$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{2R}{T}$$

Poisson equation
Unconfined aquifer

Finite difference approximation of Poisson equation

providing *dx* and *dz* is the same

$$h_1=\frac{\left(h_2+h_3+h_4+h_5-\frac{Q}{K}\right)}{4}$$

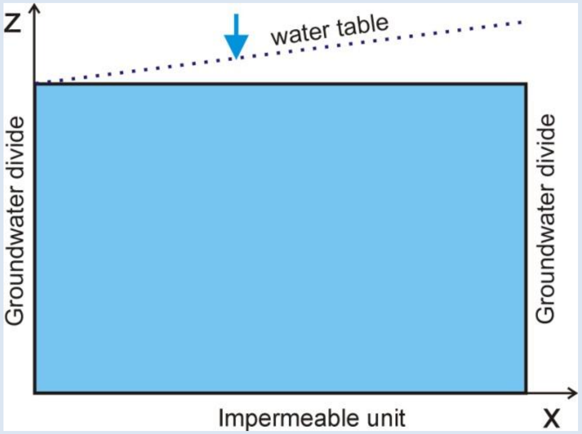


Note – from mathematical expression implies:
- *Negative sign* means recharge (e. g. injection well)
- *Positive sign* means discharge (e.g. pumping well)

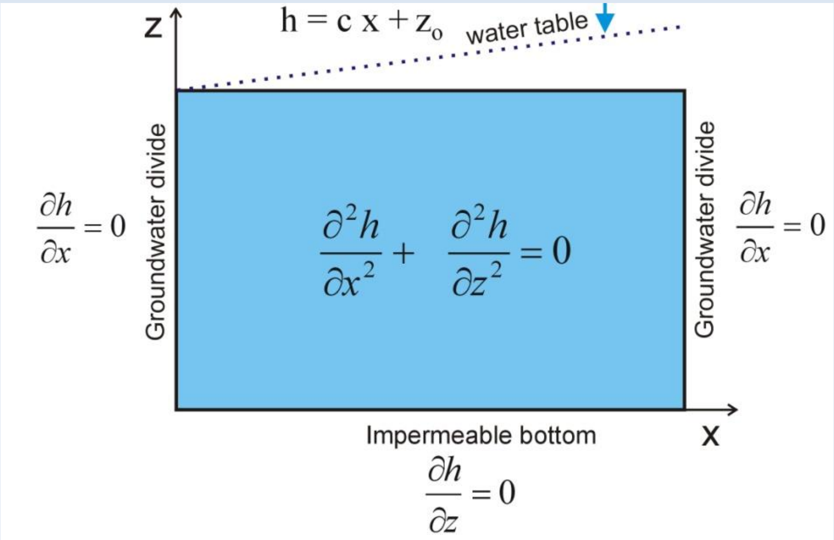
Cross-section model – Tóth problem

2-D simulation, steady-state flow
Unconfined, isotropic aquifer

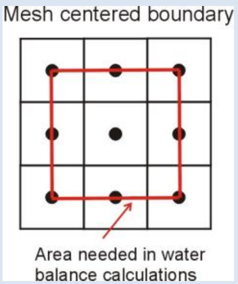
Physical model conception

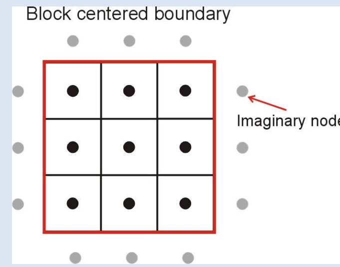


Mathematical model conception



Water balance at Mesh and Block centered grid





Grid node

Only ½*Q* at no-flow boundary condition

Cell in grid

Groundwater flow

hydraulic head (m)

