INTRODUCTION TO ALGEBRAIC TOPOLOGY

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4. Homology of CW-complexes and applications

4.1. First applications of homology. Using homology groups we can easily prove the following statements:

(1) S^n is not a retract of D^{n+1}.
(2) Every map $f : D^n \to D^n$ has a fixed point, i.e. there is $x \in D^n$ such that $f(x) = x$.
(3) If $\emptyset \neq U \subseteq \mathbb{R}^n$ and $\emptyset \neq V \subseteq \mathbb{R}^m$ are open homeomorphic sets, then $n = m$.

Outline of the proof. (1) Suppose that there is a retraction $r : D^{n+1} \to S^n$. Then we get the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = H_n(S^n) & \xrightarrow{\quad \mathrm{id} \quad} & H_n(S^n) = \mathbb{Z} \\ & {\scriptstyle i_*}\searrow \qquad \nearrow{\scriptstyle r_*} & \\ & H_n(D^{n+1}) = 0 & \end{array}$$

which is a contradiction.
(2) Suppose that $f : D^n \to D^n$ has no fixed point. Then we can define the map $g : D^n \to S^{n-1}$ where $g(x)$ is the intersection of the ray from $f(x)$ to x with S^{n-1}. However, this map would be a retraction, a contradiction with (1).
(3) The proof of the last statement follows from the isomorphisms:

$$H_i(U, U-\{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n-\{x\}) \cong \tilde{H}_{i-1}(\mathbb{R}^n-\{x\}) \cong \tilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

□

4.2. Degree of a map. Consider a map $f : S^n \to S^n$. In homology $f_* : \tilde{H}_n(S^n) \to H_n(S^n)$ has the form

$$f_*(x) = ax, \quad a \in \mathbb{Z}.$$

The integer a is called the *degree* of f and denoted by $\deg f$.
The degree has the following properties:

(1) $\deg \mathrm{id} = 1$.
(2) If $f \sim g$, then $\deg f = \deg g$.
(3) If f is not surjective, then $\deg f = 0$.
(4) $\deg(fg) = \deg f \cdot \deg g$.
(5) Let $f : S^n \to S^n$, $f(x_0, x_1, \dots, x_n) = (-x_0, x_1, \dots, x_n)$. Then $\deg f = -1$.

(6) The antipodal map $f : S^n \to S^n$, $f(x) = -x$ has $\deg f = (-1)^{n+1}$.
(7) If $f : S^n \to S^n$ has no fixed point, then $\deg f = (-1)^{n+1}$.

Proof. We outline only the proof of (5) and (7). The rest is not difficult and left as an exercise.

We show (5) by induction on n. The generator of $\tilde{H}_0(S^0)$ is $1 - (-1)$ and f_* maps it in $(-1) - 1$. Hence the degree is -1. Suppose that the statement is true for n. To prove it for $n+1$ we use the diagram with rows coming from a suitable Mayer-Vietoris exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\cong} & \tilde{H}_n(S^n) & \longrightarrow & 0 \\ & & \downarrow f_* & & \downarrow (f/S^n)_* & & \\ 0 & \longrightarrow & \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\cong} & \tilde{H}_n(S^n) & \longrightarrow & 0 \end{array}$$

If $(f/S^n)_*$ is a multiplication by -1, so is f_*.

To prove (7) we show that f is homotopic to the antipodal map through the homotopy

$$H(x,t) = \frac{tf(x) - (1-t)x}{\|tf(x) - (1-t)x\|}.$$

□

Corollary. *S^n has a nonzero continuous vector field if and only if n is odd.*

Proof. Let S^n has such a field $v(x)$. We can suppose $\|v(x)\| = 1$. Then the identity is homotopic to antipodal map through the homotopy

$$H(x,t) = \cos t\pi \cdot x + \sin t\pi \cdot v(x).$$

Hence according to properties (2) and (6)

$$(-1)^{n+1} = \deg(-\operatorname{id}) = \deg(\operatorname{id}) = 1.$$

Consequently, n is odd.

On the contrary, if $n = 2k+1$, we can define the required vector field by prescription

$$v(x_0, x_1, x_2, x_3, \ldots, x_{2k}, x_{2k+1}) = (-x_1, x_0, -x_3, x_2, \ldots, -x_{2k+1}, x_{2k}).$$

□

Exercise. Prove the properties (3), (4) and (6) of the degree.

4.3. Local degree. Consider a map $f : S^n \to S^n$ and $y \in S^n$ such that $f^{-1}(y) = \{x_1, x_2, \ldots, x_m\}$. Let U_i be open disjoint neighbourhoods of points x_i and V a neighbourhood of y such that $f(U_i) \subseteq V$. Then

$$(f/U_i)_* : H_n(U_i, U_i - \{x_i\}) \cong H_n(S^n, S^n - \{x_i\}) = \mathbb{Z} \\ \longrightarrow H_n(V, V - \{y\}) \cong H_n(S^n, S^n - \{y\}) = \mathbb{Z}$$

is a multiplication by an integer which is called a *local degree* and denoted by $\deg f|x_i$.