200	200.5	201	201.5	202
201.[illegible]735	201.[illegible]023	201.4462	201.7787	202.1511
201.[illegible]459	201.[illegible]893	201.8040	202.0173	202.2845
202.[illegible]604	202.[illegible]049	202.0632	202.2021	202.3917
202.[illegible]455	2[illegible]2.[illegible]068	202.2418	202.3363	202.4719
202.[illegible]573	202.3350	202.3610	202.4292	202.5287
202.4243	202.4149	202.4379	202.4908	202.5670
202.4626	202.4625	202.4849	202.5291	202.5910
202.4818	202.4874	202.5101	202.5499	202.6040
202.4886	202.4953	202.5180	202.5565	202.6082

Water Balance

T (m2/s)

0.0001

Discharge area

Q(m3/s)

-0.00006	-0.00007	-0.00004	-0.00003	-0.00002
-0.00003	-0.00005	-0.00004	-0.00002	-0.00001
-0.00002	-0.00003	-0.00003	-0.00002	-0.00001
-0.00001	-0.00002	-0.00002	-0.00001	-0.00001
-0.00001	-0.00001	-0.00001	-0.00001	-0.00001
0.00000	-0.00001	-0.00001	-0.00001	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000

Grid node

Imaginary nodes in no-flow boundaries

Cell in grid

Groundwater flow

hydraulic head (m)

	200	200.5	201	201.5	20
200.7190	200.[illegible]190	200.[illegible]651	201.3058	201.6882	202.
201.1918	201.[illegible]918	201.[illegible]355	201.5699	201.8568	202.
201.5210	201.[illegible]210	201.[illegible]152	201.7817	201.9970	202.
2[illegible]1.7561	201.[illegible]561	2[illegible]1.[illegible]225	201.9446	202.1078	202.
201.9249	201.9249	201.[illegible]740	202.0663	202.1921	202.
202.0444	202.0444	202.08[illegible]4	202.1546	202.2540	202.
202.1260	202.1260	202.1568	202.2155	202.2970	202.
202.[illegible]768	202.1768	202.2032	202.2537	202.3240	202.
202.2011	202.2011	202.2255	202.2721	202.3371	202.
202.2011	202.2011	202.2255	202.2721	202.3371	202.

Water Balance

T (m2/s)

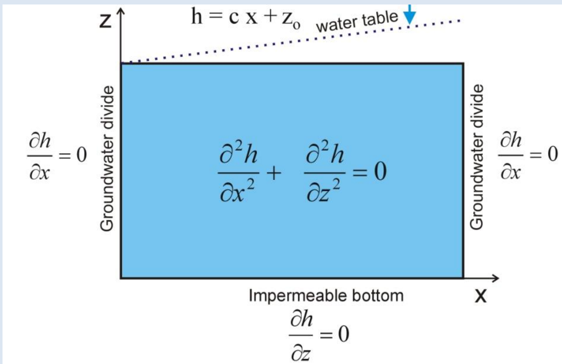
0.0001

Q(m3/s)

-0.0000719	-0.0000465	-0.0000306	-0.0000188	-0.000
-0.0000473	-0.0000370	-0.0000264	-0.0000169	-0.000
-0.0000329	-0.0000280	-0.0000212	-0.0000140	-0.000
-0.0000235	-0.0000207	-0.0000163	-0.0000111	-0.000
-0.0000169	-0.0000152	-0.0000122	-0.0000084	-0.000
-0.0000120	-0.0000108	-0.0000088	-0.0000062	-0.000
-0.0000082	-0.0000074	-0.0000061	-0.0000043	-0.000
-0.0000051	-0.0000046	-0.0000038	-0.0000027	-0.000
-0.0000024	-0.0000022	-0.0000018	-0.0000013	-0.000

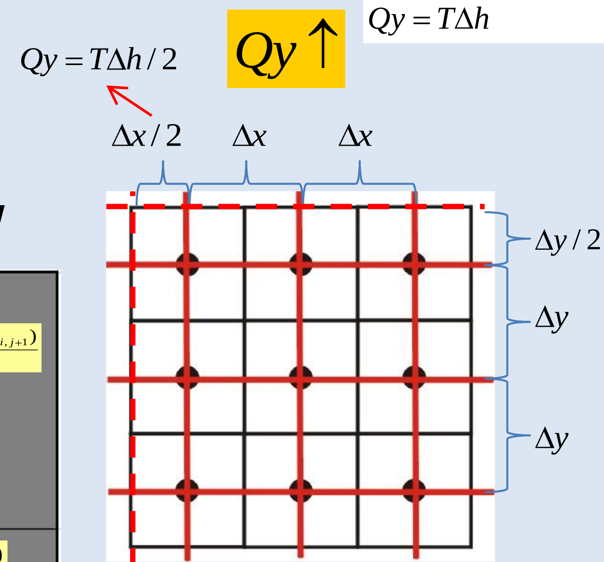
Cross-section model – Tóth problem

2-D simulation, steady-state flow
Unconfined, isotropic aquifer



Water balance in mesh centered grid

$$Qy = k(\Delta x b)(\Delta h - \Delta y)$$
$$\Delta x = \Delta y$$
$$Qy = T\Delta h$$



$Qx = T\Delta h / 2$

$Qx \rightarrow$

$$Qx = k(\Delta y b)(\Delta h - \Delta x)$$
$$\Delta x = \Delta y$$
$$Qx = T\Delta h$$

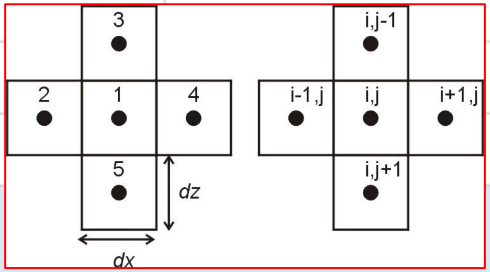
Spreadsheet approach – groundwater flow

$h_{i,j} = \frac{(2h_{i+1,j} + h_{i,j-1} + h_{i,j+1})}{4}$	$h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}}{4}$ Well $h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - \frac{Q}{K}}{4}$	$h_{i,j} = \frac{(2h_{i-1,j} + h_{i,j-1} + h_{i,j+1})}{4}$
$h_{i,j} = \frac{(2h_{i,j-1} + 2h_{i+1,j})}{4}$	$h_{i,j} = \frac{(2h_{i-1,j} + h_{i-1,j} + h_{i+1,j})}{4}$	$h_{i,j} = \frac{(2h_{i-1,j} + 2h_{i,j-1})}{4}$
(blue)	active, flow area	
(grey)	inactive, no-flow area	

2-D flow in Excel spreadsheet

$h_{i,j} = \frac{(2h_{i+1,j} + h_{i,j-1} + h_{i,j+1})}{4}$	$h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}}{4}$ Well: $h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - \frac{Q}{K}}{4}$	$h_{i,j} = \frac{(2h_{i-1,j} + h_{i,j-1} + h_{i,j+1})}{4}$
$h_{i,j} = \frac{(2h_{i,j-1} + 2h_{i+1,j})}{4}$	$h_{i,j} = \frac{(2h_{i-1,j} + h_{i-1,j} + h_{i+1,j})}{4}$	$h_{i,j} = \frac{(2h_{i-1,j} + 2h_{i,j-1})}{4}$

(blue)	active, flow area
(grey)	inactive, no-flow area



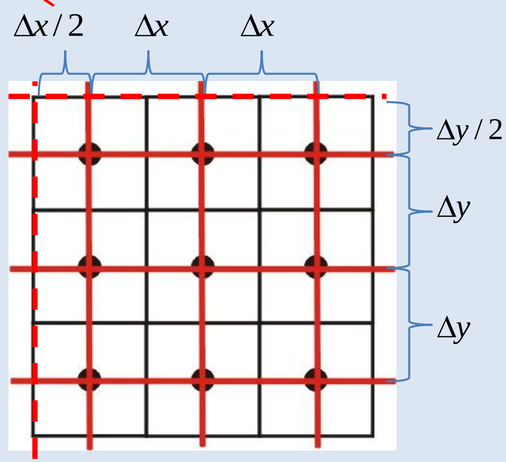
2-D water balance in Excel spreadsheet

Water balance in mesh centered grid

$$Qy = k(\Delta xb)(\Delta h - \Delta y)$$
$$\Delta x = \Delta y$$
$$Qy = T\Delta h$$