Theorem A. *Let* $f : S^n \to S^n$, $y \in S^n$ *and* $f^{-1}(y) = \{x_1, x_2, \ldots, x_m\}$. *Then*

$$\deg f = \sum_{i=1}^{m} \deg f|x_i.$$

For the proof see [Hatcher], Proposition 2.30, page 136.

The suspension Sf of a map $f : X \to Y$ is given by the prescription $Sf(x,t) = (f(x), t)$.

Theorem B. $\deg Sf = \deg f$ *for any map* $f : S^n \to S^n$.

Proof. f induces $Cf : CS^n \to CS^n$. The long exact sequence for the pair (CS^n, S^n) and the fact that $SS^n = CS^n/S^n$ give rise to the diagram

$$\begin{array}{ccccc}
\tilde{H}_{n+1}(S^{n+1}) & \xrightarrow[\cong]{} & \tilde{H}_{n+1}(CS^n, S^n) & \xrightarrow[\cong]{\partial_*} & \tilde{H}_n(S^n) \\
\downarrow{\scriptstyle Sf_*} & & \downarrow{\scriptstyle Cf_*} & & \downarrow{\scriptstyle f_*} \\
\tilde{H}_{n+1}(S^{n+1}) & \xrightarrow[\cong]{} & \tilde{H}_{n+1}(CS^n, S^n) & \xrightarrow[\cong]{\partial_*} & \tilde{H}_n(S^n)
\end{array}$$

which implies the statement. □

Corollary. *For any* $n \geq 1$ *and given* $k \in \mathbb{Z}$ *there is a map* $f : S^n \to S^n$ *such that* $\deg f = k$.

Proof. For $n = 1$ put $f(z) = z^k$ where $z \in S^1 \subset \mathbb{C}$. Using the computation based on local degree as above, we get $\deg f = k$. The previous theorem implies that the degree of $S^{n-1}f : S^n \to S^n$ is also k. □

4.4. Computations of homology of CW-complexes. If we know a CW-structure of a space X, we can compute its cohomology relatively easily. Consider the sequence of Abelian groups and its morphisms

$$(H_n(X^n, X^{n-1}), d_n)$$

where d_n is the composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_n(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}, X^{n-2}).$$

Theorem. *Let* X *be a CW-complex.* $(H_n(X^n, X^{n-1}), d_n)$ *is a chain complex with homology*

$$H_n^{CW}(X) \cong H_n(X).$$

Proof. First, we show how the groups $H_k(X^n, X^{n-1})$ look like. Put $X^{-1} = \emptyset$ and $X^0/\emptyset = X^0 \sqcup \{*\}$. Then

$$H_k(X^n, X^{n-1}) = \tilde{H}_k(X^n/X^{n-1}) = \tilde{H}_k(\bigvee S^n_\alpha) = \begin{cases} \bigoplus_\alpha \mathbb{Z} & n = k, \\ 0 & n \neq k. \end{cases}$$

Now we show that

$$H_k(X^n) = 0 \quad \text{for } k > n.$$

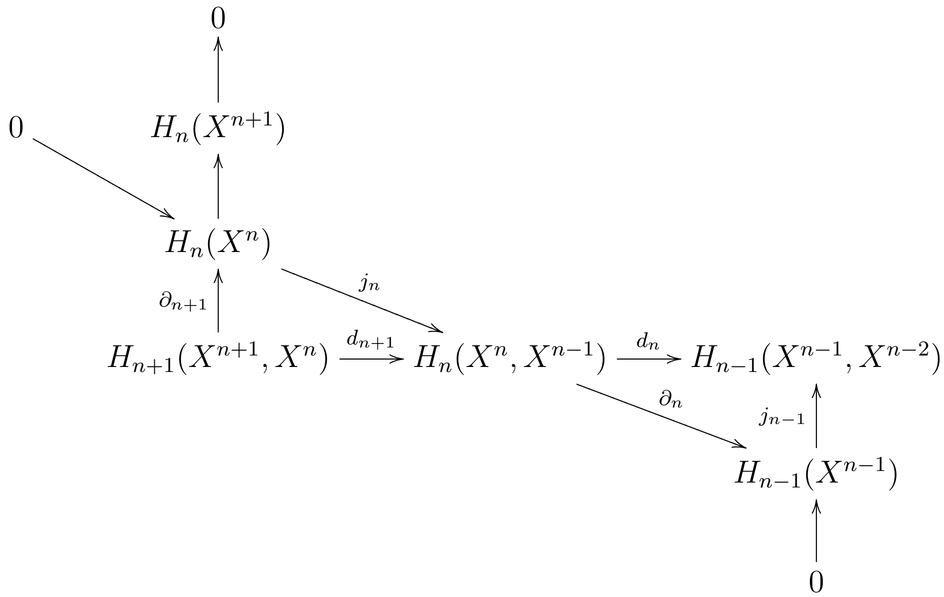
From the long exact sequence of the pair (X^n, X^{n-1}) we get $H_k(X^n) = H_k(X^{n-1})$. By induction $H^k(X^n) = H_k(X^{-1}) = 0$.