$$Qy \uparrow$$

$$Qy = T\Delta h / 2$$



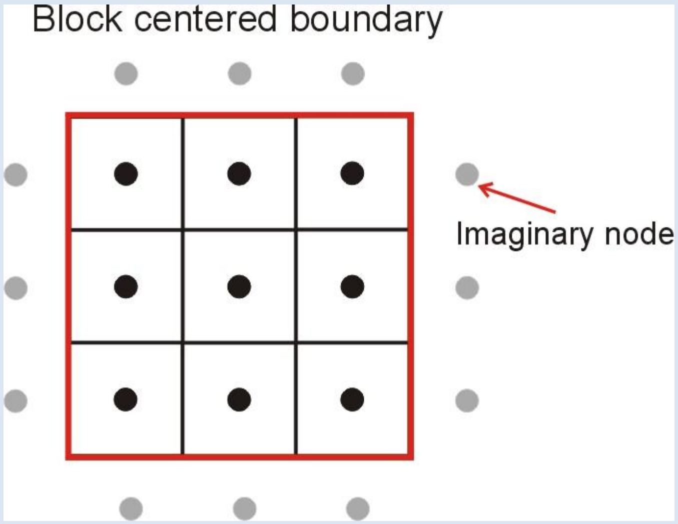
$$Qx = T\Delta h / 2$$

$$Qx \rightarrow$$

$$Qx = k(\Delta yb)(\Delta h - \Delta x)$$
$$\Delta x = \Delta y$$
$$Qx = T\Delta h$$

Difference between input and output = **error of numerical approximation -** related to (among others) Convergence criteria and grid spacing

Block-centered grid in Excel spreadsheet



Groundwater flow

hydraulic head (m)

	200	200.5	201	201.5	202	202.5	203	203.5	204	204.5	205	
200.7190	200.7190	200.9651	201.3058	201.6882	202.0901	202.5000	202.9099	203.3118	203.6942	204.0349	204.2810	204.2810
201.1918	201.1918	201.3355	201.5699	201.8568	202.1721	202.5000	202.8279	203.1432	203.4301	203.6645	203.8082	203.8082
201.5210	201.5210	201.6152	201.7817	201.9970	202.2416	202.5000	202.7584	203.0030	203.2183	203.3848	203.4790	203.4790
201.7561	201.7561	201.8225	201.9446	202.1078	[illegible]	[illegible]	[illegible]	202.8922	203.0554	203.1775	203.2439	203.2439
201.9249	201.9249	201.9740	202.0663	202.1921	[illegible]	[illegible]	[illegible]	202.8079	202.9337	203.0260	203.0751	203.0751
202.0444	202.0444	202.0824	202.1546	202.2540	[illegible]	[illegible]	[illegible]	202.7460	202.8454	202.9176	202.9556	202.9556
202.1260	202.1260	202.1568	202.2155	202.2970	202.3944	202.5000	202.6056	202.7030	202.7845	202.8432	202.8740	202.8740
202.1768	202.1768	202.2032	202.2537	202.3240	202.4084	202.5000	202.5916	202.6760	202.7463	202.7968	202.8232	202.8232
202.2011	202.2011	202.2255	202.2721	202.3371	202.4152	202.5000	202.5848	202.6629	202.7279	202.7745	202.7989	202.7989
202.2011	202.2011	202.2255	202.2721	202.3371	202.4152	202.5000	202.5848	202.6629	202.7279	202.7745	202.7989	202.7989

$$h_{i,j} = \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}}{4}$$

References

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