Next we prove that

$$H_k(X^n) = H_k(X) \quad \text{for } k \leq n-1.$$

From the long exact sequence for the pair (X^{n+1}, X^n) we obtain $H_k(X^n) = H_k(X^{n+1})$. By induction $H_k(X^n) = H_k(X^{n+m})$ for every $m \geq 1$. Since the image of each singular chain lies in some X^{n+m} we get $H_k(X^n) = H_k(X)$.

To prove Theorem we will need the following diagram with parts of exact sequences for the pairs (X^{n+1}, X^n), (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}).



From it we get

$$d_n d_{n+1} = j_{n-1}(\partial_n j_n)\partial_{n+1} = j_{n-1}(0)\partial_{n+1} = 0.$$

Further,

$$\operatorname{Ker} d_n = \operatorname{Ker} \partial_n = \operatorname{Im} j_n \cong H_n(X^n)$$

and

$$\operatorname{Im} d_{n+1} \cong \operatorname{Im} \partial_{n+1},$$

since j_{n-1} and j_n are monomorphisms. Finally,

$$H_n^{CW}(X) = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}} \cong \frac{H_n(X^n)}{\operatorname{Im} \partial_{n+1}} \cong H_n(X^{n+1}) \cong H_n(X).$$

□

Example. $H_n(X) = 0$ for CW-complexes without cells in dimension n.

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{for } k \leq 2n \text{ even,} \\ 0 & \text{in other cases.} \end{cases}$$

4.5. Computation of d_n. Let e^n_α and e^{n-1}_β be cells in dimension n and $n-1$ of a CW-complex X, respectively. Since

$$H_n(X^n, X^{n-1}) = \bigoplus_\alpha \mathbb{Z}, \quad H_{n-1}(X^{n-1}, X^{n-2}) = \bigoplus_\beta \mathbb{Z},$$

they can be considered as generators of these groups. Let $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ be the attaching map for the cell e^n_α. Then

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the following composition

$$S^{n-1} = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \to X^{n-1}/X^{n-2} \to X^n/(X^{n-2} \cup \bigcup_{\gamma\neq\beta} e^{n-1}_\gamma) = S^{n-1}.$$

For the proof we refer to [Hatcher], pages 140 and 141.

Exercise. Compute homology groups of various 2-dimensional surfaces (torus, Klein bottle, projective plane) using their CW-structure with only one cell in dimension 2.

4.6. Homology of real projective spaces. The real projective space $\mathbb{RP}^n$ is formed by cell $e^0, e^1, \dots, e^n$, one in each dimension from 0 to n. The attaching map for the cell e^{k+1} is the projection $\varphi : S^k \to \mathbb{RP}^k$. So we have to compute the degree of the composition

$$f : S^k \xrightarrow{\varphi} \mathbb{RP}^k \to \mathbb{RP}^k/\mathbb{RP}^{k-1} = S^k.$$

Every point in S^k has two preimages x_1, x_2. In a neihbourhood U_i of x_i f is a homeomorphism, hence its local degree $\deg f|x_i = \pm 1$. Since f/U_2 is the composition of the antipodal map with f/U_1, the local degrees $\deg f|x_1$ and $\deg f|x_1$ differs by the multiple of $(-1)^{k+1}$. (See the properties (4) and (6) in 4.2.) According to 4.3

$$\deg f = \pm 1(1 + (-1)^{k+1}) = \begin{cases} 0 & \text{for } k+1 \text{ odd,} \\ \pm 2 & \text{for } k+1 \text{ even.} \end{cases}$$

So we have obtained the chain complex for computation of $H^{CW}_*(\mathbb{RP}^n)$. The result is

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{for } k=0 \text{ and } k=n \text{ odd,} \\ \mathbb{Z}_2 & \text{for } k \text{ odd }, 0<k<n, \\ 0 & \text{in other cases.} \end{cases}$$

4.7. Euler characteristic. Let X be a finite CW-complex. The *Euler characteristic* of X is the number

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^k \operatorname{rank} H_k(X).$$

Theorem. *Let X be a finite CW-complex with c_k cells in dimension k. Then*

$$\chi(X) = \sum_{k=0}^{\infty}(-1)^k c_k.$$

Proof. Realize that $c_k = \operatorname{rank} H_k(X^k, X^{k-1}) = \operatorname{rank}\operatorname{Ker} d_k + \operatorname{rank}\operatorname{Im} d_{k+1}$ and that $\operatorname{rank} H_k(X) = \operatorname{rank}\operatorname{Ker} d_k - \operatorname{rank}\operatorname{Im} d_{k+1}$. Hence

$$\begin{aligned}\chi(X) &= \sum_{k=0}^{\infty}(-1)^k \operatorname{rank} H_k(X) = \sum_{k=0}^{\infty}(-1)^k(\operatorname{rank}\operatorname{Ker} d_k - \operatorname{rank}\operatorname{Im} d_{k+1})\\ &= \sum_{k=0}^{\infty}(-1)^k \operatorname{rank}\operatorname{Ker} d_k + \sum_{k=0}^{\infty}(-1)^k \operatorname{rank}\operatorname{Im} d_k = \sum_{k=0}^{\infty}(-1)^k c_k.\end{aligned}$$

□

Example. 2-dimensional oriented surface of genus g (the number of handles attached to the 2-sphere) has the Euler characteristic $\chi(M_g) = 2 - 2g$.

2-dimensional nonorientable surface of genus g (the number of Möbius bands which replace discs cut out from the 2-sphere) has the Euler characteristic $\chi(N_g) = 2 - g$.

4.8. Lefschetz Fixed Point Theorem. Let G be a finitely generated Abelian group and $h : G \to G$ a homomorphism. The trace $\operatorname{tr} h$ is the trace of the homomorphism

$$\mathbb{Z}^n \cong G/\operatorname{Torsion} G \to G/\operatorname{Torsion} G \cong \mathbb{Z}^n$$

induced by h.

Let X be a finite CW-complex. The *Lefschetz number* of a map $f : X \to X$ is

$$L(f) = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr} H_i f.$$

Notice that $L(\mathrm{id}_X) = \chi(X)$. Similarly as for the Euler characteristic we can prove

Lemma. *Let $f_n : (C_n, d_n) \to (C_n, d_n)$ be a chain homomorphism. Then*

$$\sum_{i=0}^{\infty}(-1)^i \operatorname{tr} H_i f = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr} f_i$$

whenever the right hand side is defined.

Theorem (Lefschetz Fixed Point Theorem)**.** *If X is a finite simplicial complex or its retract and $f : X \to X$ a map with $L(f) \neq 0$, then f has a fixed point.*

For the proof see [Hatcher], Chapter 2C. Theorem has many consequences.

Corollary A (Brouwer Fixed Point Theorem)**.** *Every continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. The Lefschetz number of f is 1. □

In the same way we can prove

Corollary B. *If n is even, then every continuous map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point.*

Corollary C. *Let M be a smooth compact manifold in $\mathbb{R}^n$ with nonzero vector field. Then $\chi(M) = 0$.*

The converse of this statement is also true.

Outline of the proof. If M has a nonzero vector field, there is a continuous map $f : M \to M$ which is a "small shift in the direction of the vector field". Since such a map has no fixed point, its Lefschetz number has to be zero. Moreover, f is homotopic to identity and hence

$$\chi(M) = L(\mathrm{id}_X) = L(f) = 0.$$

□

4.9. Homology with coefficients. Let G be an Abelian group. From the singular chain complex $(C_n(X), \partial_n)$ of a space X we make the new chain complex

$$C_n(X; G) = C_n(X) \otimes G, \quad \partial_n^G = \partial_n \otimes \mathrm{id}_G\,.$$

The homology groups of X with coefficients G are

$$H_n(X; G) = H_n(C_*(X; G), \partial_*^G).$$

The homology groups defined before are in fact the homology groups with coefficients $\mathbb{Z}$. The homology groups with coefficients G satisfy all the basic general properties as the homology groups with integer coefficients with the exception that

$$H_n(;G) = \begin{cases} 0 & \text{for } n \neq 0, \\ G & \text{for } n = 0. \end{cases}$$

If the coefficient group G is a field (for instance $G = \mathbb{Q}$ or $\mathbb{Z}_p$ for p a prime), then homology groups with coefficients G are vector spaces over this field. It often brings advantages.

The computation of homology with coefficients G can be carried out again using a CW-complex structure. For instance, we get

$$H_k(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq k \leq n, \\ 0 & \text{in other cases.} \end{cases}$$

For an application of $\mathbb{Z}_2$-coefficients see the proof of the following theorem in [Hatcher], pages 174–176.

Theorem (Borsuk-Ulam Theorem)**.** *Every map $f : S^n \to S^n$ satisfying*

$$f(-x) = -f(x)$$

has an odd degree.

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